A Hybrid, Dynamic Logic for Hybrid-Dynamic Information Flow

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Abstract

Information-flow security is important to the safety and privacy of cyber-physical systems (CPSs) across many domains: information leakage can both violate user privacy and reveal vulnerabilities to physical attacks. CPSs face the challenge that information can flow both in discrete cyber channels and in continuous real-valued physical channels ranging from time to motion to (e.g.) electrical currents. We call these *hybrid-dynamic information flows* (HDIFs) and introduce dHL, the first logic for verifying HDIFs in hybrid-dynamical models of CPSs. Our logic extends differential dynamic logic (dL) for hybrid-dynamical systems with hybrid-logical features for explicit program state representation, supporting relational reasoning used for information flow arguments. By verifying HDIFs, we ensure security even under a strong attacker model wherein an attacker can observe time and physical values continuously. We present a Hilbert-style proof calculus for dHL, prove it sound, and compare the expressive power of dHL with dL. We develop a hybrid system model based on the smart electrical grid FREEDM, with which we showcase dHL. We prove that the naïve model has a previously unknown information flow vulnerability, which we verify is resolved in a revised model. This is the first information flow proof both for HDIFs and for a hybrid-dynamical model in general.

1 Introduction

Cyber-physical systems (CPSs), which feature discrete computer control interacting with a continuous physical environment, are ubiquitous. They include critical infrastructure such as electricity, natural gas, and petroleum transportation grids, medical devices such as pacemakers and insulin pumps, and transportation systems including aircraft, trains, and automobiles. Because these systems are critical, it is essential to ensure their safe, correct operation, and formal methods for safety (e.g., collision-freedom) in CPS have had important successes [37, 16, 21].

There is less work on formal methods for CPS (information-flow) security, which is often as critical as physical safety. We focus on *nondeducibility* [4] of information flow, which captures the notion that an attacker cannot infer private information for certain in a system with non-observable nondeterminism. We choose this focus because for many nondeterministic systems, nondeducibility is the strongest property one can hope to achieve. As consumer-facing infrastructure such as electrical and telecommunication networks are increasingly computerized, the risk of leaking confidential customer information increases. Beyond leaking customer information, it has been suggested [3] that information flow leaks have the potential to aid attackers in identifying vulnerable infrastructure targets. Computerized medical devices risk leaking medical records protected by law (e.g., HIPAA), which can enable attacks that have life-threatening results, such as ventricular fibrillation [20]. Information leakage concerns are also significant in the transportation domain, e.g., position-spoofing attacks in aircraft have been proposed that could cause major disruptions to air-traffic control [43].

The common feature is that information flows in both computer communication channels and physical channels such as transmission lines, pipelines, the human body [41], and roadways. Because information flows through both computation and physics, CPSs demand a notion of flow which accounts for both. While information flow in CPS has been explored [1, 3, 19, 49, 25], prior works either model physics discretely or ignore physics altogether [3].

These abstractions constitute a significant *model gap*: Formal analyses of any model can only be trusted to the extent that the model is faithful to reality and to the abilities of attackers. Event-based models such as the prior model of the FREEDM grid [3], for example, assume attackers cannot observe time or exact physical quantities. Our hybrid (i.e., mixed continuous and discrete) dynamics narrow the gap greatly, modeling attackers that observe continuous time and real-valued physical quantities. In doing so, our FREEDM model reveals a leak undetected by the prior model.

A discrete-time model would also leave a smaller gap than and and reveal more bugs than an event model. A key advantage of hybrid models is they support continuous time, so we know for sure that we have accounted for the timing abilities of all possible attackers, while a discrete-time model would leave us uncertain whether the model is precise enough to reveal all practical attacks.

We investigate properties of *hybrid-dynamic information flows* (HDIFs), which combine discrete and continuous flows. We provide an approach for verifying HDIF security by introducing the logic dHL. The dHL logic features two forms of hybridness which should not be confused: it extends differential dynamic logic (dL) for reachability of hybrid-dynamical *systems* with first-order hybrid *logic* [8], which provides first-class representation of program states. This combination of hybrid dynamics with first-class program states enables us to verify HDIF security. In capturing physical and temporal phenomena, hybrid dynamics also provide a flexible framework for model-

ing side-channels and verifying them with the same techniques as other cyber-physical channels.

The distinguishing feature of information flow (vs. safety and liveness properties) is that it is not a trace property, but a 2-trace hyperproperty [13] (i.e., a property of pairs of program traces). This poses a hurdle for program verification calculi: Hoare calculi (and typical dynamic logics [44]), for example, cannot verify hyperproperties without significant source-level transformations such as self-composition [6]. These source transformations have been noted [46] to make verification tasks needlessly difficult in practice by inflating their size. Relational calculi [7] reduce verification complexity, but also generality.

Our novel use of hybrid logic not only reduces complexity but also maintains generality by allowing first-class representation of program states, which makes both direct statements and proofs of information-flow properties straightforward. In the process, we display a novel connection with hybrid logic that extends beyond information flow to general hyperproperties. Beyond its aesthetic appeal, this generality promises to enable verification of numerous related hyperproperties in one common logic, without having to adapt the logic to: i) more notions of security such as non-interference ii) more hyperproperties such as robustness [13]. We introduce a Hilbert-style proof calculus for dHL and prove it sound, then derive high-level rules for bisimulation. We relate the complexity of proofs in dHL vs. dL: a reduction is possible for a significant fragment of dHL, but impractical: i) it has limitations when applied to the general case, interfering with advanced proof techniques such as refinement [26] for modular verification, ii) the reduction causes quadratic formula size blowup in the worst case, and iii) the reduction is surprisingly subtle, suggesting that a proof by reduction to dL would be needlessly long and unintuitive.

As an example application, we give the first hybrid-dynamical model of a smart grid controller with concrete dynamics for distributed energy generation/storage and load-balancing [2] based on published descriptions of the FREEDM grid [22]. This contrasts with prior models [3], which consider only the high-level structure of event-based interactions between components. Our model shows the importance of dynamical-level modeling by revealing an information flow bug uncaught by higher-level models. We then prove a revised model secure even in the presence of HDIFs.

Our model of FREEDM captures its essential hybrid dynamics and our proof demonstrates important features of proofs in dHL: i) The well-understood principle of proof by bisimulation translates naturally to dHL proofs, ii) dHL provides an effective mechanism to tease apart the interactions between discrete transitions and continuous flow, enabling verification to scale to the complex interactions found in CPSs, and iii) typical CPSs have sufficiently complex information flows to warrant the deductive approach.

These traits are typical across different domains of CPS, showing that our approach holds promise for verifying information flow of applications in various domains beyond smart grids.

Security of cyber-physical systems other than smart-grids has been investigated, though rarely through the lens of formal logic. Water canal systems have been shown to be vulnerable to attacks that steal water while avoiding detection [5]. Smart homes can leak private information about activities of daily living, which can be HIPAA-protected, e.g. in assisted-living scenarios [45]. In (automotive) vehicular ad-hoc networks (VANETs), information leaks can compromise private information such as travel history [40], which in turn enables crimes [14] such as vehicular theft and abduction.

2 The Logic dHL

We present the complete syntax and semantics of the logic dHL, extending the dynamic logic dL with explicit hybrid-logical representation of program states. Our calculus, as with modern implementations [18] and machine-checked correctness proofs [10] for dL, is based on uniform substitution [12, §35][38]: symbols ranging over predicates, programs, etc. are explicitly represented in the syntax. This improves the ease with which dHL can be implemented and its soundness proof checked mechanically in future work.

The expressions of dHL consist of real-valued terms θ, world-valued terms w, programs α, and formulas ϕ. We write Θ for an arbitrary term θ or w, and write e when an expression can be either a term Θ or a formula ϕ, but not a program α.

Definition 1 (Real-valued terms of dHL)**.**

$$\theta ::= c \mid x \mid f(\vec{\theta}) \mid F \mid \theta + \theta \mid \theta \cdot \theta \mid @_i\theta$$

Here $c \in \mathbb{Q}$ is a literal and x is a real-valued *program variable*, said to be *flexible* because it can be bound in quantifiers. Their *rigid* counterparts are *nullary function symbols* $f(), g()$ that cannot be bound. The meaning of a function symbol $f(\vec{\theta})$ depends on an arbitrary number of real-valued arguments. Functionals F are a generalization of functions whose meaning depends on all flexible symbols. Functions and functionals are used to express axioms in Section 5. Terms in dHL add to dL at-terms $@_w\theta$ denoting the value of term θ in the state denoted by the world-valued term w.

Definition 2 (World-valued terms of dHL)**.**

$$w ::= s \mid \overline{n}$$

The language of world-valued terms w is simple, consisting only of *world variables* s, t and *nominals* $\overline{n}, \overline{m}$, which differ only in that world variables are flexible while nominals are rigid.

Definition 3 (Programs of dHL)**.**

$$\alpha, \beta ::= \ ?(\phi) \mid x := \theta \mid x := * \mid x' = \theta \,\&\, \psi \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid a$$

The hybrid program constructs of dHL are simply those of dL. Hybrid programs combine discrete programming constructs with differential equations to provide a program representation of hybrid systems. The atomic dL programs are tests $?(\phi)$ that abort execution if formula ϕ is false, assignments $x := \theta$ and $x := *$ which update program variable x to the value of term θ or a nondeterministic value, differential equation evolution $x' = \theta \,\&\, \psi$, and object-level program constants a which range over fixed, arbitrary programs. They should not be confused with the similar-looking program metavariables α used in schemata and theorems. Differential equations are the defining feature of dL; the effect of $x' = \theta \,\&\, \psi$ is to evolve the differential equation $x' = \theta$ nondeterministically for any duration, but only so long as the formula ψ is always true.

They are composed with nondeterministic choice $\alpha \cup \beta$ that runs exactly one of α or β, sequential composition $\alpha; \beta$, and nondeterministic iteration α^* that runs α any finite number of times

sequentially. Traditional deterministic programming constructs can be derived from the nondeterministic hybrid program constructs, e.g., $\texttt{if}\ (\phi)\{\alpha\}\ \texttt{else}\ \{\beta\} \equiv (?(\phi);\alpha) \cup (?(\neg\phi);\beta)$.

As an introductory (toy) example of a hybrid system, consider the following simplistic model of a diesel generator.

Example 1 (Hybrid System for a Diesel Generator).

$$\alpha_{\text{gen}} \overset{\text{def}}{\equiv} ((p := 0 \ \cup\ (p := *;\ ?(\mathit{Fuel} > 0 \ \wedge\ 0 \leq p \leq \mathit{pmax}));\\ \{\mathit{Fuel}' = -p,\ \mathit{gr}' = p \,\&\, \mathit{Fuel} \geq 0\})^*$$

The controller features two branches ($\cup$), which control the **p**ower output p: The first branch says we can always choose to turn the generator off ($p := 0$) while the second branch lets us choose any value ($p := *$) so long as there is fuel left ($\mathit{Fuel} > 0$) and the power level is within the generator's capability ($0 \leq p \leq \mathit{pmax}$). The plant is a system of differential equations where the fuel decreases continuously in proportion to the power level $\mathit{Fuel}' = -p$ and the total energy sent to the grid (gr) increases continuously in proportion to power level ($\mathit{gr}' = p$) but never so long that fuel would become negative ($\mathit{Fuel} \geq 0$). The controller and plant are repeated in a loop an arbitrary number of times.

Definition 4 (Formulas of dHL)**.**

$$\phi, \psi ::= \ \phi \wedge \psi \mid \neg\phi \mid \exists x : \mathbb{R}\ \phi \mid \theta_1 \geq \theta_2 \mid \langle\alpha\rangle\phi \\ \mid \exists s : \mathcal{W}\ \phi \mid w \mid @_w\phi \mid \downarrow s\ \phi \mid p(\vec{\Theta}) \mid P$$

Formulas $\phi \wedge \psi, \neg\phi, \exists x : \mathbb{R}\ \phi$, and $\theta_1 \geq \theta_2$ are as in first-order logic. As in dL, the diamond modality $\langle\alpha\rangle\phi$ says there exists an execution of the (nondeterministic) program α where formula ϕ holds in the ending state. Its dual, the box modality $[\alpha]\phi$, says *all* end states satisfy ϕ, and is derived: $[\alpha]\phi \equiv \neg\langle\alpha\rangle\neg\phi$. These modalities are commonly used to express partial correctnesss assertions ($P \rightarrow [\alpha]Q$) and total correctness assertions ($P \rightarrow \langle\alpha\rangle Q$) familiar from Hoare logic. We give example safety and liveness properties of Ex. 1.

Example 2 (A Safety Property for the Generator). The formula

$$\mathit{Fuel} \geq 0 \rightarrow [\alpha_{\text{gen}}]\mathit{Fuel} \geq 0$$

that the fuel level shall never go negative so long as it is initially nonnegative. Note the word "safety" is used in a technical sense to mean any property of shape $[\alpha]Q$, not only those that align with an intuitive notion of making a system safe.

Example 3 (A Liveness Property for the Generator). The formula

$$\mathit{Fuel} > 0 \wedge \mathit{pmax} > 0 \rightarrow \langle\alpha_{\text{gen}}\rangle\mathit{Fuel} = 0$$

says that assuming we start with fuel in the tank and nonzero maximum power, it is always possible to empty the tank by running the generator long enough.

The above examples are formulas of the base logic dL. In dHL, we extend dL with the following features from first-order hybrid logic. The quantifier $\exists s:\mathcal{W}\ \phi$ says there exists a world (program state) s in which ϕ holds (where ϕ can mention the world variable s). We will also use the universal quantifier $\forall s:\mathcal{W}\ \phi$, which is a derived construct by the duality $\forall s:\mathcal{W}\ \phi \equiv \neg\exists s:\mathcal{W}\ \neg\phi$. We support nominal *propositions* w that hold in exactly the one state denoted by the world-valued term w. These allow testing equality of the current state against w. Note the same syntax is used regardless whether w appears as a term or formula; these usages are distinguished by syntactic context. The hybrid *satisfaction modality* $@_w\phi$ says that ϕ is true at the unique state named by w. In addition to the typical existential and universal quantifiers, hybrid logic features the *local* quantifier $\downarrow s\ \phi$ which binds the *current* state to the world variable s within the formula ϕ, whereas the universal quantifier $\forall s:\mathcal{W}\ \phi$ binds an *arbitrary* state to s. The local quantifier $\downarrow s\ \phi$ can be derived as $\downarrow s\ \phi \leftrightarrow \exists s:\mathcal{W}\ (s \wedge \phi)$ or equivalently $\forall s:\mathcal{W}\ (s \rightarrow \phi)$. We present this quantifier in its entirety regardless, because it is important to information-flow applications and may be unfamiliar to the reader. The connectives $\downarrow s\ \phi$ and $@_w\phi$ can be understood computationally as well, as storing or loading the current state to s or from w, respectively.

The *predicate symbols* $p(\vec{\Theta})$ range over both real-valued terms θ and nominal expressions w, and are used in axioms to stand for propositions. Beyond axioms, predicates will be used widely in bisimulation arguments for information-flow: $R(i,j)$ denotes a binary predicate over nominals. Predicationals P simply stand for arbitrary formulas and are used in axioms in Section 5.

3 Information-Flow Example: FREEDM Grid

In this section, we set aside the toy example, Ex. 1, and introduce two variants of a smart grid model based on NSF FREEDM [22], a *microgrid* which controls a local section of the power grid and interacts with the surrounding *macrogrid*. Our model is the first hybrid-systems model of FREEDM and follows the published algorithm [2], incorporating detailed dynamics not present in prior models [3]. We show how information-flow security properties and their negations are stated in dHL. We prove them in Secs. 8 and 9 once the proof calculus is introduced.

Smart grids like FREEDM use computer control to make electrical grids more robust, efficient, and cost-effective in face of increasingly diverse power loads and supplies. Computer control in grids makes joint cyber-physical security of this critical infrastructure essential. Not only can information flow violations compromise private consumer information, but it has been suggested they can aid attackers [3] in identifying targets for physical attacks.

3.1 Scenario

We look at an exchange (depicted in Fig. 1) that migrates power between two neighboring transformers T_1 and T_2 connected to a macrogrid gr over a shared line. Variable names indicate units: energy is uppercase, power (derivative of energy, e.g. $B_i' = b_i$) is lowercase, and migration rates (derivative of power, e.g. $b_i' = bm_i$) end in m. Each transformer T_i carries power p_i and is connected to a renewable energy resource r_i, to a household which demands power d_i, and to a energy storage device B_i. The transformers are connected by a communication *Link*. While real instances

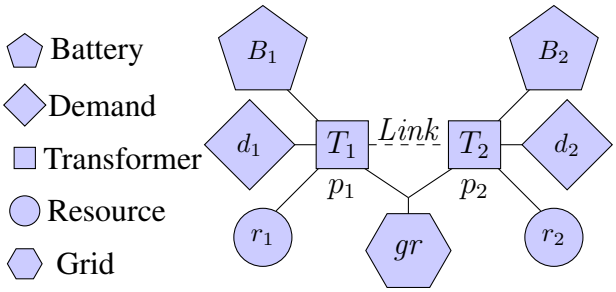

Figure 1: FREEDM load balancing

of the FREEDM grid have many transformers, each migration involves exactly two transformers, so the two-transformer case provides important insight for the general case.

Each transformer can be in one of three demand states: *Low Demand*, *Normal Demand*, or *High Demand*. The algorithm [2] states:

- Net demand n_i is the difference of gross power demand d_i and the sum of power draw p_i with generation r_i.
- A transformer is in *Low Demand* if it has net demand $n_i < 0$, *High Demand* if net demand exceeds a provided threshold $n_i \geq thresh > 0$ or *Normal* otherwise.
- If any transformer i is *Low* (has excess power) while the other (written $\bar{i}$) is *High*, power migrates at a provided constant rate $m := maxm$ until at least one of them is *Normal*.
- Any excess power supply $-n_i > 0$ not used in migration is accumulated as energy in battery i subject to $0 \leq B_i \leq B_{\max}$.
- Any excess demand $n_i > 0$ not met by migration is drawn from the battery B_i with power b_i and migration rate bm_i, or sold to the grid if the battery is full ($B_i = B_{\max}$).
- If T_i's corresponding battery B_i is empty, it draws power gr (with migration rate $gr' = grm$) from the macrogrid instead.

The grid is modeled in Fig. 2 as a hybrid program α_F, which contains the controller (ctrl) and physical model (plant). The controller migrates power (migrate) and operates the battery (bat), which has two implementations: a deterministic implementation bat_I of the above algorithm, which we show to be *insecure*, and a nondeterministic version bat_S, which we show to be *secure*. We write $\alpha_I \equiv \alpha_{F\,\mathsf{bat}}^{\mathsf{bat}_I}$ and $\alpha_S \equiv \alpha_{F\,\mathsf{bat}}^{\mathsf{bat}_S}$ to instantiate α_F with the insecure or secure battery, respectively.

Our treatment of d_i and r_i is general, assuming only that they are non-negative and can change countably often. Time $t' = 1$ is not used for control, but will factor into our proofs because it is observed by attackers and we prove that, e.g., observing the ODE duration does not leak the continuous variables p_i, B_i, b_i.

$$\alpha_F \equiv (\mathsf{ctrl};\mathsf{plant})^* \qquad \mathsf{ctrl} \equiv \mathsf{migrate};\mathsf{bat}$$

$$\begin{aligned}
\mathsf{migrate} \equiv \{\ & d_i := *;\ ?(d_i \geq 0); \quad r_i := *;\ ?(r_i \geq 0); \quad n_i := d_i - (r_i + p_i); \\
& \texttt{if}\ (n_i \geq \mathit{thresh} \wedge n_{\bar{i}} < 0) \quad \{\ m := (-1)^i \cdot \mathit{maxm}\} \\
& \texttt{else} \qquad\qquad\qquad\quad \{\ m := 0\}\}
\end{aligned}$$

$$\mathsf{plant} \equiv \{p_i' = -1^i \cdot m,\ B_i' = b_i,\ b_i' = bm_i,\ gr' = grm,\ t' = 1 \,\&\, B_i \geq 0\}$$

$$\begin{aligned}
\mathsf{bat}_I \equiv \{\ & gr := 0;\ bm_i := 0;\ grm := 0; \\
& \mathbf{if}\ ((\mathbf{n_i} \leq \mathbf{0}\ \wedge\ B_i < B_{\max}) \vee (n_i > 0 \wedge B_i > 0))\{ \\
& \quad \{\ b_i := -n_i;\ bm_i := bm_i + (-1^{i+1}) \cdot m\} \\
& \mathbf{else}\ \{\ b_i := 0;\ gr := gr + n_i;\ grm := grm + (-1)^{i+1} \cdot m\}\}
\end{aligned}$$

$$\begin{aligned}
\mathsf{bat}_S \equiv \{\ & gr := 0;\ bm_i := 0;\ grm := 0; \\
& \big(\mathbf{?}(B_i < B_{\max}) \vee (n_i > 0 \wedge B_i > 0); \\
& \quad b_i := -n_i;\ bm_i := bm_i + (-1)^{i+1} \cdot m\big) \\
& \cup \big(b_i := 0;\ gr := gr + n_i;\ grm := grm + (-1)^{i+1} \cdot m\big)\ \}
\end{aligned}$$

Figure 2: FREEDM model with insecure and secure batteries

3.2 Defining Information Flow.

We give the general formulation:

Definition 5 (Nondeducibility Information-Flow Security)**.** Let α be a program and let L be the set of publicly observable expressions, e.g., $L = \{gr, t\}$ for FREEDM with publicly-known time t and macrogrid flow gr. At its simplest, binary relation $R(i,j)$ says worlds i and j agree on all public expressions L, and is defined by:

$$R(i,j) \ \equiv\ \left(\bigwedge_{\theta \in L} (@_i\theta = @_j\theta)\right) \wedge \left(\bigwedge_{\phi \in L} (@_i\phi \leftrightarrow @_j\phi)\right)$$

In practice, $R(i,j)$ often also states that i and j satisfy any problem-specific constraints, e.g. on the range of system parameters. Now we define nondeducibility information flow:

$$\forall i_1, i_2, o_1 : \mathcal{W}\ \big(@_{i_1}\langle\alpha\rangle o_1 \wedge R(i_1,i_2) \rightarrow\ @_{i_2}\langle\alpha\rangle{\downarrow}o_2\ R(o_1,o_2)\big)$$

Nondeducibility means an observer can never deduce anything about the input value $@_{i_1}x$ of a private variable $x \notin L$ from the final values of public expressions $@_{o_1}(\theta$ or $\phi)$, when the observer does not directly observe how nondeterminism is resolved. This is the case when Def. 5 holds because it says *every* input state i_2 that agrees on public terms ($R(i_1,i_2)$) has at least one program path $@_{i_2}\langle\alpha\rangle o_2$ (where o_2 is the final state of α, as bound by the quantifier $\downarrow o_2$) that would explain the final values of public expressions at o_1. Because all inputs would have made the output possible, it is impossible to deduce anything about the input state. As we will see in Section 9, the core challenge in proving this property is identifying *which* program path explains any given final public

value: for a nondeterministic program α, only by carefully resolving nondeterminism will we find a path that ensures secure flow.

In Section 8 we will also prove bat_I is nondeducibility *insecure*, by proving the negation of nondeducibility security, i.e.:

$$\exists i_1, i_2, o_1 : \mathcal{W} \left(@_{i_1} \langle \alpha \rangle o_1 \wedge R(i_1, i_2) \wedge @_{i_2} [\alpha_I] \downarrow o_2 \, \neg R(o_1, o_2) \right)$$

4 Semantics

We return to developing dHL. Its semantic development begins with our semantic building blocks. The building blocks are *worlds* ω, μ, ν, *galaxies* g, h, and *interpretations* I, J. Interpretations give meaning to *rigid* symbols, whose meanings are fixed throughout a formula, such as functions, predicates, nominal constants, and program constants. The *flexible symbols*, whose meaning can change throughout a program or formula, are real-valued *program variables* x, y and world-valued *world variables* s, t. The program variables receive their values from the active world ω while world variables receive their values from the galaxy g, which contains an infinitude of worlds, one for each world variable. We write ω_x^r for the state that updates ω's value of program variable x to $r \in \mathbb{R}$ and likewise g_s^ω for updated galaxies.

Program and world variables are both drawn from a countable variable set $\mathcal{V}$ isomorphic to $\mathbb{N}$. The set of all worlds $\mathcal{W}$ is isomorphic to $\mathbb{R}^{\mathcal{V}}$. The set of all galaxies $\mathcal{G}$ is isomorphic to $\left(\mathbb{R}^{\mathcal{V}}\right)^{\mathcal{V}}$.

We write $I(f), I(p)$, etc. for the interpretation of a given symbol and $\mathcal{I}$ for the set of all interpretations. The types of each component are as follows, where $\wp(S)$ is the power set of S:

$$\begin{aligned} I(f) &\colon (\mathbb{R} \cup \mathcal{W}) \to \mathbb{R} \\ I(p) &\colon \wp(\mathbb{R} \cup \mathcal{W}) \\ I(F) &\colon \mathcal{G} \times \mathcal{W} \to \mathbb{R} \\ I(P) &\colon \mathcal{G} \to \wp(\mathcal{W}) \\ I(a) &\colon \mathcal{G} \to \wp(\mathcal{W} \times \mathcal{W}) \end{aligned}$$

That is, we present f and p as untyped and unary, as polyadic generalizations are straightforward.

Definition 6 (Real term semantics)**.** Let $\omega \in \mathcal{W}, g \in \mathcal{G}, I \in \mathcal{I}$, then:

$$[\![x]\!] I g \omega = \omega(x) \tag{1}$$
$$[\![c]\!] I g \omega = c \tag{2}$$
$$[\![\theta_1 + \theta_2]\!] I g \omega = [\![\theta_1]\!] I g \omega + [\![\theta_2]\!] I g \omega \tag{3}$$
$$[\![\theta_1 \cdot \theta_2]\!] I g \omega = [\![\theta_1]\!] I g \omega \cdot [\![\theta_2]\!] I g \omega \tag{4}$$
$$[\![@_w \theta]\!] I g \omega = [\![\theta]\!] I g \nu \text{ where } \nu = [\![w]\!] I g \omega \tag{5}$$
$$[\![f(\theta)]\!] I g \omega = I(f)([\![\theta]\!] I g \omega) \tag{6}$$
$$[\![F]\!] I g \omega = I(F)(g\omega) \tag{7}$$

Equations (1)–(4) describe polynomial terms as in dL. The new term construct of dHL is the at-term $@_w \theta$, whose meaning is the meaning of real term θ at the state denoted by world term w.

The language of world terms is quite simple, containing only rigid nominals $\bar{n}$ and flexible world variables s, deriving their meaning from the interpretation I or galaxy g, respectively. We write $[\![w]\!]Ig\omega$ for symmetry with real terms θ even though ω is unused.

Definition 7 (World term semantics)**.** Let $g \in \mathcal{G}, I \in \mathcal{I}$, then:

$$[\![\bar{n}]\!]Ig\omega = I(n)$$
$$[\![s]\!]Ig\omega = g(s)$$

Definition 8 (Program semantics)**.** The program semantics are as in dL, but with the addition of galaxies g. Let $\omega, \nu \in \mathcal{W}, g \in \mathcal{W}, I \in \mathcal{I}$, then:

$$(\omega,\omega) \in [\![?\phi]\!]Ig \text{ iff } \omega \in [\![\phi]\!]Ig \tag{8}$$
$$(\omega,\nu) \in [\![x := \theta]\!]Ig \text{ iff } \nu = \omega_x^r \text{ for } r = [\![\theta]\!]Ig\omega \tag{9}$$
$$(\omega,\nu) \in [\![x := *]\!]Ig \text{ iff } \nu = \omega_x^r \text{ for some } r \in \mathbb{R} \tag{10}$$
$$(\omega,\nu) \in [\![x' = \theta \,\&\, \psi]\!]Ig \text{ iff } (\omega,\nu) = (\varphi(0),\varphi(t)) \text{ and } \varphi \text{ solves}$$
$$x' = \theta \text{ on } [0,t] \text{ and } \varphi(s) \in [\![\psi]\!]Ig \text{ for all } s \in [0,t] \tag{11}$$
$$(\omega,\nu) \in [\![\alpha \cup \beta]\!]Ig \text{ iff } (\omega,\nu) \in [\![\alpha]\!]Ig \text{ or } (\omega,\nu) \in [\![\beta]\!]Ig \tag{12}$$
$$(\omega,\nu) \in [\![\alpha;\beta]\!]Ig \text{ iff } (\omega,\nu) \in ([\![\alpha]\!]Ig) \circ ([\![\beta]\!]Ig) \tag{13}$$
$$(\omega,\nu) \in [\![\alpha^*]\!]Ig \text{ iff } (\omega,\nu) \in ([\![\alpha]\!]Ig)^* \tag{14}$$
$$(\omega,\nu) \in [\![a]\!]Ig \text{ iff } (\omega,\nu) \in I(a)(g) \tag{15}$$

Galaxy g is untouched by execution. Equations (8)–(11) are the atomic hybrid programs. Differential equations (11) evolve according to the solution of the ODE for any duration $t \geq 0$, but must stop while the formula ψ still holds. In (13), $\circ$ is composition. In (14), $([\![\alpha]\!]Ig)^*$ is the reflexive, transitive closure of $[\![\alpha]\!]Ig$. Program constants (15) receive their meaning from the interpretation (and galaxy, since formulas inside programs can mention nominals).

Definition 9 (Formula semantics)**.**

$$\omega \in [\![\phi \wedge \psi]\!]Ig \text{ iff } \omega \in [\![\phi]\!]Ig \text{ and } \omega \in [\![\psi]\!]Ig \tag{16}$$
$$\omega \in [\![\neg\phi]\!]Ig \text{ iff } \omega \notin [\![\phi]\!]Ig \tag{17}$$
$$\omega \in [\![\exists x:\mathbb{R}\ \phi]\!]Ig \text{ iff } \omega_x^r \in [\![\phi]\!]Ig \text{ for some } r \in \mathbb{R} \tag{18}$$
$$\omega \in [\![\theta_1 \geq \theta_2]\!]Ig \text{ iff } [\![\theta_1]\!]Ig\omega \ \geq\ [\![\theta_2]\!]Ig\omega \tag{19}$$
$$\omega \in [\![\langle\alpha\rangle\phi]\!]Ig \text{ iff } \nu \in [\![\phi]\!]Ig \text{ for some } \nu \text{ s.t. } (\omega,\nu) \in [\![\alpha]\!]Ig \tag{20}$$
$$\omega \in [\![\exists s:\mathcal{W}\ \phi]\!]Ig \text{ iff } \omega \in [\![\phi]\!]Ig_s^\nu \text{ for some } \nu \in \mathcal{W} \tag{21}$$
$$\omega \in [\![@_w\phi]\!]Ig \text{ iff } \nu \in [\![\phi]\!]Ig \text{ for } \nu = [\![w]\!]Ig \tag{22}$$
$$\omega \in [\![\downarrow s\ \phi]\!]Ig \text{ iff } \omega \in [\![\phi]\!]Ig_s^\omega \tag{23}$$
$$\omega \in [\![w]\!]Ig \text{ iff } [\![w]\!]Ig\omega = \omega \tag{24}$$
$$\omega \in [\![p(\Theta)]\!]Ig \text{ iff } [\![\Theta]\!]Ig\omega \in I(p) \tag{25}$$
$$\omega \in [\![P]\!]Ig \text{ iff } \omega \in I(P)(g) \tag{26}$$

Equations (16)–(19) are a standard definition of first-order logic connectives, wherein we write $[\![\theta_i]\!]Ig\omega\colon \mathbb{R}$ for the denotation of real-valued term θ_i (Def. 6). Equation (20) defines the diamond modality $\langle\alpha\rangle\phi$: we employ a Kripke semantics where possible worlds are program states (Def. 8). Equations (21)–(24) define the hybrid-logical operators, where $[\![w]\!]Ig\colon \mathcal{W}$ (Def. 7) is the denotation of a world term. Equations (25)–(26) say predicates and predicationals derive their meaning from the interpretation I, with the difference that predicates depend only on their arguments while predicationals depend on all flexible symbols. We say formula ϕ is *valid* when the relation $\omega \in [\![\phi]\!]Ig$ holds for all states ω, galaxies g, and interpretations I.

5 Proof Calculus

We present a sound proof calculus for dHL, which is used for deductive verification. The calculus is given in Hilbert style, i.e., axioms are used wherever possible, with a minimal number of proof rules. Axioms are instantiated with a uniform substitution [38][12, §35] rule: from the validity of ϕ we can conclude validity of $\sigma(\phi)$ where substitution σ specifies concrete replacements for some or all rigid symbols in a formula ϕ:

$$\text{US}\ \frac{\phi}{\sigma(\phi)}$$

The side-conditions determining which substitutions σ are sound are non-trivial, and make up much of the soundness proof in Section 6, with the benefit that soundness arguments and implementation for individual axioms are greatly simplified.

5.1 Program Axioms

The axioms for programs in diamond modalities in Fig. 3 are as in dL [34]. Axioms for box modalities $[\alpha]\phi$ can be derived by duality (axiom $\langle\cdot\rangle$, Fig. 3). With the exception of the loop axioms, these axioms can be read off directly from the semantics of hybrid programs. The axiom $\langle'\rangle$ replaces a differential equation with a global solution, represented here by the expression[1] $y(t)$. Loops can be finitely unfolded with the axiom $\langle*\rangle$. More often, we reason by induction using axiom I or its derived rules.

5.2 Modal Axioms and Hilbert Rules

Generic modal axioms and Hilbert rules are as in dL [34] and are listed in Fig. 3. The axiom $\langle\cdot\rangle$ relates the diamond and box modalities, and is used to derive axioms for box modalities $[\alpha]\phi$. As is typical for Hilbert calculus, axioms are combined with rules G, US, and MP. Axiom V says nullary predicates $p()$ are preserved under program execution because they depend on no program variables.

[1]The presentation of this axiom is simplified for clarity. In reality, differential equation solving is implemented with the combination of several axioms [38].

$$\langle;\rangle \quad \langle a;b\rangle P \leftrightarrow \langle a\rangle\langle b\rangle P$$

$$\langle\cup\rangle \quad \langle a\cup b\rangle P \leftrightarrow (\langle a\rangle P \vee \langle b\rangle P)$$

$$\langle ?\rangle \quad \langle ?P\rangle Q \leftrightarrow (P\wedge Q)$$

$$\langle :=\rangle \quad \langle x:=f()\rangle p(x) \leftrightarrow p(f())$$

$$\langle :*\rangle \quad \langle x:=*\rangle p(x) \leftrightarrow \exists x:\mathbb{R}\, p(x)$$

$$\langle '\rangle \quad \langle x'=F\,\&\,q(x)\rangle p(x) \leftrightarrow \exists t{\geq}0(p(y(t)) \wedge \forall 0{\leq}s{\leq}t\; q(y(s)))$$

$$\langle *\rangle \quad \langle a^*\rangle P \leftrightarrow (P \vee \langle a\rangle\langle a^*\rangle P)$$

$$\text{I} \quad [a^*]P \leftrightarrow (P \wedge [a^*](P \rightarrow [a]P))$$

$$\text{G}\ \frac{P}{[a]P} \qquad \text{M}\ \frac{P\rightarrow Q}{\langle a\rangle P\rightarrow\langle a\rangle Q} \qquad \langle\cdot\rangle \quad \langle a\rangle P \leftrightarrow \neg[a]\neg P$$

$$\text{US}\ \frac{\phi}{\sigma(\phi)} \qquad \text{MP}\ \frac{P\rightarrow Q \quad P}{Q} \qquad \text{V} \quad p() \rightarrow [a]p()$$

$$\text{K} \quad [a](P\rightarrow Q)\rightarrow[a]P\rightarrow[a]Q$$

$$@\text{id} \quad @_{\overline{n}}\overline{n} \qquad \forall I_@ \ \frac{q(t)}{\forall s:\mathcal{W}\; q(s)}\text{(t fresh)} \qquad @{\leftrightarrow} \quad @_{\overline{n}}\overline{m} \rightarrow (p(\overline{n}) \leftrightarrow p(\overline{m}))$$

$$\exists W \quad \exists s:\mathcal{W}\; s \qquad @\text{I} \quad (\overline{n}\wedge P)\rightarrow @_{\overline{n}}P \qquad \forall E_@ \quad (\forall s:\mathcal{W}\; p(s))\rightarrow p(\overline{n})$$

$$\text{G}_@\ \frac{P}{@_{\overline{n}}P} \qquad @@ \quad @_{\overline{n}}@_{\overline{m}}P \leftrightarrow @_{\overline{m}}P \qquad \langle\overline{n}\rangle \quad ([a]{\downarrow}s\; p(s)\wedge\langle a\rangle\overline{n})\rightarrow p(\overline{n})$$

$$\downarrow \quad {\downarrow}s\; p(s) \leftrightarrow \exists s:\mathcal{W}\; (s\wedge p(s)) \qquad @\text{hom} \quad @_{\overline{n}}p(F_1,\ldots F_m) \leftrightarrow p(@_{\overline{n}}F_1,\ldots,@_{\overline{n}}F_m)$$

$$\text{K}_@ \quad @_{\overline{n}}(P\rightarrow Q)\rightarrow(@_{\overline{n}}P\rightarrow@_{\overline{n}}Q) \qquad \text{BW} \quad \langle\alpha\rangle\exists s:\mathcal{W}\; P \leftrightarrow \exists s:\mathcal{W}\;\langle\alpha\rangle P \quad (s\notin \text{FV}(\alpha))$$

Figure 3: dL axioms and rules, hybrid rules and axioms

5.3 Hybrid rules and axioms

Our hybrid modality and quantifier axioms come from first-order hybrid logic [8] and Combinatory Dynamic Logic [31] and are listed in Fig. 3. Axiom @id says nominal constant formulas $\overline{n}$ are satisfied at the state named by $\overline{n}$. Axiom $\exists W$ introduces a name for the current state. Rule $\mathrm{G}_@$ and axiom $\mathrm{K}_@$ are the Gödel generalization rule and Kripke axiom for the @ modality. Axioms $\forall I_@$ and $\forall E_@$ are Skolemization and elimination for universal world quantifiers. Axiom @I introduces an $@_{\overline{n}}\phi$ modality when $\overline{n}$ is the current state. Axiom @@ collapses nested @ modalities to the inner modality. Axiom $@\leftrightarrow$ replaces equal states in context. Axiom $\langle\overline{n}\rangle$ introduces an @ modality for any state $\overline{n}$ reachable by a program a. Axiom $\downarrow$ reduces the local quantifier to its definition in terms of the existential operator. Homomorphism axiom family @hom completely captures the meaning of at-terms. Barcan axiom schema BW swaps a quantifier $\exists s : \mathcal{W}$ with a program α where s does not appear *free* (s is never *bound* in *any* program). To make schematic program α into a constant a and make BW a concrete axiom, it would suffice to prohibit nominals inside programs.

6 Theory

We now develop the theory of dHL, showing that the proof calculus is sound and comparing the expressiveness of dHL with other logics. Complete definitions and proofs are in App. A.

6.1 Soundness

We show soundness of dHL by extending the soundness proof for the uniform substitution calculus for dL [38]. Uniform substitution allows for a modular soundness proof: the soundness proof is separated into proving that a finite list of dHL axioms are valid and that uniform substitution and the remaining Hilbert rules preserve validity. We prove that all valid dL formulas are valid dHL formulas, and thus dL axioms are also sound in dHL automatically.

6.1.1 Substitution

The US rule in dHL is analogous to that in dL:

$$\text{US}\quad \frac{\phi}{\sigma(\phi)}$$

In dL, the US rule is sound when the substitution σ does not introduce free references to bound variables. Such substitutions are called *admissible*, a condition which can be checked syntactically.

We generalize: in dHL, admissible substitutions do not introduce free references to any bound *flexible symbol*, world variables included. Admissibility conditions are checked recursively by the substitution algorithm (Fig. 5). We give the new cases and their admissibility conditions here. The *free-variable* function $\mathrm{FV}(e)$ recursively computes which flexible symbols might influence e. The novel cases of $\mathrm{FV}(e)$ are given in Fig. 4; the full algorithm is given in App. A. The admissibility checks also use a notion of *U-admissibility*:

Case	Equals
$\text{FV}(f(w))$	$= \text{FV}(w)$
$\text{FV}(\exists s:\mathcal{W}\ \phi,\ \forall s:\mathcal{W}\ \phi)$	$= \text{FV}(\phi) \setminus \{s\}$
$\text{FV}(\downarrow s\ \phi)$	$= (\text{FV}(\phi) \cup \mathcal{V}_{\mid\mathbb{R}}) \setminus \{s\}$
$\text{FV}(@_s\phi) = \text{FV}(@_s\theta)$	$= \text{FV}(\phi \text{ or } \theta)_{\mid\mathbb{W}} \cup \{s\}$
$\text{FV}(@_{\bar{s}}\phi) = \text{FV}(@_{\bar{s}}\theta)$	$= \text{FV}(\phi \text{ or } \theta)_{\mid\mathbb{W}}$
$\text{FV}(s)$	$= \mathcal{V}_{\mid\mathbb{R}} \cup \{s\}$
$\text{FV}(\overline{n})$	$= \mathcal{V}_{\mid\mathbb{R}}$

Figure 4: Free variable function (new cases)

Case	Replacement	Admissible when
$\sigma(@_w\theta)$	$= @_{\sigma(w)}\sigma(\theta)$	σ is $\mathcal{V}$-admissible for θ
$\sigma(@_w\phi)$	$= @_{\sigma(w)}\sigma(\phi)$	σ is $\mathcal{V}$-admissible for ϕ
$\sigma(\forall s:\mathcal{W}\ \phi)$	$= \forall s:\mathcal{W}\ \sigma(\phi)$	σ is $\{s\}$-admissible for ϕ
$\sigma(\exists s:\mathcal{W}\ \phi)$	$= \exists s:\mathcal{W}\ \sigma(\phi)$	σ is $\{s\}$-admissible for ϕ
$\sigma(\downarrow s\ \phi)$	$= \downarrow s\ \sigma(\phi)$	σ is $\{s\}$-admissible for ϕ
$\sigma(\overline{n})$	$= \overline{n}$, if $\overline{n} \notin \sigma$	
$\sigma(\overline{n})$	$= \sigma\overline{n}$, if $\overline{n} \in \sigma$	
$\sigma(s)$	$= s$	

Figure 5: Uniform substitution algorithm (new cases)

Definition 10 (U-admissibility)**.** We say a substitution σ is U-admissible for an expression e with a flexible symbol set U iff $\bigcup_{sym \in \sigma_{\mid\Sigma(e)}} \text{FV}(\sigma sym) \cap U = \emptyset$ where $\sigma_{\mid\Sigma(e)}$ is the restriction of σ that replaces only symbols occurring in e and where sym is an arbitrary rigid.

6.1.2 Examples

The admissibility conditions in Fig. 5 are instantiations of the principle that substitution should not introduce new free references under a binder. Yet, it can be surprisingly subtle both why these conditions are necessary for soundness and why the resulting calculus is sufficiently complete. To see why they are necessary for soundness, consider for example the instance of axiom @hom for $\sigma_C \overset{\text{def}}{\equiv} \{F_C \mapsto x,\ p(\cdot) \mapsto x = \cdot\}$, a substitution which *clashes* and which, if it were permitted,

would result in an invalid formula:

$$@_{\overline{n}}(x = x) \leftrightarrow x = @_{\overline{n}}x$$

where the LHS is a tautology and the RHS is false almost everywhere. The fundamental problem is that the modality $@_{\overline{n}}$ modifies the meaning of x: on the left it refers to $@_{\overline{n}}x$ while on the right it refers to x in the present state. In short, the meaning of σp is *non-uniform*, so substitution with σ would be unsound. The admissibility check for $@_w$ prohibits such substitutions, ensuring uniformity and thus soundness of rule US.

It is equally subtle why rule US allows axiom @hom to be instantiated at all, because the $@_{\overline{n}}$ modality binds the entire state. The key is that predicate argument $\cdot$ does not contribute to the substitution's free variables. For example, substitution $\sigma_A \overset{\text{def}}{\equiv} \{F_1 \mapsto x^2,\ p(\cdot) \mapsto \cdot \geq 0\}$ is admissible because i) F_1 already depends on all variables, so $\sigma_A(F_1)$ introduces no free variables and ii) the argument $\cdot$ is rigid, so $\sigma_A(p)$ introduces no free variables. Intuitively, σ_A should be admissible for formula $p(F_1)$ because the *argument* F_1 is a functional that can depend on all variables, yet σ_C clashes since $\sigma_C(p)$ does not defer to the argument (written $\cdot$), causing @hom to incorrectly use simply x on both sides.

We proceed to the soundness theorem, following the same structure as previous work [38]. We begin with lemmas on the correctness of free variable and signature computations, where the signature $\Sigma(e)$ is the analog of $\mathrm{FV}(e)$ for *rigid* symbols. The coincidence lemmas say that expressions depend only on their signature and free variables. We extend coincidence for terms and formulas with new cases for hybrid-logical constructs:

Lemma 1 (Coincidence).

1. If $\omega = \tilde{\omega}$ on $\mathrm{FV}(\theta)$, $g = h$ on $\mathrm{FV}(\theta)$, and $I = J$ on $\Sigma(\theta)$, then $[\![\theta]\!]Ig\omega = [\![\theta]\!]Jh\tilde{\omega}$.
2. If $\omega = \tilde{\omega}$ on $\mathrm{FV}(\phi)$, $g = h$ on $\mathrm{FV}(\phi)$, and $I = J$ on $\Sigma(\phi)$, then $\omega \in [\![\phi]\!]Ig$ iff $\tilde{\omega} \in [\![\phi]\!]Jh$.
3. If $\omega = \tilde{\omega}$, $g = h$ on $V \supseteq \mathrm{FV}(\alpha)$, $I = J$ on $\Sigma(\alpha)$, and $(\omega, \nu) \in [\![\alpha]\!]Ig$, then exists $\tilde{\nu}$ s.t. $(\tilde{\omega}, \tilde{\nu}) \in [\![\alpha]\!]Ih$ and $\nu = \tilde{\nu}$ on V.

Axioms of dL need not be reproved because dHL contains dL:

Proposition 2 (dHL contains dL). If ϕ is a dL formula, then validity in dL semantics and validity in dHL semantics coincide for ϕ.

Theorem 3 (dHL soundness). All dHL rules are sound and all axioms valid, thus all provable dHL formulas are valid.

Proof Sketch (App. D). Soundness of US is proven inductively, appealing to Lemma 1. The dL axioms are valid in dL [38] and (by Proposition 2) dHL, and by US so are their instances, even instances containing hybrid connectives. Validity of the new dHL axioms is by direct proof. □

6.2 Reducibility

We compare the expressive power of dHL to that of dL in order to determine when and in what sense dHL is necessary or especially beneficial compared to dL. The comparison is surprisingly subtle, and finds that while dHL is reducible to dL, its specialized hybrid-logical rules make direct proof in dHL preferable for practical purposes. The core idea is to emulate each world variable from dHL with a finite number of program variables in dL, resulting in an equivalent, finite dL formula. This approach is subtle mainly since dHL states contain infinitely many program variables:

1. The size of worlds is not a cosmetic decision, but rather affects validity of some formulas. Consider the formula:
$$\phi \equiv \langle x := * \rangle s$$
This is valid iff $\mathcal{V} \equiv \{x\}$, i.e. iff reaching all values of x suffices to reach all states. We consider this behavior too surprising, and eliminate it by requiring infinite states. The formula is then invalid because program $x := *$ cannot, e.g., transition between any ω and ν where $\omega(r) \neq \nu(r)$ for $r \neq x$.

2. Program constants and predicationals depend on every variable, in order to capture the notion that programs and formulas can use arbitrary variable names. Since there are infinitely many variables, they have infinitely many dependencies.

We show that for formulas without program constants and predicationals (called *concrete* formulas), infinite worlds are not an obstacle, while for formulas with program constants or predicationals (called *abstract* formulas), they are. Concrete formulas are reducible: even though each state contains infinitely many variables, it suffices to employ a single *fresh* variable r (consider example above). To make this claim formal, we introduce a notion of finite domains for states, galaxies and interpretations.

Definition 11 (Finite-domain states)**.** State ω has *finite domain* $S \subseteq \mathcal{V}$ if S is finite and $\omega(x) = 0$ for all $x \notin S$. Galaxies and interpretations are analogous. A formula ϕ is *finite-domain valid* for domain S if $\omega \in [\![\phi]\!] I g$ for all ω, I, g with finite domain S.

We outline the proof here, with full proofs in App. B. The proof proceeds by showing that in both dHL and dL, validity and finite-domain agree for concrete formulas, then showing that our reduction preserves finite-domain validity. Thus our reduction preserves validity for concrete formulas. The converse also holds because dL is a fragment of dHL (Proposition 2).

Lemma 4 (Finitization). Let ϕ be any concrete dHL formula. Then let $r \notin \mathrm{FV}(\phi) \cup \mathrm{BV}(\phi)$ (where $\mathrm{BV}(\phi)$ is everything bound in ϕ). Then ϕ is valid iff ϕ is finite-domain valid with domain $\{r\} \cup (\mathrm{FV}(\phi) \cup \mathrm{BV}(\phi))$.

Lemma 5 (Finite-domain reducibility). There exists a computable reduction $T(\phi)$ such every dHL formula ϕ is finite-domain valid iff the dL formula $T(\phi)$ is finite-domain valid.

Proof Sketch (App. B). We present the translation $T(\phi)$. In this translation we write $\vec{x}$ for the vector of all variables $x_1, \ldots, x_n$ in the domain S, and $e^{\vec{\theta}}_{\vec{x}}$ for the vectorial substitution of all θ_i for the

corresponding x_i in e. For each of the finitely-many world terms w in ϕ, let $\vec{x}^w$ be a vector of $|S|$ fresh symbols implementing $@_w\vec{x}$. When convenient, we implicitly assume $S \supset \mathrm{FV}(\phi) \cup \mathrm{BV}(\phi)$.

$$T(@_w\theta) = T(\theta)^{\vec{x}^w}_{\vec{x}} \tag{27}$$
$$T(@_w\phi) = [\vec{x} := \vec{x}^w]T(\phi) \tag{28}$$
$$T(w) = (\vec{x} = \vec{x}^w) \tag{29}$$
$$T(\forall s : \mathcal{W}\ \phi) = \forall \vec{x}^s : \mathbb{R}\ T(\phi) \tag{30}$$
$$T(\exists s : \mathcal{W}\ \phi) = \exists \vec{x}^s : \mathbb{R}\ T(\phi) \tag{31}$$
$$T(\downarrow s\ \phi) = [\vec{x}^s := \vec{x}]T(\phi) \tag{32}$$
$$T(\otimes(e_1, \ldots, e_2)) = \otimes(T(e_1), \ldots, T(e_n)) \tag{33}$$

In Equation (33), the notation $\otimes(e_1, \ldots, e_n)$ stands for any of the other dL connectives. In Equation (29), vector equality $\vec{x} = \vec{y}$ stands for conjunction $\bigwedge_i x_i = y_i$. The result is by induction. □

Lemma 6 (De-finitization). A concrete dL formula ϕ is valid iff it is finite-domain valid over domain $\mathrm{FV}(\phi) \cup \mathrm{BV}(\phi)$.

Theorem 7 (Concrete reducibility). Concrete dHL (i.e., with no rigid symbols) reduces to concrete dL. That is, for all concrete dHL formulas, the concrete dL formula $T(\phi)$ is valid iff ϕ is.

Proposition 8 (Complexity of T). $|T(\phi)| \in \Theta(|\phi|^2)$ for concrete ϕ.

Proof Sketch (App. B). To prove the upper bound, note the function $T(\phi)$ expands ϕ by at most a factor of $|S|$. By Lemma 4, $|S| \in O(|\phi|)$ suffices for finitely-valid ϕ. To prove the lower bound, simply observe there exist formulas where the number of program and world variables are both linear in the size. We give a concrete example:

$$\phi_n \equiv \exists s_1 : \mathcal{W}\ \cdots \exists s_n : \mathcal{W}\ (@_{s_1}x_1 > 0 \wedge \cdots \wedge @_{s_n}x_n > 0)$$

Now observe that applying T with $S = \{r\} \cup \mathrm{FV}(\phi_n) \cup \mathrm{BV}(\phi_n)$ results in $|T(\phi_n)| \in \Omega(n^2)$. □

We note these theorems *do not* entail (infinite) validity reduction for abstract dHL formulas. Theorem 5 does however preserve finite validity even in the presence of abstract formulas. In summary, reduction fails iff abstract constants are allowed to introduce arbitrary new variables.

Corollary 9 (Relative semi-decidability). Concrete dHL is semi-decidable relative to properties of differential equations.

Proof. By relative semi-decidability [34] of dL and Theorem 7. □

We reflect on the practical implications of the reducibility results. The reduction requires a finite variable domain, but the natural domain for abstract formulas is infinite. This means verification by reduction is especially ill-suited for proofs using advanced proof techniques like refinement [26] which rely on abstract formulas. Most dHL axioms are also abstract, and so cannot be translated to concrete dL axioms! Even on concrete formulas, the reduction exhibits quadratic blowup and obscures the more convenient proof techniques available in dHL. Thus, direct proof in dHL is strongly preferable to dL reduction for practical purposes.

$$\text{@ind}\quad \frac{@_{i_1}o_1 \to p(o_1) \quad @_{m_1}\langle\alpha\rangle o_1 \wedge p(m_1) \to p(o_1)}{@_{i_1}\langle\alpha^*\rangle o_1 \to p(o_1)}$$

$$\text{BS}^*\quad \frac{@_{i_1}\langle\alpha\rangle o_1 \wedge R(i_1,i_2) \to @_{i_2}\langle\alpha\rangle{\downarrow}o_2\ R(o_1,o_2)}{@_{i_1}\langle\alpha^*\rangle o_1 \wedge R(i_1,i_2) \to @_{i_2}\langle\alpha^*\rangle{\downarrow}o_2\ R(o_1,o_2)}$$

$$\text{BS}'\quad \frac{@_{i_1}\langle\text{ASGN}\rangle o_1 \wedge R(i_1,i_2) \to @_{i_2}[\text{ASGN}]{\downarrow}o_2\ R(o_1,o_2)}{@_{i_1}\langle\text{ODE}\rangle o_1 \wedge R(i_1,i_2) \to @_{i_2}\langle\text{ODE}\rangle{\downarrow}o_2\ R(o_1,o_2)}$$

$$\text{BS;}\quad \frac{\begin{array}{c}@_{i_1}\langle\alpha\rangle m_1 \wedge R_i(i_1,i_2) \to @_{i_2}\langle\alpha\rangle{\downarrow}m_2\ R_m(m_1,m_2)\\ @_{m_1}\langle\alpha\rangle o_1 \wedge R_m(m_1,m_2) \to @_{m_2}\langle\alpha\rangle{\downarrow}o_2\ R_o(o_1,o_2)\end{array}}{@_{i_1}\langle\alpha;\beta\rangle o_1 \wedge R_i(i_1,i_2) \to @_{i_2}\langle\alpha;\beta\rangle{\downarrow}o_2\ R_o(o_1,o_2)}$$

$$\text{BS}\cup\quad \frac{\begin{array}{c}@_{i_1}\langle\alpha\rangle o_1 \wedge R_i(i_1,i_2) \to @_{i_2}\langle\alpha\rangle{\downarrow}o_2\ R_o(o_1,o_2)\\ @_{i_1}\langle\beta\rangle o_1 \wedge R_i(i_1,i_2) \to @_{i_2}\langle\beta\rangle{\downarrow}o_2\ R_o(o_1,o_2)\end{array}}{@_{i_1}\langle\alpha\cup\beta\rangle o_1 \wedge R_i(i_1,i_2) \to @_{i_2}\langle\alpha\cup\beta\rangle{\downarrow}o_2\ R_o(o_1,o_2)}$$

Figure 6: Bisimulation: Derived rules (m_i fresh)

7 Derived Rules for Bisimulation

The proof calculus of Section 5 provides a hybrid-logical core for hyperproperty verification. We connect nondeducibility to this core by deriving a library of rules for information-flow proofs which, being derived, lie outside the core calculus. Our derived rules show that bisimulation, the core proof technique for information flow, derives from nominals in hybrid logic. Because information flow arguments specifically equate values from initial and ending states, we also derive rules for equalities over at-terms. In Section 8 and Section 9, we apply our library to our smart grid example and see it raises the level of abstraction. Derivations are given in the App. D.

7.1 Bisimulation Rules

In Fig. 6, R refers to a relation over world expressions, i.e., $R(i_1,i_2)$ means that worlds i_1 and i_2 are related in R, and m_1, m_2 refer to any **m**iddle states. The rules proceed by destructing a trace on the left; any nondeterminism is resolved identically on the right. Rule @ind is an auxiliary rule for loop induction with nominals, derived from loop induction axiom I. It is in turn used to derive Rule BS*, which says any relation R is a bisimulation for loop α^* any time it is (in every state) a bisimulation for α. Rule BS; is Hoare-style composition reasoning raised to the bisimulation level: we can reason about $\alpha;\beta$ by establishing a relation R_m that holds in the intermediate state. Rule BS$\cup$ says a nondeterministic choice maintains a bisimulation if each branch does. In rule BS$'$, ODE is a differential equation of form $\{x' = \theta, t' = 1 \,\&\, \psi\}$ (any model can be trivially extended to this form by adding a fresh variable t) and $\text{ASGN} \equiv (t := @_{o_1}t; x := y(t - @_{i_1}t))$ simplifies ODEs to assignments implementing their solutions, plugging in the same duration $@_{o_1}t - @_{i_1}t$ as in the trace $@_{i_1}\langle\text{ODE}\rangle o_1$.

$$\text{NTV}\quad @_i\langle\alpha\rangle j \to @_i\theta = @_j\theta \qquad (\mathrm{FV}(\theta)\cap\mathrm{BV}(\alpha)=\emptyset)$$

$$\text{NT:=}\quad @_i\langle x := F\rangle j \to @_i F = @_j x$$

$$\text{NT;}\quad \frac{@_i\langle\alpha\rangle m \to @_i F = @_m H \quad @_m\langle\beta\rangle j \to @_m H = @_j G}{@_i\langle\alpha;\beta\rangle j \to @_i F = @_j G}$$

$$\text{NT}\cup\quad \frac{@_i\langle\alpha\rangle j \to @_i F = @_j G \quad @_i\langle\beta\rangle j \to @_i F = @_j G}{@_i\langle\alpha\cup\beta\rangle j \to @_i F = @_j G}$$

$$\text{NT}'\quad @_i\langle x' = F, t' = 1 \,\&\, \psi\rangle j \to @_i y((@_j t) - t) = @_j x$$

$$\text{NT}*\quad \frac{@_s\langle\alpha\rangle t \to @_s F = @_t F}{@_i\langle\alpha^*\rangle j \to @_i F = @_j F}$$

Figure 7: At-terms: Derived rules (n, m, s, t fresh)

7.2 At-Terms

Fig. 7 derives rules for at-term equalities. Vacuity rule NTV says a term θ is unchanged if its variables never appear bound in α. The remaining rules are derived from the program axioms and capture the effect of each program on a term. In rule NT′, $y(t)$ is a global solution to $x' = f(x)$.

8 FREEDM: Proving Vulnerability Existence

We now prove that the naïve deterministic controller bat_I based on the published algorithm for FREEDM [2] is insecure: our quantitative dynamical model reveals a bug obscured by the finite event-based abstraction in previous models [3]. Information leaks because when $gr > 0$ is true we can infer $B_i = B_{\max}$ for some i, meaning we have leaked the information that some battery is at capacity. In principle, this could be useful to an attacker, because high charge is associated with vulnerability to (explosive) thermal runaways [15]!

To prove that a system is nondeducibility-*insecure*, we prove the negation of nondeducibility security, i.e., we prove:

Proposition 10 (bat_I is insecure). Let α_I be the insecure version of the grid α_F and then let $R(i,j) \equiv (@_i t = @_j t \wedge @_i gr = @_j gr)$. Then the following formula is valid according to dHL semantics: $\exists i_1, i_2, o_1 : \mathcal{W}\big(R(i_1,i_2) \wedge @_{i_1}\langle\alpha_I\rangle o_1 \wedge @_{i_2}[\alpha_I]{\downarrow}o_2\ \neg R(o_1,o_2)\big)$.

Proof Sketch (App. F). We begin by constructing the states i_1, i_2, o_1. First, let i be an arbitrary state. Then let i_1 be the unique state such that $@_i\langle B_i := B_{\max}; t := 0; gr := 0\rangle i_1$ and i_2 the unique state such that $@_i\langle B_i := \frac{1}{2}B_{\max}; t := 0; gr := 0\rangle i_2$. Let o_1 be any state such that $@_{i_1}\langle\alpha_I\rangle o_1$ and such that $@_{o_1}(t = 0 \wedge gr > 0)$. We know such a state exists by running α_I for exactly one iteration, setting $r_i = \max(0, -p_i)$ and $d_i = 1 + \max(0, p_i)$ which always results in $N_i = 1$. Thus after evolving the differential equation for time 0 we arrive at $gr > 0$. By induction we show that all traces of α_I maintain the invariant $J \equiv (t{\geq}0 \wedge (t{=}0 \to gr \leq 0 \wedge B_i = \frac{1}{2}B_{\max}))$, which can be

proven by mechanically applying program axioms, then checking first-order real arithmetic at the leaves. The result follows from $@_{o_1}(t = 0 \land gr > 0) \land @_{o_2} J \to \neg R(o_1, o_2)$, which itself follows from a simpler arithmetic argument: $@_{o_1} gr \neq @_{o_2} gr$. □

9 FREEDM: Ensuring and Proving Security

We learned that bat_I leaks $B_i = B_{\max}$, ultimately because it is too deterministic: If $gr > 0$ we learn for a fact some $B_i = B_{\max}$. The simplest solution, as taken in bat_S of Fig. 2, is to add the nondeterministic option to use the macrogrid even when a battery has capacity. The macrogrid is then always an option, so an attacker who observes $gr > 0$ cannot infer $B_i = B_{\max}$ for certain.

We now prove bat_S nondeducibility secure, i.e., an attacker observing only gr and t deduces nothing else. We instantiate Def. 5 to arrive at the theorem statement:

Proposition 11 (Nondeducibility for FREEDM). First, we define the relation $R(i, j)$ by the equivalence $R(i, j) \equiv (@_i t = @_j t \land @_i gr = @_j gr \land pre(i) \land pre(j))$ and define the predicate $pre(i)$ by $pre(i) \equiv @_i(maxm > 0 \land B_{\max} > 0 \land thresh > 0)$. Then formula

$$\forall i_1, i_2, o_1 : \mathcal{W} \left(@_{i_1} \langle \alpha_S \rangle o_1 \land R(i_1, i_2) \to @_{i_2} \langle \alpha_S \rangle \downarrow o_2 \; R(o_1, o_2) \right)$$

is valid, where α_S is the secure version of the grid α_F.

Proof Sketch (App. E). Recall from Section 3.2 that the heart of the proof is choosing a trace $@_{i_2} \langle \alpha_S \rangle o_2$ which shows the public outputs $@_{o_1} gr$ and $@_{o_1} t$ of trace

$$@_{i_2} \langle \alpha_S \rangle o_2$$

are possible from all related input states i_2. We apply loop rule BS* with $R(i, j)$ defined by $R(i, j) \equiv @_i t = @_j t \land @_i gr = @_j gr \land pre(i) \land pre(j)$. The key proof observation is that for the final values of gr to agree, it suffices that the values agree for both gr and grm at the start of the ODE. We perform this reasoning formally using the composition rule BS; with R_P defined as $R_P(i, j) \equiv R(i, j) \land @_i gr = @_j gr \land @_i grm = @_j grm \land @_i t = @_j t$. This gives two proof obligations: one for the control and one for the physics. We split into four cases for the controller using Lemma 12, according to whether each transformer chooses to migrate.

Lemma 12 (Controller cases). The following dHL formula is valid, where *ctrl* is as in Fig. 2:

$$\begin{aligned} @_{i_1} \langle ctrl \rangle m_1 \to @_{m_1} \big(& (gr = 0 \land grm = 0) \lor (gr = n_1 \land grm = m) \\ & \lor (grm = n_2 \land grm = -m) \lor (gr = n_1 + n_2 \land grm = 0) \big) \end{aligned}$$

The second case is representative, the rest are in App. E.

Case $@_{m_1}(gr = n_1 \land grm = m)$: By inspection, it suffices to set $n_1 = @_{m_1} n_1$ and to set $n_2 = 0, m = 0$. If $@_{m_1} n_1 - @_{i_2} p_1 \geq 0$ then we set $r_1 = -@_{m_1} n_1 - @_{i_2} p_1$ and $d_1 = 0$ else we set $r_1 = 0$ and $d_1 = @_{i_2} p_1 + @_{m_1} n_1$. By algebra, in each case $@_{m_2} n_1 = @_{m_1} n_1$. Then for $i = 2$ if $@_{i_2} p_2 \geq 0$ then set $r_2 = -@_{i_2} p_2$ and $d_2 = 0$ else set $r_2 = 0, d_2 = @_{i_2} p_2$. By algebra, $@_{m_2} n_2 = 0$. Executing the load balancer, we get $m = 0$ because $n_2 = 0$ (thus T_2 is *Normal*). Each case takes the second branch of the battery controller, getting $gr = n_1 + n_2 = n_1$ and $grm = m \cdot -1^2 + m \cdot -1^1 = 0 = m$ as desired. This completes the proof of Proposition 11. □

We have shown how to use dHL to find and resolve hybrid-dynamic information flow (HDIF) vulnerabilities in CPSs. We reflect on how verified safety of a model can contribute to the safety of real-world implementations as well.

Our first model was insecure because a deterministic branch in the battery controller leaked information. This was fixed by introducing a nondeterministic branch; this fact that determinism can make a model less secure is colloquially known as the "refinement paradox". Implementations can approximate this added nondeterminism, e.g., with randomized branching. The exact extent of security in the implementation would depend on the probability with which each branch is taken. Our models taught us that the battery controller needs such measures, but the load balancing controller, in contrast, is secure even with deterministic control. This knowledge is helpful in practice because measures like randomization typically reduce operational suitability of a controller, so verifying that a deterministic controller is secure enables using the efficient deterministic controller with confidence.

In comparison with many other formal security models, our approach is especially well-suited to verifying security in the presence of *side-channels*. Many would-be side-channels for cyber systems (time, electrical flow, etc.) are *primary* channels in CPS and, as shown in our models, are modeled naturally as hybrid systems. Once these channels are modeled in our hybrid system, they can be verified with the same techniques as any other HDIF!

10 Derivable Extensions

It is question of natural theoretical interest: which logical features are fundamentally new, and which can be derived from each other? When we formalize the theory of a logic in a theorem prover (as we have done for the base logic dL [10]) , this becomes a question of practical interest as well: if we derive as many features as possible, this allows us to minimize the complexity of the *core* language and thus minimize formalization effort. It turns out that dynamic logic and hybrid logic are a potent combination; we discuss here some of the features that can be derived from the core provided by dHL.

10.1 Differential Refinement Logic

The refinement formula $\Gamma \vdash \alpha \leq \beta$ says that under the assumption that all formulas in the context (list) Γ are true, α refines β, i.e. α only reaches a (non-strict) subset of states reachable by β. Refinements have been developed for hybrid systems in the logic dRL [26], a variant of dL. Refinements are useful, for example, because they can reduce the amount of user effort required for a proof. This is especially true when (as is common in practice) we wish to develop a series of progressively more complex models, in which case dRL helps modularly decompose the proof effort.

There is no obvious reduction from dRL to dL in the presence of program constants, for the same reason as dHL: Refinements describe entire (infinite) program states, which are not easily reduced to finite dL formulas. In contrast, their definition in dHL is straightforward:

$$\Gamma \vdash (\alpha \leq \beta) \equiv (\wedge_\Gamma \rightarrow \forall s : \mathcal{W}\ (\langle\alpha\rangle s \rightarrow \langle\beta\rangle s)) \tag{34}$$

As is typical for refinement logics, program equivalence is definable from refinement:

$$\Gamma \vdash (\alpha = \beta) \equiv (\Gamma \vdash \alpha \leq \beta) \wedge (\Gamma \vdash \beta \leq \alpha) \tag{35}$$

Just as refinement reasoning helps with modular verification of realistic models, equivalence reason enables contextual reasoning for programs, which typically enables simpler proofs.

Because dRL is derivable from dHL, we can conclude that it is safe to omit dRL from the core logic. As was the case with the rules of Sec. 7, we could derive the dRL refinement rules from dHL to provide a new proof of soundness for dRL.

10.2 Universal and Existential Modalities

Other commonly-used modalities in modal logic include the *universal* and *exisitential* modalities, which we write $\boxtimes\phi$ and $\diamondsuit\phi$, and which say the formula ϕ is true in *all* or *some* states, respectively. These are easily defined in dHL:

$$\boxtimes\phi \equiv \forall s : \mathcal{W}\ @_s\phi \quad \text{and} \quad \diamondsuit\phi \equiv \exists s : \mathcal{W}\ @_s\phi \tag{36}$$

It is perhaps not surprising that they can be so defined, but these modalities may be of interest for implementation reasons. The dL calculus as implemented in KeYmaera X [18, 38] deals not only in valid formulas, but primarily in locally-sound (axiomatic) proof rules (i.e., a proof shows that validity of one formula is derivable from validity of others). The universal modality allows internalizing the local soundness of an axiomatic proof rule as the validity of a formula. This is not only useful as a conceptual bridge, but would also enable broader application of substitution reasoning [38], because the substitution rules available in dL can by applied to *individual formulas* but not (soundly) to most *proof rules*. Applications of such substitutions may include modular proof techniques similar to those pursued in the development of dRL [26].

Frame Classes. The universal operator can also be used to define further extensions to dHL. There are many propositional logics that differ in which Kripke frames are allowed in interpretations of modal operators, i.e., what axioms the modalities must obey. As hybrid programs can express many behaviors, the only common modal axioms that hold for all modalities $[\alpha]\phi$ are those of the modal system K:

$$\text{K} \quad [\alpha](A \to B) \to [\alpha]A \to [\alpha]B$$

$$\text{G} \quad \frac{A}{[\alpha]A}$$

However, one can imagine wanting to freely mix traditional modalities from various propositional dynamic logics with the dynamic-logical reasoning of dHL. The universal modality makes it easy to introduce (by axiomatization) new program constants that implement a desired traditonal modality. For example, we could add the System K4 box modality by introducing a program constant named K_4 and assuming it obeys the transitivity axioms $[K_4]\phi \to [K_4][K_4]\phi$ of System K4 (for all

instances ϕ of axiom K4 used in the proof). We axiomatize K_4 in a proof of an arbitrary theorem ψ by instead proving the formula:

$$\boxtimes([K_4]\phi \to [K_4][K_4]\phi) \to \psi$$

This accomplishes is a reduction from validity in dHL +K4 to validity in dHL. As with other applications of the universal modality, we could express this theorem as soundness of an *axiomatic proof rule* [18] in KeYmaera X, but can not write it as a (conceptually simpler) single formula without a universal modality.

11 Related Work

11.1 Dynamic Logics and Hybrid Logics.

The logic dHL is a hybrid version of the dynamic logic dL, adding the ability to verify hyperproperties in addition to safety and liveness properties. The KeYmaera [39] and KeYmaera X [18] have been successfully applied in numerous safety case studies [27, 23, 24]. The logic $\mathsf{d}\mathcal{L}_h$ [33] was proposed as an extension of dL with propositional hybrid connectives, but lacks world quantifiers and at-terms, which are essential for information flow. Dynamic logic and first-order hybrid logic have been combined in Combinatory PDL [30], which extends Propositional Dynamic Logic (PDL) with additional set-theoretic program combinators, but has neither at-terms nor assignments, let alone differential equations as dHL does. Hybrid logic has been used in preference logics [47], reactive systems logics [28], and distributed systems logics [32] and type systems [48] as well. While many CPSs are distributed systems, distributed systems reasoning alone does not suffice to verify hybrid discrete and continuous dynamics. The logic $\mathsf{Qd}\mathcal{L}$ [35] allows verification of distributed hybrid dynamics, but is not a hybrid logic, and faces the same challenges with hyperproperties as dL does. First-order hybrid logic [8] (without dynamic-logical or continuous features) and its proof theory [11] have been studied in detail. The latter includes a treatment of *non-rigid designators* which are equivalent to at-terms of variables $@_w x$, i.e., our at-terms are a natural generalization of non-rigid designators.

11.2 Static Information Flow Security.

Logics and type systems for information-flow security have been widely studied for discrete programs. Sebelfeld and Myers [42] provide a survey of language-based security approaches. Approaches can be broadly categorized into automatic vs. interactive (or manual) approaches. Automation increases the potential user base, typically at the cost of greatly reduced completeness. When the proof is done automatically, the simplicity of the proof is of little concern, and self-composition [6] can be used to reduce information-flow proofs to a safety property suitable for Hoare and dynamic logics.

In interactive use, self-composition has been noted [46] to make proofs awkward by reducing locality: bisimulation techniques consider the local effect of each statement α on two traces, but

self-composition may move the original statement α far from its copy. For humans, a usable calculus as provided by dHL is far more important. As our smart grid example demonstrates, typical HDIFs rely on fine-grained interactions between discrete and continuous dynamics within system loops. This suggests that automated approaches would struggle and that our approach, which is amenable to interactive proof, is merited. An approach analogous to proof by reduction from dHL to dL has been implemented for the dynamic logic JAVADL in the theorem prover KeY (which supports both automatic and interactive proof, but not nominals). To avoid the awkwardness of the reduction approach, calculi meant for interactive use [7, 29] typically build in special-purpose relations for information flow. The disadvantage of such calculi is that baking in these relations prevents generalizing to other hyperproperties.

We strike a middle ground with dHL: The proof techniques we would expect of a dedicated calculus are easily implemented as derived rules, yet we maintain the generality to express arbitrary safety and liveness properties and hyperproperties as well. Our development hints that the relationship between hybrid logic and hyperproperties is general and deserves further exploration.

While we present the first HDIF result for a CPS, information flow has been verified for discrete models of several different CPSs via model-checking; e.g., Akella [3] has verified process algebra models of FREEDM and Wang [49] has verified a Petri-net model of a pipeline network. The absence of continuous dynamics amounts to a significant *model gap* between these models and reality. HDIFs greatly narrow the model gap: for example, our HDIF analysis of our FREEDM model revealed a vulnerability that was not visible in the discrete model of Akella [3]. This all goes to say there are ample options for future work for CPS security through the logical lens.

12 Conclusion and Future Work

We introduced dHL, a hybrid logic for verifying information-flow security properties of hybrid dynamical systems in order to ensure the security of critical cyber-physical systems (CPS). In contrast with previous approaches, it allows verifying cyber-physical hybrid-dynamic information flows (HDIFs), communicating information through both discrete computation and physical dynamics, so security is ensured even when attackers observe continuously-changing values in continuous time. It achieves this by combining dL, a logic for hybrid dynamical systems, with hybrid-logical features enabling explicit reference to program states. This provides a novel way to verify information flow: information flow properties are hyperproperties, which are expressed naturally in hybrid logic via its ability to refer freely to states from multiple traces simultaneously. The foundation of hybrid logic allows verification (and falsification) of security in a common system at no added complexity, and we expect the same system can support additional notions of information flow such as non-interference, as well as arbitrary hyperproperties. We introduced a calculus for dHL, proved it sound, and derived high-level bisimulation rules for information flow proofs. Our use of uniform substitution provides modularity: we can instantiate all existing dL axioms with dHL formulas and need not individually reprove that each axiom is a valid formula of dHL. Uniform substitution also provides a clear path for extending the dL theorem-prover KeYmaera X [18] and soundness formalization [10] with dHL.

We showed that dHL is capable of verifying the presence or absence of information-flow vul-

nerabilities in realistic hybrid models. As an example, we debugged and then verified a hybrid model of the FREEDM [22] smart grid controller based on the published algorithm [2] with load-balancing and distributed energy generation and storage, all important features for practical grids. Moreover, the proof demonstrates both i) the close correspondence between dHL information-flow proofs and natural-language proofs and ii) the non-trivial proof arguments that quickly arise when mixing cyber and physical dynamics.

The main places where dHL proofs require more effort than an informal proof were in introducing names for intermediate states and observing the effect of a program on an individual term. Much as the Bellerophon language has helped automate low-level steps in plain dL proofs [17], we hope to provide a proof language in an eventual KeYmaera X implementation of dHL to allow us to automate the majority of these low-level steps, making proofs efficiently match to human intuition. Our information-flow arguments depend closely on the exact semantics of the program and do not follow from, e.g., simple syntactic checks on variable dependencies, meaning the expressive power provided by dHL's deductive calculus is essential for verifying realistic CPS information flow problems. A major novelty in both the logic dHL and our model of FREEDM is the presence of HDIFs that mix discrete cyber and continuous physical flows. These cyber-physical flows arise naturally in many other critical applications, such as oil and natural gas networks, canals, smart homes, medical devices, and vehicles, which deserve future exploration.

Lastly, we wish to explore potentional uses of our refinement and universal modalities [26] in modular hybrid systems modeling and verification.

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A Uniform Substitution Algorithm

We give the complete presentation of the uniform substitution algorithm, i.e.

- The $\mathrm{FV}(e)$ function computing flexible symbols which can influence expression e
- The $\mathrm{BV}(\alpha)$ function computing flexibles which can change during program α (unchanged from prior work)
- The $\mathrm{MBV}(\alpha)$ function computing flexibles which are necessarily bound on *all* execution paths of α
- The signature $\Sigma(e)$ of rigid symbols in expression e.
- The substitution algorithm $\sigma(e)$ proper.

Here $U_{|\mathbb{R}}$ and $U_{|\mathbb{W}}$ denote the restriction of set U to only program variables or world variables, respectively. Equations (37-48) are as in previous work [38]. In Equation 48, $\mathrm{MBV}(\alpha)$ is the set of variables that are bound on *all* executions of α. Equation 49 says quantifiers remove the quantified world variable from the free variable set because references to s in ϕ refer to the value bound by the quantifier. Equations 51 and 52 say the only free variables are the world variables of θ or ϕ (and s in the world variable case). It may be surprising that the free program variables of ϕ and θ make no appearance. The reason is this: program variable references x within $@_w\phi$ or $@_w\theta$ refer to $@_s x$, which is encapsulated by the single dependency on s [2] , or to $@_{\bar{n}} x$ which is *rigid* due to the rigidity of $\bar{n}$ and thus incurs no dependencies on a flexible symbol such as x.

$$
\begin{aligned}
\mathrm{BV}(?\phi) &= \{\} \\
\mathrm{BV}(x := \theta) &= \{x\} \\
\mathrm{BV}(x := *) &= \{x\} \\
\mathrm{BV}(x' = \theta \,\&\, \psi) &= \{x, x'\} \\
\mathrm{BV}(\alpha \cup \beta) &= \mathrm{BV}(\alpha) \cup \mathrm{BV}(\beta) \\
\mathrm{BV}(\alpha; \beta) &= \mathrm{BV}(\alpha) \cup \mathrm{BV}(\beta) \\
\mathrm{BV}(\alpha^*) &= \mathrm{BV}(\alpha) \\
\mathrm{BV}(a) &= \mathcal{V}_{|\mathbb{W}} \cup \mathcal{V}_{|\mathbb{R}} \\
\hline
\mathrm{MBV}(\alpha \cup \beta) &= \mathrm{MBV}(\alpha) \cap \mathrm{MBV}(\beta) \\
\mathrm{MBV}(\alpha; \beta) &= \mathrm{MBV}(\alpha) \cup \mathrm{MBV}(\beta) \\
\mathrm{MBV}(a) = \mathrm{MBV}(\alpha^*) &= \{\} \\
\mathrm{MBV}(\alpha) &= \mathrm{BV}(\alpha)
\end{aligned}
$$

[2] One might be tempted to increase the precision of admissibility by distinguishing, e.g. dependency on $@_s x$ from $@_s y$. This would have no benefit because all *binders* of states bind them in their entirety, in which case introducing free reference to *any* $@_s x$ violates admissibility. We thus lose nothing by using the simpler dependency on s.

$$\mathrm{FV}(c) = \{\} \tag{37}$$
$$\mathrm{FV}(x) = \{x\} \tag{38}$$
$$\mathrm{FV}(\theta_1 + \theta_2) = \mathrm{FV}(\theta_1) \cup \mathrm{FV}(\theta_2) \tag{39}$$
$$\mathrm{FV}(\theta_1 \cdot \theta_2) = \mathrm{FV}(\theta_1) \cup \mathrm{FV}(\theta_2) \tag{40}$$
$$\mathrm{FV}(f(\theta)) = \mathrm{FV}(\theta) \tag{41}$$
$$\mathrm{FV}(f(w)) = \mathrm{FV}(w) \tag{42}$$
$$\mathrm{FV}(F) = \mathcal{V}_{|\mathbb{R}} \cup \mathcal{V}_{|\mathbb{W}} \tag{43}$$

$$\mathrm{FV}(\phi \wedge \psi) = \mathrm{FV}(\phi) \cup \mathrm{FV}(\psi) \tag{44}$$
$$\mathrm{FV}(\neg\phi) = \mathrm{FV}(\phi) \tag{45}$$
$$\mathrm{FV}(\exists x : \mathbb{R}\ \phi) = \mathrm{FV}(\phi) \setminus \{x\} \tag{46}$$
$$\mathrm{FV}(\theta_1 \geq \theta_2) = \mathrm{FV}(\theta_1) \cup \mathrm{FV}(\theta_2) \tag{47}$$
$$\mathrm{FV}(\langle\alpha\rangle\phi) = \mathrm{FV}(\alpha) \cup (\mathrm{FV}(\phi) \setminus \mathrm{MBV}(\alpha)) \tag{48}$$
$$\mathrm{FV}(\exists s : \mathcal{W}\ \phi,\ \forall s : \mathcal{W}\ \phi) = \mathrm{FV}(\phi) \setminus \{s\} \tag{49}$$
$$\mathrm{FV}(\downarrow s\ \phi) = (\mathrm{FV}(\phi) \cup \mathcal{V}_{|\mathbb{R}}) \setminus \{s\} \tag{50}$$
$$\mathrm{FV}(@_s\phi) = \mathrm{FV}(@_s\theta) = \mathrm{FV}(\phi \text{ or } \theta)_{|\mathbb{W}} \cup \{s\} \tag{51}$$
$$\mathrm{FV}(@_{\bar{s}}\phi) = \mathrm{FV}(@_{\bar{s}}\theta) = \mathrm{FV}(\phi \text{ or } \theta)_{|\mathbb{W}} \tag{52}$$
$$\mathrm{FV}(s) = \mathcal{V}_{|\mathbb{R}} \cup \{s\} \tag{53}$$
$$\mathrm{FV}(\overline{n}) = \mathcal{V}_{|\mathbb{R}} \tag{54}$$

$$\mathrm{FV}(?\phi) = \mathrm{FV}(\phi) \tag{55}$$
$$\mathrm{FV}(x := \theta) = \mathrm{FV}(\theta) \tag{56}$$
$$\mathrm{FV}(x := *) = \{\} \tag{57}$$
$$\mathrm{FV}(x' = \theta\,\&\,\psi) = \mathrm{FV}(\theta) \cup \mathrm{FV}(H) \tag{58}$$
$$\mathrm{FV}(\alpha \cup \beta) = \mathrm{FV}(\alpha) \cup \mathrm{FV}(\beta) \tag{59}$$
$$\mathrm{FV}(\alpha; \beta) = \mathrm{FV}(\alpha) \cup (\mathrm{FV}(\beta) \setminus \mathrm{MBV}(\alpha)) \tag{60}$$
$$\mathrm{FV}(\alpha^*) = \mathrm{FV}(\alpha) \tag{61}$$
$$\mathrm{FV}(a) = \mathcal{V}_{|\mathbb{R}} \cup \mathcal{V}_{|\mathbb{W}} \tag{62}$$

Figure 8: Free variable computation

$$\Sigma(@_{\overline{n}}\phi) = \Sigma(@_{\overline{n}}\theta) = \Sigma(\phi \text{ or } \theta) \cup \{\overline{n}\}$$
$$\Sigma(@_s\phi) = \Sigma(@_s\theta) = \Sigma(\phi \text{ or } \theta)$$
$$\Sigma(\exists x : \mathcal{W}\ \phi) = \Sigma(\forall x : \mathcal{W}\ \phi) = \Sigma(\downarrow x\ \phi) = \Sigma(\phi)$$
$$\Sigma(sym \in f, F, p, P) = \{sym\}$$
$$\Sigma(\otimes(e_1, \ldots, e_n)) = \Sigma(e_1) \cup \cdots \cup \Sigma(e_n)$$

Figure 9: Signature computation (sym is an arbitrary rigid)

Analogously to FV(e), the signature $\Sigma(e)$ indicates all *rigid* symbols which influence the meaning of e. Note that since rigid symbols, by definition, are *not* bound by the $@_w\theta$ modality, they are always counted in the signature:

Admissibility conditions are checked recursively during the substitution algorithm proper (Figure 10). These checks use an auxilliary notion called $U-$admissiblity:

Definition 12 (U-admissibility)**.** We say a substitution σ is U-admissible for an expression e with a flexible symbol set U iff $\bigcup_{sym \in \sigma_{|\Sigma(e)}} \text{FV}(\sigma sym) \cap U = \emptyset$ where $\sigma_{|\Sigma(e)}$ is the restriction of σ that replaces only symbols occurring in e.

This makes the admissibility conditions as expressed in the main paper precise. Note also in Figure 10 that the symbol $\cdot$ is a reserved function (or nominal) symbol standing for the argument.

B Reducibility Proofs

We begin with the inclusion of dL into dHL. This is intuitivitely obvious because dL is a fragment of dHL, but the technical details are finicky. The main finicky detail is that states and interpretations in dHL are larger than in dL. We write $\omega_h = \omega_d \cup S$ to say that dHL state ω_h is an extension of the dL state ω_d, where the set S contains all the additional mappings of ω_h. Likewise we say $I_d \sqsubseteq I_h$ when I_h is an extension of I_d. The $\sqsubseteq$ relation is not quite the subset relation but closely related. We write $D(\omega_h)$ for the dL part of a state and $H(\omega_h)$ for the hybrid part. It holds when:

- $I_h(a) = \{(\omega_h, \nu_h) \mid (D(\omega_h), D(\omega_\nu)) \in I_d(a) \wedge H(\omega_h) = H(\nu_h)\}$
- $I_h(f) = I_h(f)(r) = I_d(f)(r)$
- $I_h(F) = \forall(\omega_h = \omega_d \cup S) I_h(f)(\omega_h) = I_d(f)(\omega_d)$
- $I_h(p) = I_h(p)(r) = I_d(p)(r)$
- $I_h(P) = \forall(\omega_h = \omega_d \cup S) I_h(P)(\omega_h) = I_d(P)(\omega_d)$

Case	Replacement	Admissible when:	
	$\sigma(c) = c$		(63)
	$\sigma(x) = x$		(64)
	$\sigma(\theta_1 + \theta_2) = \sigma(\theta_1) + \sigma(\theta_2)$		(65)
	$\sigma(\theta_1 \cdot \theta_2) = \sigma(\theta_1) \cdot \sigma(\theta_2)$		(66)
	$\sigma(f(\theta)) = \{\cdot \mapsto \sigma(\theta)\}(\sigma f), f \in \sigma$, else $f(\sigma(\theta))$		(67)
	$\sigma(f(w)) = \{\cdot \mapsto \sigma(w)\}(\sigma f), f \in \sigma$, else $f(\sigma(w))$		(68)
	$\sigma(F) = \sigma F, F \in \sigma$		(69)
	$\sigma(F) = F, F \notin \sigma$		(70)
	$\sigma(a) = \sigma a, a \in \sigma$, else a		(71)
	$\sigma(x := \theta) = x := \sigma(\theta)$		(72)
	$\sigma(x := *) = x := *$		(73)
	$\sigma(?(\phi)) = ?(\sigma(\phi))$		(74)
	$\sigma(\{x' = \theta \,\&\, \psi\}) = \{x' = \sigma(\theta) \,\&\, \sigma(\psi)\}$	σ $\{x, x'\}$-admiss. for θ, ψ	(75)
	$\sigma(\alpha; \beta) = \sigma(\alpha); \sigma(\beta)$	σ BV$(\sigma(\alpha))$-admiss. for β	(76)
	$\sigma(\alpha \cup \beta) = \sigma(\alpha) \cup \sigma(\beta)$		(77)
	$\sigma(\alpha^*) = \sigma(\alpha)^*$	σ BV$(\sigma(\alpha))$-admiss. for α	(78)
	$\sigma(\theta_1 \geq \theta_2) = \sigma(\theta_1) \geq \sigma(\theta_2)$		(79)
	$\sigma(p(\theta)) = \{\cdot \mapsto \sigma(\theta)\}(\sigma p), p \in \sigma$,else $p(\sigma(\theta))$		(80)
	$\sigma(p(w)) = \{\cdot \mapsto \sigma(w)\}(\sigma p), p \in \sigma$,else $p(\sigma(w))$		(81)
	$\sigma(P) = (\sigma P), P \in \sigma$, else P		(82)
	$\sigma(\neg\phi) = \neg\sigma(\phi)$		(83)
	$\sigma(\phi \wedge \psi) = \sigma(\phi) \wedge \sigma(\psi)$		(84)
	$\sigma(\exists x : \mathbb{R}\ \phi) = \exists x : \mathbb{R}\ \sigma(\phi)$	σ $\{x\}$-admiss. for ϕ	(85)
	$\sigma(\langle\alpha\rangle\phi) = \langle\sigma(\alpha)\rangle\sigma(\phi)$	σ BV$(\sigma(\alpha))$-admiss. for ϕ	(86)
	$\sigma(@_w\phi) = @_{\sigma(w)}\sigma(\phi)$	σ $\mathcal{V}$-admiss. for ϕ	(87)
	$\sigma(@_w\theta) = @_{\sigma(w)}\sigma(\theta)$	σ $\mathcal{V}$-admiss. for θ	(88)
	$\sigma(\forall s : \mathcal{W}\ \phi) = \forall s : \mathcal{W}\ \sigma(\phi)$	σ $\{s\}$-admiss. for ϕ	(89)
	$\sigma(\exists s : \mathcal{W}\ \phi) = \exists s : \mathcal{W}\ \sigma(\phi)$	σ $\{s\}$-admiss. for ϕ	(90)
	$\sigma(\downarrow s\ \phi) = \downarrow s\ \sigma(\phi)$	σ $\{s\}$-admiss. for ϕ	(91)
	$\sigma(\overline{n}) = \overline{n},$ if $\overline{n} \notin \sigma$, else $\sigma\overline{n}$		(92)
	$\sigma(s) = s$		(93)

Figure 10: Uniform Substitution Algorithm

We now establish a series of lemmas leading up to the reduction

Lemma 13 (Surjectivity). For all ω_h, I_h exist ω_d, S, I_d such that $\omega_h = \omega_d \cup S$ and $I_d \sqsubseteq I_h$

Proof. By straightforward construction, let $\omega_d = D(\omega_h), S = (\omega_h \setminus \omega_d), I_d = D(I_h)$. □

Lemma 14 (Term inclusion). For all $\omega_h = \omega_d \cup S$ and $I_d \sqsubseteq I_h$ and all θ, $[\![\theta]\!]\omega_d I_d = [\![\theta]\!]\omega_h I_h$.

Proof. By induction on the term θ.

- **Case** $c \in \mathbb{Q} : [\![c]\!]\omega_d I_d = c = [\![c]\!]\omega_h I_h$.
- **Case** $x \in \mathcal{V} : [\![x]\!]\omega_d I_d = \omega_d(x) = \omega_h(x) = [\![x]\!]\omega_h I_h$ because $\omega_d \subseteq \omega_h$.
- **Case** $\theta_1 + \theta_2 : [\![\theta_1 + \theta_2]\!]\omega_d I_d = [\![\theta_1]\!]\omega_d I_d + [\![\theta_2]\!]\omega_d I_d = [\![\theta_1]\!]\omega_h I_h + [\![\theta_2]\!]\omega_h I_h = [\![\theta_1 + \theta_2]\!]\omega_h I_h$ by IH.
- **Case** $\theta_1 \cdot \theta_2 : [\![\theta_1 \cdot \theta_2]\!]\omega_d I_d = [\![\theta_1]\!]\omega_d I_d \cdot [\![\theta_2]\!]\omega_d I_d = [\![\theta_1]\!]\omega_h I_h \cdot [\![\theta_2]\!]\omega_h I_h = [\![\theta_1 \cdot \theta_2]\!]\omega_h I_h$ by IH.
- **Case** $f(\theta) : [\![f(\theta)]\!]\omega_d I_d = I_d(f)([\![\theta]\!]\omega_d I_d) = I_h(f)([\![\theta]\!]\omega_d I_d) = I_h(f)([\![\theta]\!]\omega_h I_h) = [\![f(\theta)]\!]\omega_h I_h$ by IH and by $I_d \sqsubseteq I_h$.
- **Case** $F : [\![F]\!]\omega_d I_d = I_d(F)(\omega_d) = I_h(F)(\omega_h) = [\![F]\!]\omega_h I_h$ by $I_d \sqsubseteq I_h$.

□

Lemma 15 (Program inclusion). For all $\omega_d, \nu_d, S, I_d, I_h$ if $(\omega_d, \nu_d) \in [\![\alpha]\!] I_d$ and $I_d \sqsubseteq I_h$ then $(\omega_h, \nu_h) \in [\![\alpha]\!] I_h$ for $\omega_h = \omega_d \cup S$ and $\nu_h = \nu_d \cup S$.

Proof. By induction on α, in simultaneous induction with Lemma 16.

- **Case** $x := \theta : (\omega_d, \nu_d) \in [\![x := \theta]\!] I_d$ implies $\nu_d = \omega_d{}_x^{[\![\theta]\!]\nu_d I_d}$ equals $\omega_d{}_x^{[\![\theta]\!]\omega_h I_h}$ by Lemma 14. By semantics, $(\omega_h, \omega_h{}_x^{[\![\theta]\!]\omega_h I_h}) \in [\![x := \theta]\!] I_h$, then note by assumptions and set arithmetic, have $\nu_h = \omega_h{}_x^{[\![\theta]\!]\omega_h I_h}$, completing the case.
- **Case** $x := * : (\omega_d, \nu_d) \in [\![x := *]\!] I_d$ implies $\nu_d = \omega_d{}_x^r$ for some $r \in \mathbb{R}$. By semantics, $(\omega_h, \omega_h{}_x^r) \in [\![x := *]\!] I_h$ then note by assumptions and set arithmetic have $\nu_h = \omega_h{}_x^r$, completing the case.
- **Case** $?(\phi) : (\omega_d, \nu_d) \in [\![?\phi]\!] Ig$ implies $\omega_d = \nu_d$ and $\omega_d \in [\![\phi]\!] I_d$. By assumptions then $\omega_h = \nu_h$. By Lemma 16 have $\omega_h \in [\![\phi]\!] I_h$, completing the case.
- **Case** $x' = \theta \,\&\, \psi : (\omega_d, \nu_d) \in [\![x' = \theta \,\&\, \psi]\!] I_d$ implies $\omega_d = \varphi_d(0), \nu_d = \varphi_d(t), t \geq 0$ and for all $s \in [0, t] I_d \in [\![H]\!] \varphi_d(s)$ where φ_d solves the ODE $x' = \theta$ on interval $[0, t]$. Now construct a new solution $\varphi_h(r)(x) = \varphi_d(r)(x)$ for program variables x and $\varphi_h(r)(s) = \omega_h(s) = \nu_h(s)$. Then φ_h must also be a solution because φ_d is a solution and an ODE can only bind program variables anyway. We still have $t \geq 0$ and by Lemma 16 we also have $\forall s \in [0, t] I_d \in [\![H]\!] \varphi_d(s)$, which suffices to show $(\omega_h, \nu_h) \in [\![x' = \theta \,\&\, \psi]\!] I_h$ because $\varphi_h(0) = \omega_h$ and $\varphi_h(t) = \nu_h$ by construction of φ_h and the assumptions on ω_h, ν_h.

- **Case** $\alpha \cup \beta$: $(\omega_d, \nu_d) \in [\![\alpha \cup \beta]\!] I_d$ implies $(\omega_d, \nu_d) \in [\![\alpha]\!] I_d$ and $(\omega_d, \nu_d) \in [\![\beta]\!] I_d$ so by IH $(\omega_h, \nu_h) \in [\![\alpha]\!] I_h$ and $(\omega_h, \nu_h) \in [\![\beta]\!] I_h$ so $(\omega_h, \nu_h) \in [\![\alpha \cup \beta]\!] I_h$.
- **Case** $\alpha; \beta : (\omega_d, \nu_d) \in [\![\alpha; \beta]\!] I_d$ implies there exists μ_d such that $(\omega_d, \mu_d) \in [\![\alpha]\!] I_d$ and $(\mu_d, \nu_d) \in [\![\beta]\!] I_d$. Let $\mu_h = \mu_d \cup S$. Then we can apply the IHs getting $(\omega_h, \mu_h) \in [\![\alpha]\!] I_h$ and $(\mu_h, \nu_h) \in [\![\beta]\!] I_h$ so $(\omega_h, \nu_h) \in [\![\alpha; \beta]\!] I_h$.
- **Case** α^* : $(\omega_d, \nu_d) \in [\![\alpha^*]\!] I_d$ implies $(\omega_d, \nu_d) \in ([\![\alpha]\!] I_d)^*$ implies $(\omega_d, \nu_d) \in ([\![\alpha]\!] I_d)^k$ for some $k \in \mathbb{N}$, where $([\![\alpha]\!] I_d)^0 = \{(\omega, \omega) \mid \omega \in \mathcal{W}\}$ and $([\![\alpha]\!] I_d)^{k+1} = \{(\omega, \nu) \mid (\omega, \mu) \in [\![\alpha]\!] I_d \text{ and } (\mu, \nu) \in ([\![\alpha]\!] I_d)^k \text{ for some } \mu\}$. Proceed by induction on k. In the case $k = 0$ then $\nu_d = \omega_d$ and $\nu_h = \omega_h$ and $(\omega_h, \nu_h) \in [\![\alpha^*]\!] I_h$. In the case $k + 1$ by have (1) $(\omega_d, \mu_d) \in [\![\alpha]\!] I_d$ and (2) $(\mu_d, \nu_d) \in [\![\alpha^*]\!] I_h$. From (2) by inner IH, $(\mu_h, \nu_h) \in [\![\alpha^*]\!] I_h$ where $\mu_h = \mu_d \cup S$. From (1) by outer IH have $(\omega_h, \mu_h) \in [\![\alpha]\!] I_h$, thus $(\omega_h, \nu_h) \in [\![\alpha^*]\!] I_h$.
- **Case** a : $(\omega_d, \nu_d) \in [\![a]\!] I_d$ implies $(\omega_d, \nu_d) \in I_d(a)$ so $(\omega_h, \nu_h) \in I_h(a)$ by $I_d \sqsubseteq I_h$ so $(\omega_h, \nu_h) \in [\![a]\!] I_h$.

□

Lemma 16 (Formula Inclusion). If $\omega_d \subseteq \omega_h$ and $I_d \sqsubseteq I_h$ then for all $\phi \in \mathsf{dL}$ have $I_d \in [\![\phi]\!] \omega_d$ iff $I_h \in [\![\phi]\!] \omega_d$.

Proof. By induction on ϕ, in mutual induction with Lemma 15.

- **Case** $\theta_1 \sim \theta_2 : I_d \in [\![\theta_1 \sim \theta_2]\!] \omega_d$ iff $[\![\theta_1]\!] \omega_d I_d \sim [\![\theta_2]\!] \omega_d I_d$ iff (by Lemma 14) $[\![\theta_1]\!] \omega_h I_h \sim [\![\theta_2]\!] \omega_h I_h$ iff $[\![\theta_1 \sim \theta_2]\!] \omega_d I_d$.
- **Case** $\phi \wedge \psi : I_d \in [\![\phi \wedge \psi]\!] \omega_d$ iff $I_d \in [\![\phi]\!] \omega_d$ and $I_d \in [\![\psi]\!] \omega_d$ iff (by IH) $I_h \in [\![\phi]\!] \omega_h$ and $I_h \in [\![\psi]\!] \omega_h$ iff $I_h \in [\![\phi \wedge \psi]\!] \omega_h$.
- **Case** $\neg\phi : I_h \in [\![\neg\phi]\!] \omega_h$ iff not $I_h \in [\![\phi]\!] \omega_h$ iff (by IH) not $I_h \in [\![\phi]\!] \omega_h$ iff $I_h \in [\![\neg\phi]\!] \omega_h$.
- **Case** $\exists x : \mathbb{R}\ \phi : I_d \in [\![\exists x : \mathbb{R}\ \phi]\!] \omega_d$ iff exists $r \in \mathbb{R}$ such that $I_d \in [\![\phi]\!] \omega_{d_x^r}$ iff (by IH) $I_h \in [\![\phi]\!] \omega_{h_x^r}$ (since $\omega_{d_x^r} \cup S = (\omega_d \cup S)_x^r$) iff $I_h \in [\![\exists x : \mathbb{R}\ \phi]\!] \omega_h$.
- **Case** $\langle\alpha\rangle\phi$: $I_d \in [\![\langle\alpha\rangle\phi]\!] \omega_d$ iff exists ν_d s.t. (1) $(\omega_d, \nu_d) \in [\![\alpha]\!] I_d$ and (2) $I_d \in [\![\phi]\!] \omega_d$ iff (1) $(\omega_h, \nu_h) \in [\![\alpha]\!] I_h$ and (2) $I_h \in [\![\phi]\!] \nu_h$ by IH 1 and 2 respectively iff $I_h \in [\![\langle\alpha\rangle\phi]\!] \omega_h$.
- **Case** $p(\theta) : I_d \in [\![p(\theta)]\!] \omega_d$ iff $I_d(f)([\![\theta]\!] \omega_d I_d)$ iff $I_d(f)([\![\theta]\!] \omega_h I_h)$ (by Lemma 14) iff (by $I_d \sqsubseteq I_h$) $\omega_h \in [\![f(\theta)]\!] I_h$.
- **Case** $P : I_d \in [\![P]\!] \omega_d$ iff $I_d(P)(\omega_d)$ iff (by $I_d \sqsubseteq I_h$ and $\omega_d \subseteq \omega_h$) $I_h(P)(\omega_h)$ iff $I_h \in [\![P]\!] \omega_h$.

□

Having completed the lemmas we can complete the main proof of reducibility:

Theorem 17 (dHL contains dL). For all $\phi \in \mathsf{dL}$, ϕ is valid in dL iff ϕ is valid in dHL

Proof. To show ϕ is valid in dHL, fix an interpretation I_h and state ω_h and show $I_h \in [\![\phi]\!] \omega_h$. By Lemma 13, exist I_d and ω_d, S such that $I_d \sqsubseteq I_h$ and $I_h = I_d \cup S$. By validity of ϕ in dL, have $I_d \in [\![\phi]\!] \omega_d$. Then by Lemma 16, $I_h \in [\![\phi]\!] \omega_h$. □

C Concrete Reducibility

There are two intuitions behind the concrete reduction:

1. World variables can be simulated with program variables, with nominal constants likewise simulated by constant functions.

2. Such a simulation can be done finitely despite the infinity of states because (a) dL constructs see only explicitly-mentioned variables and (b) while hybrid constructs see the unmentioned variables, they see no difference between finitely or infinitely-many unmentioned variables.

Point 2 is shown formally by showing that in both dL and dHL, validity for concrete formulas agrees with *finite-domain* validity where states, galaxies and interpretations are non-zero on only finitely many variables. Then point 1 is shown by induction because the translation preserves finite-domain validity.

In what follows we fix arbitrary bijections to $\mathbb{R}$ P_ω, P_g from $\mathbb{R}^\mathbb{N}$ and $\left(\mathbb{R}^\mathbb{N}\right)^\mathbb{N}$ respectively. We also let r always refer some canonical variable that is fresh in the translated formula ϕ. Furthermore $\mathrm{V}(\phi) = \mathrm{FV}(\phi) \cup \mathrm{BV}(\phi)$ refers to all variables mentioned in ϕ. As in Appendix A we use $S_{|\mathbb{R}}$ for the restriction of flexible symbol set S to just program variables and $S_{|\mathbb{W}}$ for the restriction to just world variables.

Definition 13 (Finite-domain world)**.** A world ω is *finite-domain* with domain $\mathrm{V}(\phi)_{|\mathbb{R}}$ iff $\{x \mid \omega(x) \neq 0\}$ is finite.

Definition 14 (Finite-domain galaxy)**.** A galaxy g is *finite-domain* with domain $\mathrm{V}(\phi)_{|\mathbb{W}}$ iff $\{s \mid g(s) \neq \omega_0\}$ is finite, where ω_0 is $\{(x, 0) \mid x \in \mathcal{V}\}$.

Definition 15 (Finite-domain interpretation)**.** An interpretation I is *finite-domain* (with domain S) iff (1) for all $\overline{n}$, $I(n)$ has domain $S_{|\mathbb{R}}$ and (2) For all p, f, w_1, w_2 if w_1 and w_2 agree on $S_{|\mathbb{R}}$ then $I(p)(w_1) = I(p)(w_2)$ and $I(f)(w_1) = I(f)(w_2)$.

Definition 16 (Finite validity)**.** A formula ϕ is *finitely-valid* iff for all finite-domain I, g, ω, $Ig \in [\![\phi]\!]\omega$

We define a bijection $(\widetilde{\omega})$ between dHL worlds and finite-domain dHL worlds. We write the inverse direction of the bijection with inverse notation $\underset{\sim}{\omega}$.

World translation in dHL:

$$(\widetilde{\omega}) = \{(x, \omega(x)) \mid x \in \mathrm{V}(\phi)_{|\mathbb{R}}\} \cup \{(x, 0) \mid x \notin \mathrm{V}(\phi)_{|\mathbb{R}}\}$$
$$\cup \{(r, P_\omega(\nu))\} \text{ for } \nu = \{(x, \omega(x)) \mid x \notin \mathrm{V}(\phi)_{|\mathbb{R}}\}$$

The fact that $(\widetilde{\omega})$ is a bijection follows directly from P_ω being a bijection.

The bijection $(\widetilde{g})$ for galaxies is a simple extension:

Galaxy translation in dHL:

$$(\widetilde{g}) = \{(s, \widetilde{(g(s))}) \mid s \in \mathrm{V}(\phi)_{|\mathbb{W}}\} \tag{94}$$

The bijection $(\widetilde{I})$ for interpretations proceeds by cases on rigid symbols. We write the inverse direction of the bijection with inverse notation $\underset{\sim}{I}$. The rigid symbols that can appear in a concrete formula are nominals $\overline{n}$ and untyped predicates p and untyped functions f.

Definition 17 (Interpretation translation in dHL)**.**

$$\begin{aligned}(\widetilde{I})(f)(x) &= I(f)(x)\\ (\widetilde{I})(p)(x) &= I(p)(x)\\ (\widetilde{I})(\overline{n}) &= (\widetilde{I(\overline{n})})\\ (\widetilde{I})(p)(w) &= I(p)(\underset{\sim}{w})\\ (\widetilde{I})(f)(w) &= I(f)(\underset{\sim}{w})\end{aligned}$$

We next define translation from finite dHL to finite dL
World+galaxy finitization

$$(\omega\overline{\times}g) = \omega \cup \{(x_n, g(n)(x)) \mid x \in S_{|\mathbb{R}}, n \in S_{|\mathbb{W}}\}$$

Interpretation finitization

$$\begin{aligned}(\widetilde{I})(f)(x) &= I(f)(x)\\ (\widetilde{I})(p)(x) &= I(p)(x)\\ (\widetilde{I})(f)(w) &= I(f)(x_n \mid x \in S_{|\mathbb{R}})\\ (\widetilde{I})(p)(w) &= I(p)(x_n \mid x \in S_{|\mathbb{R}})\\ (\widetilde{I})(x_n()) &= (\widetilde{I(\overline{n})})(x)\\ (\widetilde{I})(p)(w) &= I(p)(\underset{\sim}{w})\\ (\widetilde{I})(f)(w) &= I(f)(\underset{\sim}{w})\end{aligned}$$

The corresponding definitions in dL are simpler because there are no nominals:
World translation in dL

$$(\widetilde{\omega}) = \{(x, \omega(x)) \mid x \in \mathrm{V}(\phi)\} \cup \{(x, 0) \mid x \notin \mathrm{V}(\phi)\} \tag{95}$$

Definition 18 (Interpretation translation in dL)**.**

$$\begin{aligned}(\widetilde{I})(f)(x) &= I(f)(x)\\ (\widetilde{I})(p)(x) &= I(p)(x)\\ (\widetilde{I})(\overline{n}) &= (\widetilde{I(\overline{n})})\\ (\widetilde{I})(p)(w) &= I(p)(\underset{\sim}{w})\\ (\widetilde{I})(f)(w) &= I(f)(\underset{\sim}{w})\end{aligned}$$

Having defined the key concepts, we can state the lemmas that contain all the work of the proof:

Lemma 18 (Concrete dHL Finitization - World terms). $(\widetilde{[\![w]\!]\omega I g}) = [\![w]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g})$.

Proof. By cases on w.

- **Case** $\overline{n}$: Then $(\widetilde{[\![\overline{n}]\!]\omega Ig}) = (\widetilde{I(n)}) = (\widetilde{I})(n) = [\![\overline{n}]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g})$.
- **Case** s: Then $(\widetilde{[\![s]\!]\omega Ig}) = (\widetilde{g(s)}) = (\widetilde{g})(s) = [\![s]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g})$.

□

Lemma 19 (Concrete dHL Finitization - Real terms). $[\![\theta]\!]\omega Ig = [\![\theta]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g})$.

Proof. By induction on θ.

- **Case** c : $[\![c]\!]\omega Ig = c = [\![c]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g})$
- **Case** x : $[\![x]\!]\omega Ig = \omega(x) = (\widetilde{\omega})(x) = [\![x]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g})$ by fact $x \in \mathrm{V}(x)$ and def of $(\widetilde{\omega})$.
- **Case** $\theta_1 + \theta_2$: $[\![\theta_1 + \theta_2]\!]\omega Ig = [\![\theta_1]\!]\omega Ig + [\![\theta_2]\!]\omega Ig = [\![\theta_1]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g}) + [\![\theta_2]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g}) = [\![\theta_1 + \theta_2]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g})$
- **Case** Have $\theta_1 \cdot \theta_2$: $[\![\theta_1 \cdot \theta_2]\!]\omega Ig = [\![\theta_1]\!]\omega Ig \cdot [\![\theta_2]\!]\omega Ig = [\![\theta_1]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g}) \cdot [\![\theta_2]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g}) = [\![\theta_1 \cdot \theta_2]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g})$
- **Case** $f(\theta)$: $[\![f(\theta)]\!]\omega Ig = I(f)([\![\theta]\!]\omega Ig) = (\widetilde{I})(f)([\![\theta]\!]\omega Ig) = (\widetilde{I})(f)([\![\theta]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g})) = [\![f(\widetilde{())}]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g})$ by definition of translation for $f(x\colon \mathbb{R})$.
- **Case** $f(w)$: $[\![f(w)]\!]\omega Ig = I(f)([\![w]\!]\omega Ig)$ and

$$\begin{aligned}
[\![f(w)]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g}) = \\
(\widetilde{I})(f)([\![f(w)]\!](\widetilde{I})(\widetilde{g})) = \\
(\widetilde{I})(f)((\widetilde{[\![f(w)]\!]\omega Ig})) = \\
I(f)((\widetilde{[\![f(w)]\!]\omega Ig})) = \\
I(f)([\![f(w)]\!]\omega Ig)
\end{aligned}$$

  by IH and Lemma 18 so both sides are equal.

- **Case** $@_w\theta$: $[\![@_w\theta]\!]\omega Ig = [\![\theta]\!]([\![w]\!]Ig)Ig = [\![\theta]\!](\widetilde{[\![w]\!]Ig})(\widetilde{I})(\widetilde{g}) = [\![\theta]\!]([\![w]\!](\widetilde{I})(\widetilde{g}))(\widetilde{I})(\widetilde{g}) = [\![@_w\theta]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g})$ by IH and Lemma 18.

□

Lemma 20 (Concrete dHL Finitization - Formulas and Programs). A concrete dHL formula is valid iff it is finitely valid. Specifically, $Ig \in [\![\phi]\!]\omega$ iff $(\widetilde{I})(\widetilde{g}) \in [\![\phi]\!](\widetilde{\omega})$. By simultaneous induction we also show finitization for programs: $(\omega, \nu) \in [\![\alpha]\!]Ig$ iff $((\widetilde{\omega}), (\widetilde{\nu})) \in [\![\alpha]\!](\widetilde{I})(\widetilde{g})$.

Proof.
- **Case** Have $\theta_1 \geq \theta_2$: $Ig \in [\![\theta_1 \geq \theta_2]\!]\omega$ iff $[\![\theta_1]\!]\omega Ig \geq [\![\theta_2]\!]\omega Ig$ iff $[\![\theta_1]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g}) \geq [\![\theta_2]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g})$ iff $(\widetilde{I})g \in [\![\theta_1 \geq \theta_2]\!](\widetilde{\omega})$.

- **Case** $\phi\wedge\psi$: $Ig\in[\![\phi\wedge\psi]\!]\omega$ iff $Ig\in[\![\phi]\!]\omega$ and $Ig\in[\![\psi]\!]\omega$ iff $(\widetilde{I})(\widetilde{g})\in[\![\phi]\!](\widetilde{\omega})$ and $(\widetilde{I})(\widetilde{g})\in[\![\psi]\!](\widetilde{\omega})$ iff $(\widetilde{I})(\widetilde{g})\in[\![\phi\wedge\psi]\!](\widetilde{\omega})$.

- **Case** $\neg\phi$: $Ig\in[\![\neg\phi]\!]\omega$ iff not $Ig\in[\![\phi]\!]\omega$ iff not $(\widetilde{I})(\widetilde{g})\in[\![\phi]\!](\widetilde{\omega})$ iff $(\widetilde{I})(\widetilde{g})\in[\![\neg\phi]\!](\widetilde{\omega})$.

- **Case** $\exists x:\mathbb{R}\ \phi$: $Ig\in[\![\exists x:\mathbb{R}\ \phi]\!]\omega$ iff exists $r\in\mathbb{R}$ s.t. $Ig\in[\![\phi]\!]\omega^r_x$ iff exists $r\in\mathbb{R}$ s.t. $(\widetilde{I})(\widetilde{g})\in[\![\phi]\!](\widetilde{\omega})^r_x$ iff exists $r\in\mathbb{R}$ s.t. $(\widetilde{I})(\widetilde{g})\in[\![\phi]\!](\widetilde{\omega^r_x})$ (for $x\in \mathrm{V}(\phi)$)iff $(\widetilde{I})(\widetilde{g})\in[\![\exists x:\mathbb{R}\ \phi]\!](\widetilde{\omega})$.

- **Case** $p(\theta)$: $Ig\in[\![p(\theta)]\!]\omega$ iff $I(p)([\![\theta]\!]\omega Ig)$ iff $I(p)([\![\theta]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g}))$ iff $(\widetilde{I})(p)([\![\theta]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g}))$ iff $(\widetilde{I})(\widetilde{g})\in[\![p(\theta)]\!](\widetilde{\omega})$.

- **Case** $p(w)$: $Ig\in[\![p(w)]\!]\omega$ iff $I(p)([\![w]\!]Ig\omega)$ iff $I(p)(\widetilde{[\![w]\!]Ig\omega})$ and $(\widetilde{I})(\widetilde{g})\in[\![p(w)]\!](\widetilde{\omega})$ iff $(\widetilde{I})(p)([\![w]\!](\widetilde{I})(\widetilde{g})(\widetilde{\omega}))$ iff $I(p)([\![w]\!]\underbrace{(\widetilde{I})(\widetilde{g})}(\widetilde{\omega}))$ iff $I(p)(\underbrace{([\![w]\!]Ig\omega)})$ iff $I(p)([\![w]\!]Ig\omega)$ so both sides are equal.

- **Case** $\langle\alpha\rangle\phi$: $Ig\in[\![\langle\alpha\rangle\phi]\!]\omega$ iff there exists ν s.t. $(\omega,\nu)\in[\![\alpha]\!]Ig$ and $Ig\in[\![\phi]\!]\nu$ iff there exists ν s.t. $((\widetilde{\omega}),(\widetilde{\nu}))\in[\![\alpha]\!](\widetilde{I})(\widetilde{g})$ and $(\widetilde{I})(\widetilde{g})\in[\![\phi]\!](\widetilde{\nu})$ iff there exists $(\widetilde{\nu})$ s.t. $((\widetilde{\omega}),(\widetilde{\nu}))\in[\![\alpha]\!](\widetilde{I})(\widetilde{g})$ and $(\widetilde{I})(\widetilde{g})\in[\![\phi]\!](\widetilde{\nu})$ iff there $(\widetilde{I})(\widetilde{g})\in[\![\langle\alpha\rangle\phi]\!](\widetilde{\omega})$

- **Case** $@_w\phi$: in this case $Ig\in[\![@_w\phi]\!]\omega$ iff $Ig\in[\![\phi]\!]I([\![w]\!]Ig\omega)$ iff $(\widetilde{I})(\widetilde{g})\in[\![\phi]\!](\widetilde{[\![w]\!]Ig\omega})$ iff $(\widetilde{I})(\widetilde{g})\in[\![\phi]\!][\![w]\!](\widetilde{I})(\widetilde{g})(\widetilde{\omega})$ iff we have $(\widetilde{I})(\widetilde{g})\in[\![@_w\phi]\!](\widetilde{\omega})$

- **Case** $\exists s:\mathcal{W}\ \phi$: $Ig\in[\![\exists s:\mathcal{W}\ \phi]\!]\omega$ iff exists $\nu\in\mathcal{W}$ s.t. $Ig^\nu_s\in[\![\phi]\!]\omega$ iff exists $\nu\in\mathcal{W}$ s.t. $(\widetilde{I})(\widetilde{g^\nu_s})\in[\![\phi]\!](\widetilde{\omega})$ iff exists $\nu\in\mathcal{W}$ s.t. $(\widetilde{I})(\widetilde{g})^{(\widetilde{\nu})}_s\in[\![\phi]\!](\widetilde{\omega})$ iff exists $\nu\in\mathcal{W}$ s.t. $(\widetilde{I})(\widetilde{g})^\nu_s\in[\![\phi]\!](\widetilde{\omega})$ iff $(\widetilde{I})(\widetilde{g})\in[\![\exists s:\mathcal{W}\ \phi]\!](\widetilde{\omega})$

- **Case** $\downarrow s\ \phi$: Have $Ig\in[\![\downarrow s\ \phi]\!]\omega$ iff $Ig^\omega_s\in[\![\phi]\!]\omega$ iff $(\widetilde{I})(\widetilde{g^\omega_s})\in[\![\phi]\!](\widetilde{\omega})$ iff $(\widetilde{I})(\widetilde{g})^{(\widetilde{\omega})}_s\in[\![\phi]\!](\widetilde{\omega})$ iff $(\widetilde{I})(\widetilde{g})\in[\![\downarrow s\ \phi]\!](\widetilde{\omega})$.

- **Case** w : $Ig\in[\![w]\!]\omega$ iff $\omega=[\![w]\!]Ig$ iff $(\widetilde{\omega})=[\![w]\!](\widetilde{I})(\widetilde{g})$ iff $(\widetilde{I})(\widetilde{g})\in[\![w]\!](\widetilde{\omega})$.

- **Case** $x:=\theta$: $(\omega,\nu)\in[\![x:=\theta]\!]Ig$ iff $\nu=\omega^{[\![\theta]\!]\omega Ig}_x$ iff $(\widetilde{nu})=(\widetilde{\omega})^{[\![\theta]\!](\widetilde{\omega})(\widetilde{I})(\widetilde{g})}_x$ (by $x\in\mathrm{V}(\alpha)$) iff $((\widetilde{\omega}),(\widetilde{\nu}))\in[\![x:=\theta]\!](\widetilde{I})(\widetilde{g})$

- **Case** $x:=*$: $(\omega,\nu)\in[\![x:=*]\!]Ig$ iff exists $r\in\mathbb{R}$ s.t. $\nu=\omega^r_x$ iff exists $r\in\mathbb{R}$ s.t. $(\widetilde{\nu})=(\widetilde{\omega})^r_x$ (by $x\in\mathrm{V}(\alpha)$) iff $((\widetilde{\omega}),(\widetilde{\nu}))\in[\![x:=*]\!](\widetilde{I})(\widetilde{g})$.

- **Case** $?(\phi)$: $(\omega,\nu)\in[\![?(\phi)]\!]Ig$ iff $Ig\in[\![\phi]\!]\omega$ and $\nu=\omega$ iff $(\widetilde{I})(\widetilde{g})\in[\![\phi]\!](\widetilde{\omega})$ and $(\widetilde{\nu})=(\widetilde{\omega})$ (by bijectivity) iff we have that $((\widetilde{\omega}),(\widetilde{\nu}))\in[\![?(\phi)]\!](\widetilde{I})(\widetilde{g})$

- **Case** $\{x'=\theta\,\&\,\psi\}$: $(\omega,\nu)\in[\![\{x'=\theta\,\&\,\psi\}]\!]Ig$ iff exists $t{\geq}0$ and φ such that φ solves $x'=\theta$ on $[0,t]$ with $Ig\in[\![\psi]\!]\varphi(s)$ for all $s\in[0,t]$ iff exists $t{\geq}0$ and $(\widetilde{\varphi})$ such that $(\widetilde{\varphi})$ solves $x'=\theta$ on $[0,t]$ with $(\widetilde{I})(\widetilde{g})\in[\![\psi]\!]\varphi(s)$ for all $s\in[0,t]$ by constructing $(\widetilde{\varphi})$ as per state translation and because $x\in\mathrm{V}(\alpha)$.

- **Case** $\alpha \cup \beta$: $(\omega,\nu) \in [\![\alpha \cup \beta]\!] Ig$ iff $(\omega,\nu) \in [\![\alpha]\!] Ig$ or $(\omega,\nu) \in [\![\beta]\!] Ig$ iff $((\widetilde{\omega}),(\widetilde{\nu})) \in [\![\alpha]\!](\widetilde{I})(\widetilde{g})$ or $((\widetilde{\omega}),(\widetilde{\nu})) \in [\![(\widetilde{\beta})]\!](\widetilde{I})(\widetilde{g})$ iff we have $((\widetilde{\omega}),(\widetilde{\nu})) \in [\![\alpha \cup \beta]\!](\widetilde{I})(\widetilde{g})$.

- **Case** $\alpha;\beta$: $(\omega,\nu) \in [\![\alpha;\beta]\!] Ig$ iff exists $\mu \in \mathcal{W}$ s.t. $(\omega,\mu) \in [\![\alpha]\!] Ig$ and $(\mu,\nu) \in [\![\beta]\!] Ig$ iff exists $\mu \in \mathcal{W}$ s.t. $((\widetilde{\omega}),(\widetilde{\mu})) \in [\![\alpha]\!](\widetilde{I})(\widetilde{g})$ and $((\widetilde{\mu}),(\widetilde{\nu})) \in [\![\beta]\!](\widetilde{I})(\widetilde{g})$ iff exists $\mu \in \mathcal{W}$ s.t. $((\widetilde{\omega}),\mu) \in [\![\alpha]\!](\widetilde{I})(\widetilde{g})$ and $(\mu,(\widetilde{\nu})) \in [\![\beta]\!](\widetilde{I})(\widetilde{g})$ by bijectivity iff $((\widetilde{\omega}),(\widetilde{\nu})) \in [\![\alpha;\beta]\!](\widetilde{I})(\widetilde{g})$

- **Case** α^* : $(\omega,\nu) \in [\![\alpha^*]\!] Ig$ iff exists $k \in \mathbb{N}$ s.t. $(\omega,\nu) \in ([\![\alpha]\!] Ig)^k$ exists $k \in \mathbb{N}$ s.t. $((\widetilde{\omega}),(\widetilde{\nu})) \in \left([\![\alpha]\!](\widetilde{I})(\widetilde{g})\right)^k$ (by obvious induction on k) $((\widetilde{\omega}),(\widetilde{\nu})) \in [\![\alpha^*]\!](\widetilde{I})(\widetilde{g})$.

□

Lemma 21 (Finite Translatability). A dHL formula ϕ is finitely-valid iff its translation $(\widetilde{\phi})$ is finitely-valid. Specifically, for I, g, ω, ν with finite domain S we have:

- $[\![\theta]\!]\omega Ig = [\![\widetilde{\theta}]\!]\widetilde{I}(\omega \overline{\times} g)$
- $\omega \in [\![\phi]\!] Ig$ iff $(\omega \overline{\times} g) \in [\![\widetilde{\phi}]\!]\widetilde{I}$
- $(\omega,\nu) \in [\![\alpha]\!] Ig$ iff $((\omega \overline{\times} g),(\nu \overline{\times} g)) \in [\![\widetilde{\alpha}]\!]\widetilde{I}$

Proof.

- **Case** c : $[\![c]\!]\omega Ig = c = \widetilde{c} = [\![\widetilde{c}]\!](\omega \overline{\times} g)\widetilde{I}$
- **Case** x : $[\![x]\!]\omega Ig = \omega(x) = \omega(\widetilde{x}) = (\omega \overline{\times} g)(\widetilde{x}) = [\![\widetilde{x}]\!](\omega \overline{\times} g)\widetilde{I}$.
- **Case** $\theta_1 + \theta_2$: $[\![\theta_1 + \theta_2]\!]\omega Ig = [\![\theta_1]\!]\omega Ig + [\![\theta_2]\!]\omega Ig = [\![\widetilde{\theta_1}]\!]\widetilde{I}(\omega \overline{\times} g) + [\![\widetilde{\theta_2}]\!](\omega \overline{\times} g)\widetilde{I} = [\![\widetilde{\theta_1 + \theta_2}]\!](\omega \overline{\times} g)\widetilde{I}$
- **Case** $\theta_1 \cdot \theta_2$: Have $[\![\theta_1 \cdot \theta_2]\!]\omega Ig = [\![\theta_1]\!]\omega Ig \cdot [\![\theta_2]\!]\omega Ig = [\![\widetilde{\theta_1}]\!]\widetilde{I}(\omega \overline{\times} g) \cdot [\![\widetilde{\theta_2}]\!](\omega \overline{\times} g)\widetilde{I} = [\![\widetilde{\theta_1 \cdot \theta_2}]\!](\omega \overline{\times} g)\widetilde{I}$
- **Case** $f(\theta)$: $[\![f(\theta)]\!]\omega Ig = I(f)([\![\theta]\!]\omega Ig) = I(f)([\![\widetilde{\theta}(\omega \overline{\times} g)\widetilde{I}]\!])=$
  $\widetilde{I}(f)([\![\widetilde{\theta}(\omega \overline{\times} g)\widetilde{I}]\!])=[\![f(\theta)]\!]\omega Ig$
- **Case** $f(\overline{n})$: $[\![f(\overline{n})]\!]\omega Ig = I(f)(I(n)) = \widetilde{I}(f)(I(n)_1,\ldots,I(n)_k) = [\![\widetilde{f(\overline{n})}]\!](\omega \overline{\times} g)\widetilde{I}$
- **Case** $f(s)$: In this case $[\![f(s)]\!]\omega Ig = I(f)(g(s)) = \widetilde{I}(s)((\omega \overline{\times} g)(n_1),\ldots,(\omega \overline{\times} g)(n_k)) = [\![\widetilde{f(s)}]\!](\omega \overline{\times} g)\widetilde{I}$
- **Case** $@_{\overline{n}}\theta$: $[\![@_{\overline{n}}\theta]\!]\omega Ig = [\![\theta]\!] I(n) Ig = [\![\theta]\!](I(n) \overline{\times} g)\widetilde{I} = [\![\theta]\!]\nu_{x_i}^{I(n)_i}\widetilde{I}$ for all ν by coincidence because $x_1,\ldots,x_k \supseteq \mathrm{FV}(\theta) = [\![\theta]\!]\omega_{x_i}^{I(n)_i}\widetilde{I} = [\![\theta]\!]\omega_{x_i}^{\widetilde{I}(n_i)}\widetilde{I} = [\![\theta_{x_i}^{n_i}]\!]\omega Ig = [\![\widetilde{\theta}]\!](\omega \overline{\times} g)\widetilde{I}$
- **Case** $\theta_1 \geq \theta_2$: $\omega \in [\![\theta_1 \geq \theta_2]\!] Ig$ iff $[\![\theta_1]\!]\omega Ig \geq [\![\theta_2]\!]\omega Ig$ iff $[\![\widetilde{\theta_1}]\!](\omega \overline{\times} g)\widetilde{I} \geq [\![\widetilde{\theta_2}]\!](\omega \overline{\times} g)\widetilde{I}$ iff $(\omega \overline{\times} g) \in [\![\theta_1 \geq \theta_2]\!]\widetilde{I}$

- **Case** $\phi \wedge \psi$: $\omega \in [\![\phi \wedge \psi]\!] Ig$ iff $\omega \in [\![\phi]\!] Ig$ and $\omega \in [\![\psi]\!] Ig$ iff $(\omega \overline{\times} g) \in \widetilde{[\![\phi]\!]} \widetilde{I}$ and $(\omega \overline{\times} g) \in \widetilde{[\![\psi]\!]} \widetilde{I}$ iff $(\omega \overline{\times} g) \in \widetilde{[\![\phi \wedge \psi]\!]} \widetilde{I}$.

- **Case** $\neg\phi$: $\omega \in [\![\neg\phi]\!] Ig$ iff not $\omega \in [\![\phi]\!] Ig$ iff not $(\omega \overline{\times} g) \in \widetilde{[\![\phi]\!]} \widetilde{I}$ iff $(\omega \overline{\times} g) \in \widetilde{[\![\neg\phi]\!]} \widetilde{I}$.

- **Case** $\exists x : \mathbb{R}\ \phi$: $\omega \in [\![\exists x : \mathbb{R}\ \phi]\!] Ig$ iff $\omega_x^r \in \widetilde{[\![\phi]\!] Ig}$ for some r iff $(\omega_x^r \overline{\times} g) \in \widetilde{[\![\phi]\!]} \widetilde{I}$ for some r iff $(\omega \overline{\times} g)_x^r \in \widetilde{[\![\phi]\!]} \widetilde{I}$ for some r iff $(\omega \overline{\times} g) \in \widetilde{[\![\exists x : \mathbb{R}\ \phi]\!]} \widetilde{I}$

- **Case** $p(\theta)$: $\omega \in [\![p(\theta)]\!] Ig$ iff $I(p)([\![\theta]\!] \omega Ig)$ iff $I(p)(\widetilde{[\![\theta]\!]}(\omega \overline{\times} g) \widetilde{I})$ iff $(\omega \overline{\times} g) \in \widetilde{[\![p(\theta)]\!]} \widetilde{I}$

- **Case** $\langle\alpha\rangle\phi$: $\omega \in [\![\langle\alpha\rangle\phi]\!] Ig$ iff exists $\nu \in \mathcal{W}$ s.t. $(\omega, \nu) \in [\![\alpha]\!] Ig$ and $\omega \in [\![\phi]\!] Ig$ iff exists $\nu \in \mathcal{W}$ s.t. $((\omega \overline{\times} g), (\nu \overline{\times} g)) \in \widetilde{[\![\alpha]\!]} \widetilde{I}$ and $(\omega \overline{\times} g) \in \widetilde{[\![\phi]\!]} \widetilde{I}$

- **Case** $p(\overline{n})$: $\omega \in [\![p(\overline{n})]\!] Ig$ iff $I(p)(I(\overline{n}))$ iff $\widetilde{I}(p)(I(\overline{n}_1, \ldots, \overline{n}_k))$ iff $\widetilde{I}(p)(\widetilde{I}(\overline{n_1}, \ldots, \overline{n_k}))$ iff $(\omega \overline{\times} g) \in [\![p(\overline{n})]\!] \widetilde{I}$

- **Case** $p(s)$: In this case we have $\omega \in [\![p(s)]\!] Ig$ iff $I(p)(g(s))$ iff $\widetilde{I}(p)(g(s)_1, \ldots, g(s)_k)$ iff $\widetilde{I}(p)((\omega \overline{\times})(s_1), \ldots, (\omega \overline{\times} g)(s_n))$ iff $(\omega \overline{\times} g) \in \widetilde{[\![p(s)]\!]} \widetilde{I}$

- **Case** $@_{\overline{n}}\phi$: $\omega \in [\![@_{\overline{n}}\phi]\!] Ig$ iff $I(n) \in [\![\phi]\!] Ig$ iff $(I(n) \overline{\times} g) \in \widetilde{[\![\phi]\!]} \widetilde{I}$ iff $(\omega \overline{\times} g)_{x_i}^{I(n)_i} \in \widetilde{[\![\phi]\!]} \widetilde{I}$ by coincidence lemma iff $(\omega \overline{\times} g)_{x_i}^{\widetilde{I}(n_i)} \in \widetilde{[\![\phi]\!]} \widetilde{I}$ iff $(\omega \overline{\times} g) \in \widetilde{[\![[x_i := n_i]\phi]\!]} \widetilde{I}$ iff $(\omega \overline{\times} g) \in \widetilde{[\![@_{\overline{n}}\phi]\!]} \widetilde{I}$

- **Case** $@_s\phi$: $\omega \in [\![@_s\phi]\!] Ig$ iff $g(s) \in [\![\phi]\!] Ig$ iff $(g(s) \overline{\times} g) \in \widetilde{[\![\phi]\!]} \widetilde{I}$ iff $(\omega \overline{\times} g)_{x_i}^{g(s)_i} \in \widetilde{[\![\phi]\!]} \widetilde{I}$ by coincidence lemma iff $(\omega \overline{\times} g)_{x_i}^{(g(s) \overline{\times} g)(s_i)} \in \widetilde{[\![\phi]\!]} \widetilde{I}$ iff $(\omega \overline{\times} g) \in \widetilde{[\![[x_i := s_i]\phi]\!]} \widetilde{I}$ iff $(\omega \overline{\times} g) \in \widetilde{[\![@_s\phi]\!]} \widetilde{I}$

- **Case** $\exists s : \mathcal{W}\ \phi$: Here $\omega \in [\![\exists s : \mathcal{W}\ \phi]\!] Ig$ iff exists $\nu \in \mathcal{W}$ s.t. $\omega \in [\![\phi]\!] Ig_s^\nu$ iff exists $\nu \in \mathcal{W}$ s.t. $(\omega \overline{\times} g_s^\nu) \in \widetilde{[\![\phi]\!]} \widetilde{I}$ iff exists $r_1, \ldots, r_k$ s.t. $(\omega \overline{\times} g)_{x_i}^{r_i} \in \widetilde{[\![\phi]\!]} \widetilde{I}$ iff $(\omega \overline{\times} g) \in \widetilde{[\![\exists x_i : \mathbb{R}\ \phi]\!]} \widetilde{I}$ iff $(\omega \overline{\times} g) \in \widetilde{[\![\exists s : \mathcal{W}\ \phi]\!]} \widetilde{I}$

- **Case** $\downarrow s\ \phi$: Here $\omega \in [\![\downarrow s\ \phi]\!] Ig$ iff $\omega \in [\![\phi]\!] Ig_s^\omega$ iff $(\omega \overline{\times} g_s^\omega) \in \widetilde{[\![\phi]\!]} \widetilde{I}$ iff $(\omega \overline{\times} g)_{s_i}^{\omega(x_i)} \in \widetilde{[\![\phi]\!]} \widetilde{I}$ iff $(\omega \overline{\times} g) \in \widetilde{[\![[s_i := x_i]\phi]\!]} \widetilde{I}$ iff $(\omega \overline{\times} g) \in \widetilde{[\![\downarrow s\ \phi]\!]} \widetilde{I}$.

- **Case** $\overline{n}$: $\omega \in [\![\overline{n}]\!] Ig$ iff $\omega = I(n)$ iff $\bigwedge\limits_{x_i} (\omega(x_i) = I(n)_i)$ iff $\bigwedge\limits_{x_i} \Big((\omega \overline{\times} g)(x_i) = \widetilde{I}(n_i) \Big)$ iff $\in [\![\bigwedge\limits_{x_i} (x_i = n_i)]\!] (\omega \overline{\times} g) \widetilde{I}$ iff $(\omega \overline{\times} g) \in \widetilde{[\![\overline{n}]\!]} \widetilde{I}$

- **Case** s : $\omega \in [\![s]\!] Ig$ iff $\omega = g(s)$ iff $\bigwedge\limits_{x_i} (\omega(x_i) = g(s)_i)$ iff $\bigwedge\limits_{x_i} ((\omega \overline{\times} g)(x_i) = (\omega \overline{\times} g)(s_i))$ iff $(\omega \overline{\times} g) \in [\![\bigwedge\limits_{x_i} (x_i = s_i)]\!] \widetilde{I}$ iff $(\omega \overline{\times} g) \in \widetilde{[\![s]\!]} \widetilde{I}$

- **Case** $x := \theta$: Here $(\omega, \nu) \in [\![x := \theta]\!] Ig$ iff $\nu = \omega_x^{[\![\theta]\!] \omega Ig}$ iff $\nu = \omega_x^{[\![\theta]\!] (\omega \overline{\times} g) \widetilde{I}}$ iff $(\nu \overline{\times} g) = (\omega \overline{\times} g)_x^{[\![\theta]\!] (\omega \overline{\times} g) \widetilde{I}}$ iff $((\omega \overline{\times} g), (\nu \overline{\times} g)) \in [\![x := \theta]\!] \widetilde{I}$

- **Case** $x := * : (\omega, \nu) \in [\![x := *]\!] Ig$ iff $\nu = \omega_x^r$, some $r \in \mathbb{R}$ iff $(\nu \overline{\times} g) = (\omega \overline{\times} g)_x^r$, some $r \in \mathbb{R}$ iff $((\omega \overline{\times} g), (\nu \overline{\times} g)) \in [\![x := *]\!] \widetilde{I}$.

- **Case** $?(\phi) : (\omega, \nu) \in [\![?(\phi)]\!] Ig$ iff $\omega = \nu$ and $\omega \in [\![\phi]\!] Ig$ iff $\omega = \nu$ and $(\omega \overline{\times} g) \in [\![\widetilde{\phi}]\!] \widetilde{I}$ iff $\widetilde{\omega} = \widetilde{\nu}$ and $(\omega \overline{\times} g) \in [\![\widetilde{\phi}]\!] \widetilde{I}$ iff $((\omega \overline{\times} g), (\nu \overline{\times} g)) \in [\![\widetilde{?(\phi)}]\!] \widetilde{I}$.

- **Case** $\{x' = \theta \,\&\, \psi\} : (\omega, \nu) \in [\![\{x' = \theta \,\&\, \psi\}]\!] Ig$ iff exists $t \geq 0$ and φ solves $x' = \theta$ on $[0, t]$ with $\varphi(0) = \omega, \varphi(t) = \nu$, and $\varphi(s) \in [\![\psi]\!] Ig$ for all $s \in [0, t]$. iff exists $t \geq 0$ and $\widetilde{\varphi}$ solves $x' = \widetilde{\theta}$ on $[0, t]$ with $\widetilde{\varphi}(0) = (\omega \overline{\times} g), \widetilde{\varphi}(t) = (\nu \overline{\times} g)$, and $\widetilde{\varphi}(s) \in [\![\widetilde{\psi}]\!] \widetilde{I}$ for all $s \in [0, t]$. Construct the solution $\widetilde{\varphi}$ by applying the $\widetilde{\omega}$ translation. Then $[\![\theta]\!] \varphi(t) Ig$ and $[\![\widetilde{\theta}]\!] \widetilde{\varphi}(t) \widetilde{I}$ are identical as functions of time thus φ solves $x' = \theta$ iff $\widetilde{\varphi}$ solves $x' = \widetilde{\theta}$. iff $((\omega \overline{\times} g), (\nu \overline{\times} g)) \in [\![\widetilde{\{x' = \theta \,\&\, \psi\}}]\!] \widetilde{I}$

- **Case** $\alpha \cup \beta : (\omega, \nu) \in [\![\alpha \cup \beta]\!] Ig$ iff $(\omega, \nu) \in [\![\alpha]\!] Ig$ or $(\omega, \nu) \in [\![\beta]\!] Ig$ iff $((\omega \overline{\times} g), (\nu \overline{\times} g)) \in [\![\widetilde{\alpha}]\!] \widetilde{I}$ or $((\omega \overline{\times} g), (\nu \overline{\times} g)) \in [\![\widetilde{\beta}]\!] \widetilde{I}$ iff $((\omega \overline{\times} g), (\nu \overline{\times} g)) \in [\![\widetilde{\alpha \cup \beta}]\!] \widetilde{I}$.

- **Case** $\alpha; \beta : (\omega, \nu) \in [\![\alpha; \beta]\!] Ig$ iff $(\omega, \mu) \in [\![\alpha]\!] Ig$ and $(\mu, \nu) \in [\![\beta]\!] Ig$ for some $\nu \in \mathcal{W}$ iff $((\omega \overline{\times} g), (\mu \overline{\times} g)) \in [\![\widetilde{\alpha}]\!] \widetilde{I}$ and $((\mu \overline{\times} g), (\nu \overline{\times} g)) \in [\![\widetilde{\beta}]\!] \widetilde{I}$ for some $\nu \in \mathcal{W}$ iff $(\widetilde{\omega} g, \widetilde{\nu} g) \in [\![\widetilde{\alpha; \beta}]\!] \widetilde{I}$.

- **Case** $\alpha^* :$ Here $(\omega, \nu) \in [\![\alpha^*]\!] Ig$ iff $(\omega, \nu) \in [\![\alpha^k]\!] Ig$, for some $k \in \mathbb{N}$ iff (inducting on k) $((\omega \overline{\times} g), (\nu \overline{\times} g)) \in [\![(\widetilde{\alpha})^k]\!] \widetilde{I}$ iff $((\omega \overline{\times} g), \widetilde{\nu} g) \in [\![\widetilde{\alpha^*}]\!] \widetilde{I}$.

□

Lemma 22 (Concrete dL Finitization). A concrete dL formula is valid iff it is finitely valid. Specifically, $I \in [\![\phi]\!] \omega$ iff $(\widetilde{I}) \in [\![\phi]\!] (\widetilde{\omega})$.

Proof. Trivially by coincidence: every concrete dL formula has finitely-many free variables, so every state agrees with some finite state on its domain, and thus satisfies ϕ iff the corresponding finite state does. □

Theorem 23 (dHL reduces to dL). There exists a computable reduction $\widetilde{\phi}$ such that for all concrete $\phi \in$ dHL, ϕ is valid dHL, iff $\widetilde{\phi}$ is valid in dHL.

Proof. Fix $\phi \in$ dHL. By Lemma 20, ϕ is valid in dHL iff it is finitely-valid. Then ϕ is finitely valid iff $\widetilde{\phi}$ is finitely valid in dL by Lemma 21. Then $\widetilde{\phi}$ is finitely-valid in dL iff it is valid by Lemma 22, so ϕ is valid iff $\widetilde{\phi}$ is valid. □

D Soundness Proofs

We begin with extending the soundness proofs of uniform substitution. When proofs build on prior work [38] we present only the new cases. We show a series of coincidence lemmas.

Lemma 24 (Coincidence for Terms). If $\omega \cup g = \tilde{\omega} \cup h$ on $\mathrm{FV}(\theta)$ and $I = J$ on $\Sigma(\theta)$ then $[\![\theta]\!]\omega Ig = [\![\theta]\!]\tilde{\omega} Jh$.

Proof. Induction on θ.

- **Case** $@_s\theta$ Have $\mathrm{FV}(@_s\theta) = \{s\} \cup \{t \mid t \in \mathrm{FV}(\theta)\}$. Then have $[\![@_s\theta]\!]\omega Ig = [\![\theta]\!]g(s)Ig = [\![\theta]\!]h(s)Jh$ by IH since $\omega(s) = \tilde{\omega}(s)$ on $\mathrm{FV}(\theta)$ from assumption equal on $\{t \mid t \in \mathrm{FV}(\theta)\}$.
- **Case** $@_{\overline{n}}\theta$ Have $\mathrm{FV}(@_{\overline{n}}\theta) = \{t \mid t \in \mathrm{FV}(\theta)\}$ and $\Sigma(@_{\overline{n}}\theta) = \{\overline{n}\} \cup \Sigma(\theta)$. Then have $[\![@_{\overline{n}}\theta]\!]\omega Ig = [\![\theta]\!]I(n)Ig = [\![\theta]\!]J(n)Jh$ by IH since $I(n) = J(n)$ on $\mathrm{FV}(\theta)$ from assumption equal on $\{t \mid t \in \mathrm{FV}(\theta)\}$ and agree on all program variables since $I(n) = J(n)$ by assumption.

The remaining cases are as in prior work. □

Lemma 25 (Coincidence for Formulas). If $\omega = \tilde{\omega}$ on $\mathrm{FV}(\phi)$ and $I = J$ on $\Sigma(\phi)$ then $Ig \in [\![\phi]\!]\omega$ iff $J \in [\![\phi]\!]\tilde{\omega}$.

Proof. Induction on ϕ (and simultaneous induction on α for coincidence for programs), but we show only the new cases.

- **Case** $s\ \omega \in [\![s]\!]Ig$ iff $g(s) = \omega$ iff (because $\mathcal{V}$ and s in $\mathrm{FV}(s)$) $h(s) = \tilde{\omega}$ iff $\tilde{\omega} \in [\![s]\!]Jh$.
- **Case** $\overline{n}\ \omega \in [\![\overline{n}]\!]Ig$ iff $I(n) = \omega$ iff (because $\mathcal{V}$ in $\mathrm{FV}(\overline{n})$ and $n \in \Sigma(\overline{n})$) $J(n) = \tilde{\omega}$ iff $\tilde{\omega} \in [\![\overline{n}]\!]Jh$.
- **Case** $@_s\phi\ \omega \in [\![@_s\phi]\!]Ig$ iff $g(s) \in [\![\phi]\!]Ig$ iff (because $\{t \mid t \in \mathrm{FV}(\phi)\}$ in free vars and s in free vars) iff $h(s) \in [\![\phi]\!]Jh$ iff $\tilde{\omega} \in [\![@_s\phi]\!]Jh$.
- **Case** $@_{\overline{n}}\phi\ \omega \in [\![@_{\overline{n}}\phi]\!]Ig$ iff $I(n) \in [\![\phi]\!]Ig$ iff (because $\{t \mid t \in \mathrm{FV}(\phi)\}$ in free vars and n in signature) iff $J(n) \in [\![\phi]\!]Jh$ iff $\tilde{\omega} \in [\![@_{\overline{n}}\phi]\!]Jh$.
- **Case** $\forall s : \mathcal{W}\ \phi\ \omega \in [\![\forall s : \mathcal{W}\ \phi]\!]Ig$ iff for all worlds ν, $\omega \in [\![\phi]\!]Ig^\nu_s$ iff (since the free variables are $\mathrm{FV}(\phi) \setminus \{s\}$) for all world ν have $\tilde{\omega} \in [\![\phi]\!]Jh^\nu_s$ iff $\tilde{\omega} \in [\![\forall s : \mathcal{W}\ \phi]\!]Jh$.
- **Case** $\exists s : \mathcal{W}\ \phi\ \omega \in [\![\exists s : \mathcal{W}\ \phi]\!]Ig$ iff for some world ν, $\omega \in [\![\phi]\!]Ig^\nu_s$ iff (since free vars are $\mathrm{FV}(\phi) \setminus \{s\}$) for some world ν have $\tilde{\omega} \in [\![\phi]\!]Jh^\nu_s$ iff $\tilde{\omega} \in [\![\exists s : \mathcal{W}\ \phi]\!]Jh$.
- **Case** $\downarrow s\ \phi\ \omega \in [\![\downarrow s\ \phi]\!]Ig$ iff $\omega \in [\![\phi]\!]Ig^\omega_s$ iff (since free vars are $\{x \in \mathcal{V}\} \cup \mathrm{FV}(\phi) \setminus \{s\}$) have $\tilde{\omega} \in [\![\phi]\!]Jh^{\tilde{\omega}}_s$ iff $\tilde{\omega} \in [\![\downarrow s\ \phi]\!]Jh$.

□

Lemma 26 (Coincidence for adjoints). Adjoint interpretations $\sigma^*_\omega I$ update the interpretation I to reflect the effect of a substitution σ: the meaning of every symbol substituted by σ is updated to the meaning of its replacement in state ω. Adjoints are analogous to prior work [38].

The notion of U-admissibility used here is as in Appendix A and as in prior work [38]:

Definition 19 (U-admissibility). We say a substitution σ is U-admissible for an expression e with a flexible symbol set U iff $\bigcup_{sym\in\sigma_{|\Sigma(e)}} \mathrm{FV}(\sigma sym)$ where $\sigma_{|\Sigma(e)}$ is the restriction of σ that replaces only symbols occurring in e and where sym is an arbitrary rigid.

If $\omega = \nu$ on $\mathrm{FV}(\sigma)$, then $\sigma^*_\omega I = \sigma^*_\nu I$. If σ is U-admissible for e and $\omega = \nu$ on U^C then $[\![e]\!]\sigma^*_\omega I = [\![e]\!]\sigma^*_\nu I$.

Proof. From prior work [38]. □

Lemma 27 (Term Substitution). $[\![\sigma(\theta)]\!]\omega Ig = [\![\theta]\!]\omega\sigma^*_\omega I$.

Proof. By induction on θ. We include just the new cases.

- **Case** $@_{\bar n}\theta$, $n \in \sigma$: $[\![\sigma(@_{\bar n}\theta)]\!]\omega Ig = [\![@_{\sigma n}\sigma(\theta)]\!]\omega Ig = [\![\sigma(\theta)]\!]([\![\sigma n]\!]\omega Ig)Ig$. Then let $\nu = [\![\sigma n]\!]\omega Ig$, then $[\![\sigma(\theta)]\!]\nu Ig = [\![\theta]\!]\nu\sigma^*_\nu I$ by IH, then by Lemma 26 and admissibility assumption, $[\![\theta]\!]\nu\sigma^*_\nu I = [\![\theta]\!]\nu\sigma^*_\omega I = [\![@_{\bar n}\phi]\!]\omega\sigma^*_\omega I$ since $\nu = [\![n]\!]\omega\sigma^*_{Ig} I$ by definition of adjoints.

- **Case** $@_{\bar n}\theta$, $n \notin \sigma$: $[\![\sigma(@_{\bar n}\theta)]\!]\omega Ig = [\![@_{\bar n}\sigma(\theta)]\!]\omega Ig = [\![\sigma(\theta)]\!]I(n)Ig$. Then let $\nu = I(n)$, then $[\![\sigma(\theta)]\!]\nu Ig = [\![\theta]\!]\nu\sigma^*_\nu I$ by IH, then by Lemma 26 and admissibility assumption, $[\![\theta]\!]\nu\sigma^*_\nu I = [\![\theta]\!]\nu\sigma^*_\omega I = [\![@_{\bar n}\phi]\!]\omega\sigma^*_\omega I$ since $\nu = \omega^{\sigma^*_{Ig} I(\bar n)}_{\mathtt{state}}$ by definition of adjoints.

- **Case** $@_s\theta$: $[\![\sigma(@_s\theta)]\!]\omega Ig = [\![@_s\sigma(\theta)]\!]\omega Ig = [\![\sigma(\theta)]\!]g(s)Ig$. Then let $\nu = \omega(s)$, then $[\![\sigma(\theta)]\!]\nu Ig = [\![\theta]\!]\nu\sigma^*_\nu I$ by IH, then by Lemma 26 and admissibility assumption, $[\![\theta]\!]\nu\sigma^*_\nu I = [\![\theta]\!]\nu\sigma^*_\omega I = [\![@_s\phi]\!]\omega\sigma^*_\omega I$.

□

Lemma 28 (Formula Substitution). $\omega \in [\![\sigma(\phi)]\!]Ig$ iff $\omega \in [\![\phi]\!]\sigma^*_{\omega g} Ig$.

Proof. By induction on ϕ with simultaneous induction on programs α. We present only the new cases.

- **Case** $@_{\bar n}\phi, \bar n \in \sigma$: $\omega \in [\![\sigma(@_{\bar n}\phi)]\!]Ig$ iff $[\![\sigma n]\!]Ig\omega \in [\![\sigma(\phi)]\!]Ig$. Let $\nu = [\![\sigma n]\!]Ig\omega$ then have $\nu \in [\![\sigma(\phi)]\!]Ig$ iff $\nu \in [\![\phi]\!]\sigma^*_{\nu g} Ig$ iff (by IH) $\nu \in [\![\phi]\!]\sigma^*_{\omega g} Ig$ iff (by admissibility assumption) $\big([\![n]\!]\sigma^*_{\omega g} Ig\omega\big) \in [\![\phi]\!]Ig$ (since $\nu = [\![n]\!]\sigma^*_{\omega g} Ig\omega$ by IH again) iff $\omega \in [\![@_{\bar n}\phi]\!]Ig$ as desired.

- **Case** $@_{\bar n}\phi, \bar n \notin \sigma$: $\omega \in [\![\sigma(@_{\bar n}\phi)]\!]Ig$ iff $\omega \in [\![@_{\bar n}\sigma(\phi)]\!]Ig$ iff $I(n) \in [\![\sigma(\phi)]\!]Ig$ iff Let $\nu = I(n)$ then have $\nu \in [\![\sigma(\phi)]\!]Ig$ iff $\nu \in [\![\phi]\!]\sigma^*_{\nu g} Ig$ iff (by IH) $\nu \in [\![\phi]\!]\sigma^*_{\omega g} Ig$ iff (by admissibility assumption) $\sigma^*_{\omega g} I(n) \in [\![\phi]\!]Ig$ (since $\nu = \sigma^*_{\omega g} I(n)$ by adj def) iff $\omega \in [\![@_{\bar n}\phi]\!]Ig$ as desired.

- **Case** $@_s\phi$: $\omega \in [\![\sigma(@_s\phi)]\!] Ig$ iff $\omega \in [\![@_s\sigma(\phi)]\!] Ig$ iff $g(s) \in [\![\sigma(\phi)]\!] Ig$ iff Let $\nu = g(s)$ then have $\nu \in [\![\sigma(\phi)]\!] Ig$ iff $\nu \in [\![\phi]\!] \sigma^*_{\nu g} Ig$ iff (by IH) $\nu \in [\![\phi]\!] \sigma^*_{\omega g} Ig$ iff (by admissibility assumption) $g(s) \in [\![\phi]\!] Ig$ (since $\nu = g(s)$) iff $\omega \in [\![@_s\phi]\!] Ig$ as desired.

- **Case** $\overline{n}, \overline{n} \in \sigma$: $\omega \in [\![\sigma(\overline{n})]\!] Ig$ iff $\omega \in [\![\sigma\overline{n}]\!] Ig$ iff $\omega \in [\![\overline{n}]\!] \sigma^*_{\omega g} Ig$ (by adj def).

- **Case** $\overline{n}, \overline{n} \notin \sigma$: $\omega \in [\![\sigma(\overline{n})]\!] Ig$ iff $\omega \in [\![\overline{n}]\!] Ig$ iff $\omega \in [\![\overline{n}]\!] \sigma^*_{\omega g} Ig$ (by adj def).

- **Case** s : $\omega \in [\![\sigma(s)]\!] Ig$ iff $\omega \in [\![s]\!] Ig$ iff $\omega = g(s)$ iff $\omega \in [\![s]\!] \sigma^*_{\omega g} Ig$ (by adj def).

- **Case** $\forall s : \mathcal{W}\ \phi$: $\omega \in [\![\sigma(\forall s : \mathcal{W}\ \phi)]\!] Ig$ iff $\omega \in [\![\forall s : \mathcal{W}\ \sigma(\phi)]\!] Ig$ iff $\omega \in [\![\sigma(\phi)]\!] Ig^\nu_s$ for all worlds ν iff (by IH) $\omega \in [\![\phi]\!] \sigma^*_{\omega g^\nu_s} Ig$ for all worlds ν iff (by admissibility) $\omega^\nu_s \in [\![\phi]\!] \sigma^*_{\omega g} Ig$ for all worlds ν iff $\omega \in [\![\forall s : \mathcal{W}\ \phi]\!] \sigma^*_{\omega g} Ig$.

- **Case** $\exists s : \mathcal{W}\ \phi$: $\omega \in [\![\sigma(\exists s : \mathcal{W}\ \phi)]\!] Ig$ iff $\omega \in [\![\exists s : \mathcal{W}\ \sigma(\phi)]\!] Ig$ iff $\omega \in [\![\sigma(\phi)]\!] Ig^\nu_s$ for some world ν iff (by IH) $\omega \in [\![\phi]\!] \sigma^*_{\omega g^\nu_s} Ig$ for some world ν iff (by admissibility) $\omega \in [\![\phi]\!] \sigma^*_{\omega g} Ig$ for some world ν iff $\omega \in [\![\exists s : \mathcal{W}\ \phi]\!] \sigma^*_{\omega g} Ig$.

- **Case** $\downarrow s\ \phi$: $\omega \in [\![\sigma(\downarrow s\ \phi)]\!] Ig$ iff $\omega \in [\![\downarrow s\ \sigma(\phi)]\!] Ig$ iff $\omega \in [\![\sigma(\phi)]\!] Ig^\omega_s$ iff (by IH) $\omega \in [\![\phi]\!] \sigma^*_{\omega g^\omega_s} Ig^\omega_s$ iff (by admissibility) $\omega \in [\![\phi]\!] \sigma^*_{\omega g} Ig^\omega_s$ iff $\omega \in [\![\downarrow s\ \phi]\!] \sigma^*_{\omega g} Ig$.

□

Soundness of the uniform substitution rule follows immediately by transferring over the same proof from dL [38]. Next, note by Theorem 17 we get validity of all dL axioms for free, so it suffices to show soundness for the new axioms of dHL.

Theorem 29 (Hybrid Axiom Soundness). The hybrid axioms are valid.

Proof. We show the axioms are sound one at a time.

- Axiom $\mathrm{K}_@$ $@_c(P \to Q) \to @_cP \to @_cQ$ is valid. Fix I and ω, g. Let $\mu = I(c)$. Assume $\omega \in [\![@_c(P \to Q)]\!] Ig$, then (a) $\nu \in [\![P \to Q]\!] Ig$. Assume $\omega \in [\![@_cP]\!] Ig$, then (b) $\nu \in [\![P]\!] Ig$. By (a), (b) and modus ponens, (c) $\nu \in [\![Q]\!] Ig$ so $\omega \in [\![@_cQ]\!] Ig$.

- Axiom @id is valid. Fix I and ω, g. $\omega \in [\![@_a@_bP]\!] Ig$ iff $I(a) \in [\![@_bP]\!] Ig$ iff $I(b) \in [\![P]\!] Ig$ iff $\omega \in [\![@_bP]\!] Ig$.

- Axiom @I $a \wedge P \to @_aP$ is valid. Fix I and ω, g. Assume $\omega \in [\![a \wedge P]\!] Ig$ so (a) $I(a) = \omega$ and (b)$\omega \in I(P)$. Then $I(a) = \omega$ by (a) then $I(a) \in [\![P]\!] Ig$ by (b) and $\omega \in [\![@_aP]\!] Ig$.

- Axiom $@{\leftrightarrow}$ $@_ac \to (p(a) \leftrightarrow p(c))$ is valid. Fix interpretation I, state ω, and galaxy g. Assume (a) $\omega \in [\![@_ac]\!] Ig$, thus $I(a) = I(c)$ so (b) $I(a) = I(c)$ Then $\omega = I(p)(I(a))$ iff $\omega = I(p)(I(b))$ so $\omega \in [\![(p(a) \leftrightarrow p(c))]\!] Ig$.

- Axiom $\langle\overline{n}\rangle\ [a]{\downarrow}s\ p(s) \wedge \langle a\rangle c \to p(c)$ is valid. Fix I and ω, g. Assume (a) $\omega \in [\![[a]{\downarrow}s\ p(s)]\!] Ig$ and (b) $\omega \in [\![\langle a\rangle c]\!] Ig$. By (a), for all world ν if $(\omega, \nu) \in I(a)$ then $\nu \in I(p)$. By (b), there exists world μ where $(\omega, \nu) \in I(a)$ and (c) $\mu = I(c)$. Instantiating (a), have $\mu g \in I(p)$ Combined with (c), have $\omega \in [\![p(c)]\!] Ig$ and thus $\omega \in [\![@_cP]\!] Ig$.

- Axiom $\forall E_@\ \forall s:\mathcal{W}\ p(s) \to p(\overline{n})$ Fix I and ω, g. Assume (a) for all world ν, $\omega \in [\![p(s)]\!] I g^\nu_s$ so (b) $I(p)(\nu)$. Pick $\nu = I(n)$ then have $I(p)(I(n))$ and $\omega \in [\![p(\overline{n})]\!] I g$.

- Rule $\forall I_@\ \dfrac{q(y)}{\forall x:\mathcal{W}\ q(x)}$ is sound for fresh y. Fix I and ω. Assuming for all choices of y have (a) $I(q)(\mu(y))$ for all states μ. By (a) have (b) $\omega \in [\![q(s)]\!] I g^\mu_s$ for all μ. By (b) have $\omega \in [\![\forall s:\mathcal{W}\ q(s)]\!] I g$.

- Axiom $\downarrow$ (i.e. formula $\downarrow s\ p(s) \equiv \exists s:\mathcal{W}\ p(s)$) is valid. Fix I and ω. Then $\omega \in [\![\downarrow s\ p(s)]\!] I g$ iff $\omega \in [\![p(s)]\!] I g^\omega_s$ iff $I(p)(\omega)$ iff exists ν s.t. $\nu = \omega$ and $I(p)(\nu)$ iff $\omega \in [\![\exists s:\mathcal{W}\ s \wedge p(s)]\!] I g$.

- Axiom $\exists W\ \exists s:\mathcal{W}\ s$. Fix I and ω. Suffices to show exists world ν such that $\omega = \nu$. Pick $\nu = \omega(\texttt{state})$ by reflexivity.

- Axiom schema BW $\langle\alpha\rangle \exists s:\mathcal{W}\ P \leftrightarrow \exists s:\mathcal{W}\ \langle\alpha\rangle P$ is valid. Fix I, g, and ω.

$$
\begin{aligned}
&\omega \in [\![\langle\alpha\rangle \exists s:\mathcal{W}\ P]\!] I g \\
\equiv &\nu \in [\![\exists s:\mathcal{W}\ P]\!] I g, \text{ for some} (\omega,\nu) \in [\![\alpha]\!] I g \\
\equiv &\nu \in [\![P]\!] I g^\mu_s, \text{ for some } (\omega,\nu) \in [\![\alpha]\!] I g, \mu \in \mathcal{W} \\
\equiv &\nu \in [\![P]\!] I g^\mu_s, \text{ for some } \mu \in \mathcal{W}, (\omega,\nu) \in [\![\alpha]\!] I g \\
\equiv^* &\nu \in [\![P]\!] I g^\mu_s, \text{ for some } \mu \in \mathcal{W}, (\omega,\nu) \in [\![\alpha]\!] I g^\mu_s \\
\equiv &\nu \in [\![\langle\alpha\rangle P]\!] I g^\mu_s, \text{ for some } \mu \in \mathcal{W} \\
\equiv &\omega \in [\![\exists s:\mathcal{W}\ \langle\alpha\rangle P]\!] I g
\end{aligned}
$$

  Where the starred step holds by the coincidence lemma due to the assumption $s \notin \text{FV}(\alpha)$.

- Axiom $\text{G}_@\ \dfrac{\phi}{@_i\phi}$ is sound. Fix I,g, ω. Assume for all $\nu \in \mathcal{W}$, all $h \in \mathcal{G}$, have $\nu \in [\![\phi]\!] h$. Then instantiate $\nu = g(i), h = g$ and have $g(i) \in [\![\phi]\!] I g$ and thus $\omega \in [\![@_i g]\!] I g$. Since this held for all g and ω and I the conclusion is valid, i.e. the rule is sound.

□

Theorem 30 (At-Term soundness). The at-term axioms are valid.

Proof. We begin with the non-derived axiom @hom.

Lemma 31. Formula $@_i p(F_1, \dots F_n) \leftrightarrow p(@_i F_1, \dots, @_i F_n)$ is valid.

Proof. Semantic proof. Fix $k \in \mathbb{N}$, interpretation I and state ω. Then the sides have the same semantics:

$$\begin{aligned}
&\omega \in [\![p(@_i F_1, \dots, @_i F_k)]\!] Ig \\
\equiv &I(p)([\![@_i F_1]\!] Ig\omega, \dots, [\![@_i F_k]\!] Ig\omega) \\
\equiv &I(p)([\![F_1]\!] IgI(i), \dots, [\![F_k]\!] IgI(i)) \\
\equiv &I(i) \in [\![p(F_1, \dots, F_k)]\!] Ig \\
\equiv &\omega \in [\![@_i p(F_1, \dots, F_k)]\!] Ig
\end{aligned}$$

□

Combining axiom @hom with existing axioms and sequent rules (which are derivable from typical hilbert axioms), we derive the remaining axioms.

- Axiom NT:= is valid:

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{f() = f()}\,\text{refl}}{[x := f()]f() = x}\,[:=]}{\langle x := f()\rangle j \vdash \langle x := f()\rangle j}\,\text{id}}{i, \langle x := f()\rangle j \vdash @_j f() = x}\,\langle \overline{n} \rangle}{i, \langle x := f()\rangle j \vdash f() = @_j x}\,@\text{hom}}{i, \langle x := F\rangle j \vdash F = @_j x}\,\text{US}}{i \vdash \langle x := F\rangle j \to F = @_j x}\,\to\text{R}}{\vdash @_i(\langle x := F\rangle j \to F = @_j x)}\,@\text{I}}{@_i\langle x := F\rangle j \vdash @_i(\langle x := F\rangle j \to F = @_j x)}\,\text{hide}}{@_i\langle x := F\rangle j \to @_i F = @_j x}\,\to\text{R}}{@_i(F = @_j x)}\,@\text{hom}}{@_i\langle x := F\rangle j \vdash @_i F = @_j x}\,\text{K}_@$$

$$\dfrac{*}{@_i\langle x := F\rangle j \vdash @_i\langle x := F\rangle j}\,\text{id}$$

□

- Axiom NT; is valid:

Proof.

$$\text{cut}\,\frac{\mathbb{@}_i\langle\alpha\rangle m \vdash \mathbb{@}_i F = \mathbb{@}_m H \qquad \mathbb{@}_m\langle\beta\rangle j \vdash \mathbb{@}_m H = \mathbb{@}_j G \qquad \mathbb{R}\,\dfrac{*}{\mathbb{@}_i F = \mathbb{@}_m H, \mathbb{@}_m H = \mathbb{@}_j G \vdash \mathbb{@}_i F = \mathbb{@}_j G}}{\begin{array}{r}\text{M,V,@I}\,\dfrac{\mathbb{@}_i\langle\alpha\rangle m, \mathbb{@}_m\langle\beta\rangle j \vdash \mathbb{@}_i F = \mathbb{@}_j G}{\mathbb{@}_i\langle\alpha\rangle m \wedge \langle\beta\rangle j \vdash \mathbb{@}_i F = \mathbb{@}_j G}\\ \exists\text{L}\,\dfrac{\exists m : \mathcal{W}\ \mathbb{@}_i\langle\alpha\rangle m \wedge \langle\beta\rangle j \vdash \mathbb{@}_i F = \mathbb{@}_j G}{\exists W\text{,BW,M}\,\dfrac{\mathbb{@}_i\langle\alpha\rangle\langle\beta\rangle j \vdash \mathbb{@}_i F = \mathbb{@}_j G}{\langle;\rangle\,\dfrac{\mathbb{@}_i\langle\alpha;\beta\rangle j \vdash \mathbb{@}_i F = \mathbb{@}_j G}{}}}\end{array}}$$

□

- Axiom NT$\cup$ is valid:

Proof.

$$\vee\,\frac{\mathbb{@}_i\langle\alpha\rangle j \vdash \mathbb{@}_i F = \mathbb{@}_j G \qquad \mathbb{@}_i\langle\beta\rangle j \vdash \mathbb{@}_i F = \mathbb{@}_j G}{\begin{array}{r}\vee\text{@L}\,\dfrac{\mathbb{@}_i\langle\alpha\rangle j \vee \mathbb{@}_i\langle\beta\rangle j \vdash \mathbb{@}_i F = \mathbb{@}_j G}{\langle\cup\rangle\,\dfrac{\mathbb{@}_i\langle\alpha\rangle j \vee \langle\beta\rangle j \vdash \mathbb{@}_i F = \mathbb{@}_j G}{\mathbb{@}_i\langle\alpha\cup\beta\rangle j \vdash \mathbb{@}_i F = \mathbb{@}_j G}}\end{array}}$$

□

- Axiom NT$'$ is valid:

Proof.

$$\text{K}_{\mathbb{@}}\,\frac{\text{id}\,\dfrac{*}{\mathbb{@}_i\langle x' = f(x)\,\&\,q(x)\rangle j \vdash \mathbb{@}_i\langle x' = f(x)\,\&\,q(x)\rangle j} \qquad \begin{array}{r}\text{@I}\,\dfrac{\mathbb{@}_i t{\geq}0 \wedge (\forall 0{\leq}s{\leq}tq(y(s))) \vdash \mathbb{@}_i f(x) = g(y(t))}{i, t{\geq}0, (\forall 0{\leq}s{\leq}tq(y(s))) \vdash f(x) = g(y(t))}\\ \text{hide}\,\dfrac{i, t{\geq}0, \mathbb{@}_j x = y(t), (\forall 0{\leq}s{\leq}tq(y(s))) \vdash f(x) = g(\mathbb{@}_j x)}{\text{US}\,\dfrac{i, t{\geq}0, \mathbb{@}_j x = y(t), (\forall 0{\leq}s{\leq}tq(y(s))) \vdash f(x) = \mathbb{@}_j g(x)}{\text{NT:=}\,\dfrac{i, t{\geq}0, [x := y(t)]j, (\forall 0{\leq}s{\leq}tq(y(s))) \vdash f(x) = \mathbb{@}_j g(x)}{\exists\text{L,}\wedge\text{L}\,\dfrac{i, \exists t{\geq}0 [x := y(t)]j \wedge (\forall 0{\leq}s{\leq}tq(y(s))) \vdash f(x) = \mathbb{@}_j g(x)}{\langle'\rangle\,\dfrac{i, \langle x' = f(x)\,\&\,q(x)\rangle j \vdash f(x) = \mathbb{@}_j g(x)}{\text{@I}\,\dfrac{\mathbb{@}_i\langle x' = f(x)\,\&\,q(x)\rangle j \vdash f(x) = \mathbb{@}_j g(x)}{}}}}}\end{array}}{\begin{array}{r}\text{@hom}\,\dfrac{\mathbb{@}_i\langle x' = f(x)\,\&\,q(x)\rangle j \vdash \mathbb{@}_i(f(x) = \mathbb{@}_j g(x))}{\to\text{R}\,\dfrac{\mathbb{@}_i\langle x' = f(x)\,\&\,q(x)\rangle j \vdash \mathbb{@}_i f(x) = \mathbb{@}_j g(x)}{\mathbb{@}_i\langle x' = f(x)\,\&\,q(x)\rangle j \to \mathbb{@}_i f(x) = \mathbb{@}_j g(x)}}\end{array}}$$

□

- Axiom NT$*$ is valid:

  Proof. Recall in this proof Rule FP is the fixpoint rule for loops, interderivable from loop induction I.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{j \vdash @_j F = @_j F}\,\text{id}}{j \vdash @_j(F = @_j F)}\,@\text{hom}}{j \vdash F = @_j F}\,@\text{I} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{@_s\langle\alpha\rangle t \vdash @_s F = @_t F}{\forall s,t : \mathcal{W}\ @_s\langle\alpha\rangle t \vdash @_s F = @_t F}\,\forall\text{I}}{@_k\langle\alpha\rangle \ell \vdash @_k F = @_\ell F}\,\text{cut} \qquad \dfrac{\dfrac{*}{@_k F = @_\ell F, @_\ell F = @_j F \vdash @_k F = @_j F}\,\mathbb{R}}{@_k F = @_\ell F, @_\ell(F = @_j F) \vdash @_k(F = @_j F)}\,\exists W}{k, @_k\langle\alpha\rangle \ell, @_\ell(F = @_j F) \vdash @_k(F = @_j F)}\,\text{G},\exists W,\text{BW},\exists\text{L}}{k, @_k\langle\alpha\rangle(F = @_j F) \vdash @_k(F = @_j F)}\,@\text{I},\exists W,\exists\text{L}}{\langle\alpha\rangle(F = @_j F) \vdash F = @_j F}}{(j \vee \langle\alpha\rangle(F = @_j F)) \vdash F = @_j F}\,\text{FP}}{\langle\alpha^*\rangle j \vdash F = @_j F}\,\text{G}_@,@\text{hom}}{@_i\langle\alpha^*\rangle j \vdash (@_i F) = (@_j F)}$$

  □

- Axiom NTV is valid:

  Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{k, @_k\langle\alpha\rangle j \vdash F = F}\,\text{refl}}{k, @_k\langle\alpha\rangle j \vdash @_k(F = F)}\,@\text{I}}{k, @_k\langle\alpha\rangle j \vdash F = @_k F}\,@\text{hom}}{k, @_k\langle\alpha\rangle j \vdash [\alpha](F = @_k F)}\,\text{V}}{k, @_k\langle\alpha\rangle j \vdash [\alpha](F = @_k F)}\,@\text{I}}{@_i\langle\alpha\rangle j \vdash [\alpha](F = @_i F)}\,\forall E_@,\exists W}{@_i\langle\alpha\rangle j \vdash @_j(F = @_i F)}\,\langle\overline{n}\rangle}{@_i\langle\alpha\rangle j \to @_i F = @_j F}\,@\text{hom},{\to}\text{R}$$

  □

□

Theorem 32 (Bisimulation Soundness). The bisimulation (derived) rules are sound.

Proof.

- The auxilliary loop elimination rule @ind is derived:

$$\frac{@_{i_1} o_1 \to p(o_1) \quad @_{m_1}\langle\alpha\rangle o_1 \wedge p(m_1) \to p(o_1)}{@_{i_1}\langle\alpha^*\rangle o_1 \to p(o_1)}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{@_{i_1} s \to p(s)}}{@_{i_1}(s \wedge p(s))}\,\text{prop}}{@_{i_1}\exists s : \mathcal{W}\ (s \wedge p(s))}\,\forall I_@}{@_{i_1}{\downarrow}s\ p(s)}\,\downarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_2}{(s \wedge p(s) \wedge \langle\alpha\rangle t) \to p(t)}}{s \wedge p(s) \wedge \langle\alpha\rangle(t \wedge \neg p(t)) \to \bot}\,\text{M,prop}}{s \wedge p(s) \to [\alpha](t \to p(t))}\,\langle\cdot\rangle}{s \wedge p(s) \to \forall t : \mathcal{W}\ [\alpha](t \to p(t))}\,\text{skolem}}{s \wedge p(s) \to [\alpha]\forall t : \mathcal{W}\ (t \to p(t))}\,\text{BW}}{s \wedge p(s) \to [\alpha]\exists t : \mathcal{W}\ (t \wedge p(t))}\,\text{prop}}{{\downarrow}s\ p(s) \to [\alpha]{\downarrow}t\ p(t)}\,\forall I_@}{@_{i_1}{\downarrow}s\ p(s) \to @_{i_1}[\alpha^*]{\downarrow}t\ p(t)}\,\text{I}}{@_{i_1}[\alpha^*]{\downarrow}s\ p(s)}\,\langle\overline{n}\rangle}{@_{i_1}\langle\alpha^*\rangle o_1 \to p(o_1)}$$

□

- Loop bisimulation rule BS* is derived: $\dfrac{@_{i_1}\langle\alpha\rangle o_1 \wedge R(i_1,i_2) \to @_{i_2}\langle\alpha\rangle{\downarrow}o_2\ R(o_1,o_2)}{@_{i_1}\langle\alpha^*\rangle o_1 \wedge R(i_1,i_2) \to @_{i_2}\langle\alpha^*\rangle{\downarrow}o_2\ R(o_1,o_2)}$

Proof. Below, $\mathcal{D}$ is the open goal.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{@_{i_1} o_1, R(i_1,i_2) \to R(i_1,i_2)}\,\text{id}}{@_{i_1} o_1, R(i_1,i_2) \to R(o_1,i_2)}\,@{\leftrightarrow}}{@_{i_1} o_1, R(i_1,i_2) \to @_{i_2} R(o_1,i_2)}\,\exists W,{\downarrow}}{@_{i_1} o_1, R(i_1,i_2) \to @_{i_2}{\downarrow}o_2\ R(o_1,o_2)}\,{\downarrow}}{@_{i_1} o_1, R(i_1,i_2) \to @_{i_2}{\downarrow}o_2\ R(o_1,o_2)}\,@{\leftrightarrow}}{@_{i_1} o_1, R(i_1,i_2) \to @_{i_2}\langle\alpha^*\rangle{\downarrow}o_2\ R(o_1,o_2)}\,\langle*\rangle \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}}{@_{m_1}\langle\alpha\rangle o_1 \wedge R(m_1,m_2) \to @_{m_2}\langle\alpha\rangle{\downarrow}o_2\ R(o_1,o_2)}}{@_{m_1}\langle\alpha\rangle o_1 \wedge @_{i_2}\langle\alpha^*\rangle m_2 \wedge R(m_1,m_2) \to @_{m_2}\langle\alpha\rangle{\downarrow}o_2\ R(o_1,o_2)}\,\text{hide}}{@_{m_1}\langle\alpha\rangle o_1 \wedge @_{i_2}\langle\alpha^*\rangle m_2 \wedge R(m_1,m_2) \to @_{i_2}\langle\alpha^*\rangle\langle\alpha\rangle{\downarrow}o_2\ R(o_1,o_2)}\,@\text{I, M}}{@_{m_1}\langle\alpha\rangle o_1 \wedge @_{i_2}\langle\alpha^*\rangle m_2 \wedge R(m_1,m_2) \to @_{i_2}\langle\alpha^*\rangle{\downarrow}o_2\ R(o_1,o_2)}\,\langle*\rangle}{@_{m_1}\langle\alpha\rangle o_1 \wedge @_{i_2}\langle\alpha^*\rangle{\downarrow}m_2\ R(m_1,m_2) \to @_{i_2}\langle\alpha^*\rangle{\downarrow}o_2\ R(o_1,o_2)}\,\exists W,{\downarrow}}{@_{i_1}\langle\alpha^*\rangle o_1, R(i_1,i_2) \vdash @_{i_2}\langle\alpha^*\rangle{\downarrow}o_2\ R(o_1,o_2)}\,@\text{ind}$$

□

- Sequential composition bisimulation rule BS; is derived:

$$\frac{\begin{array}{c}@_{i_1}\langle\alpha\rangle m_1 \wedge R_i(i_1,i_2) \to @_{i_2}\langle\alpha\rangle\downarrow m_2\ R_m(m_1,m_2)\\ @_{m_1}\langle\alpha\rangle o_1 \wedge R_m(m_1,m_2) \to @_{m_2}\langle\alpha\rangle\downarrow o_2\ R_o(m_1,m_2)\end{array}}{@_{i_1}\langle\alpha;\beta\rangle o_1 \wedge R_i(i_1,i_2) \to @_{i_2}\langle\alpha;\beta\rangle\downarrow o_2\ R_o(o_1,o_2)}$$

Proof. Let $\Gamma \equiv @_{i_1}\langle\alpha\rangle m_1, @_{m_1}\langle\beta\rangle o_1, R(i_1,i_2)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, @_{i_2}\langle\alpha\rangle\downarrow m_2\ R(m_1,m_2), @_{m_2}\langle\beta\rangle\downarrow o_2\ R(o_1,o_2) \vdash @_{m_2}\langle\beta\rangle\downarrow o_2\ R(o_1,o_2)}\,*}{\Gamma, @_{i_2}\langle\alpha\rangle\downarrow m_2\ R(m_1,m_2), @_{m_2}\langle\beta\rangle\downarrow o_2\ R(o_1,o_2) \vdash @_{i_2}\langle\alpha;\beta\rangle\downarrow o_2\ R(o_1,o_2)}\,\text{id}}{\Gamma, @_{i_2}\langle\alpha\rangle\downarrow m_2\ R(m_1,m_2) \vdash @_{i_2}\langle\alpha;\beta\rangle\downarrow o_2\ R(o_1,o_2)}\,@\text{I,M}}{@_{i_1}\langle\alpha\rangle m_1, @_{m_1}\langle\beta\rangle o_1, R(i_1,i_2) \vdash @_{i_2}\langle\alpha;\beta\rangle\downarrow o_2\ R(o_1,o_2)}\,\text{cut},\mathcal{D}_\infty}{@_{i_1}\langle\alpha\rangle\exists m_1 : \mathcal{W}\ (m_1 \wedge \langle\beta\rangle o_1), R(i_1,i_2) \vdash @_{i_2}\langle\alpha\rangle\langle\beta\rangle\downarrow o_2\ R(o_1,o_2)}\,\text{BW},@\text{I},@@,\exists\text{L}}{@_{i_1}\langle\alpha\rangle\langle\beta\rangle o_1, R(i_1,i_2) \vdash @_{i_2}\langle\alpha\rangle\langle\beta\rangle\downarrow o_2\ R(o_1,o_2)}\,\exists W,\text{M}}{@_{i_1}\langle\alpha;\beta\rangle o_1, R(i_1,i_2) \vdash @_{i_2}\langle\alpha;\beta\rangle\downarrow o_2\ R(o_1,o_2)}\,\text{cut},\mathcal{D}_\in\ \langle;\rangle$$

□

- Choice composition bisimulation rule BS$\cup$ is derived:

$$\frac{\begin{array}{c}@_{i_1}\langle\alpha\rangle o_1 \wedge R_i(i_1,i_2) \to @_{i_2}\langle\alpha\rangle\downarrow o_2\ R_o(o_1,o_2)\\ @_{i_1}\langle\beta\rangle o_1 \wedge R_i(i_1,i_2) \to @_{i_2}\langle\beta\rangle\downarrow o_2\ R_o(o_1,o_2)\end{array}}{@_{i_1}\langle\alpha\cup\beta\rangle o_1 \wedge R_i(i_1,i_2) \to @_{i_2}\langle\alpha\cup\beta\rangle\downarrow o_2\ R_o(o_1,o_2)}$$

Proof.

$$\dfrac{\dfrac{\dfrac{@_{i_1}\langle\alpha\rangle o_1 \wedge R_i(i_1,i_2) \to @_{i_2}\langle\alpha\rangle\downarrow o_2\ R_o(o_1,o_2)}{@_{i_1}\langle\alpha\rangle o_1 \wedge R_i(i_1,i_2) \to @_{i_2}\langle\alpha\cup\beta\rangle\downarrow o_2\ R_o(o_1,o_2)}\,\langle\cup\rangle \quad \dfrac{@_{i_1}\langle\beta\rangle o_1 \wedge R_i(i_1,i_2) \to @_{i_2}\langle\beta\rangle\downarrow o_2\ R_o(o_1,o_2)}{@_{i_1}\langle\beta\rangle o_1 \wedge R_i(i_1,i_2) \to @_{i_2}\langle\alpha\cup\beta\rangle\downarrow o_2\ R_o(o_1,o_2)}\,\langle\cup\rangle}{(@_{i_1}\langle\alpha\rangle o_1 \vee @_{i_1}\langle\beta\rangle o_1) \wedge R_i(i_1,i_2) \to @_{i_2}\langle\alpha\cup\beta\rangle\downarrow o_2\ R_o(o_1,o_2)}\,\vee\text{L}}{@_{i_1}\langle\alpha\cup\beta\rangle o_1 \wedge R_i(i_1,i_2) \to @_{i_2}\langle\alpha\cup\beta\rangle\downarrow o_2\ R_o(o_1,o_2)}\,\langle\cup\rangle$$

□

□

E Full proof of secure smart grid model

Additional Notations We use some additional concepts and notations in the proof appendices, in order to make proofs simpler.

While the presentation given in the body of the paper is Hilbert-style, this is easily extended to a sequent-style calculus by adding standard propositional sequent calculus rules and contextual equivalence rules which carry over directly from the base uniform substitution calculus for dL [38]. We write sequents $\Gamma \vdash \Delta$ where Γ (the *antecedent*) and Δ (the *succedent*) are both lists of formulas, and theh sequent $\Gamma \vdash \Delta$ is true in exactly the same states as the formula $\left(\bigwedge_{\phi\in\Gamma} \phi\right) \rightarrow \left(\bigvee_{\psi\in\Delta} \Delta\right)$.

We mark steps that close a proof branch with an asterisk (*). When a derivation is too large to fit in its entirety, we introduce variables standing for unfinished branches of the derivation which are then proved separately. These variables typically start with $\mathcal{D}$ for *derivation*.

We use tuple notation for conciseness, e.g. $@_i(x,y) = @_j(x,y)$ is shorthand for $@_i x = @_j x \wedge @_i y = @_j y$. Likewise, $x, y := \theta_x, \theta_y$ is shorthand for the program $x := \theta_x; y := \theta_y$. For long sequences of assignments, we elide irrelevant assignments with dots (. . .). We also write transitive equalities $x = y = z$ with the typical meaning $x = y \wedge y = z$.

As in the main paper we use e^{θ}_{x} for the substitution of θ for x in e. We will also sometimes construct named substitutions σ and substitute with the notation $\sigma(e)$ when convenient, as in the uniform substitution algorithm.

Lastly, we use straightforward derived constructs for clarity, specifically truth $\top$ and falsehood $\bot$ which are trivially derived as $0 < 1$ and $0 < 0$, respectively.

Derived Rules In this proof we will use a few derived rules which were ignored in the main text for clarity of presentation.

$$\text{M}[\cdot]\quad \frac{P \rightarrow Q}{[a]P \rightarrow [a]Q}$$

$$\text{FP}\quad \frac{(P \vee \langle a\rangle Q) \rightarrow Q}{\langle a^*\rangle P \rightarrow Q}$$

Rule M[·] derives trivially from K, G, and MP and is the box analog of M. Rule FP has been shown inter-derivable with loop induction axiom I in prior work [36].

Proof. In this proof we decompose the model into pieces, letting α_N stand for the nondeterministic assignments

$$\begin{aligned} &d_i := *;\ ?(d_i \geq 0);\\ &r_i := *;\ ?(r_i \geq 0);\\ &n_i := d_i - (r_i + p_i); \end{aligned}$$

α_L stand for the load balancer

$$\begin{aligned} &\texttt{if}(n_i \geq \mathit{thresh} \wedge N_{\bar{i}} < 0)\\ &\quad m := M \cdot (-1)^i\\ &\texttt{else}\\ &\quad m := 0 \end{aligned}$$

α_B stand for the battery controller

$$\begin{aligned} &gr := 0;\ bm_i := 0;\ gm := 0;\\ &\quad \big(?(B_i < B_{max}) \vee (n_i > 0 \wedge B_i > 0);\\ &\qquad b_i := -n_i;\ bm_i := bm_i + m \cdot (-1)^{i+1}\big)\\ &\cup \big(b_i := 0;\ gr := gr + n_i;\ gm := gm + m \cdot (-1)^{i+1}\big) \end{aligned}$$

α_P stand for the differential equation:

$$\{p_i' = m \cdot -1^i, B_i' = b_i, b_i' = bm_i, gr' = gm, t' = 1 \,\&\, B_i \geq 0\}$$

and combine letters in subscripts to denote, compositions, e.g. $\alpha_{NLBP} \equiv (\alpha_N; \alpha_L; \alpha_B; \alpha_P)$. Here we define the relations $R(i,j) \equiv @_i t = @_j t \wedge @_i gr = @_i J \wedge thresh \geq 0$ and $R_m(i,j) \equiv R(i,j) \wedge @_i gm = @_j gm \wedge m > 0 \wedge B_{max} > 0$. The main proof, where $\mathcal{D}_c$ stands for the controller correctness lemma:

$$\text{lemma}\ \frac{\mathcal{D}_c}{R(i_1,i_2), @_{i_1}\langle\alpha_{NLB}\rangle o_1 \vdash @_{i_2}\langle\alpha_{NLB}\rangle{\downarrow}o_2\ R(o_1,o_2)} \qquad \mathbb{R},R_m\ \frac{*}{R_m(m_1,m_2), @_{m_1}\langle\alpha_P\rangle o_1 \vdash @_{m_2}[t := @_{o_1}t; gr, \ldots := gr + gm(t - @_{m_1}t)]@_{o_1}t = @_{o_1}t \wedge gr = @_{m_1}gr + gm(@_{o_1}t - @_{m_1}t)}$$

$$\text{NT}'\ \frac{R_m(m_1,m_2), @_{m_1}\langle\alpha_P\rangle o_1 \vdash @_{m_2}[t := @_{o_1}t; gr, \ldots := gr + gm(t - @_{m_1}t)]((@_{o_1}t = @_{o_1}t) \wedge (gr = @_{o_1}gr))}{[:=]\ \frac{R_m(m_1,m_2), @_{m_1}\langle\alpha_P\rangle o_1 \vdash @_{m_2}[t := @_{o_1}t; gr, \ldots := gr + gm(t - @_{m_1}t)](t = @_{o_1}t \wedge gr = @_{o_1}gr)}{@\text{hom},{\downarrow}\ \frac{R_m(m_1,m_2), @_{m_1}\langle\alpha_P\rangle o_1 \vdash @_{m_2}[t := @_{o_1}t; gr, \ldots := gr + gm(t - @_{m_1}t)]{\downarrow}o_2\ R(o_1,o_2)}{\text{BS}'\ \frac{R_m(m_1,m_2), @_{m_1}\langle\alpha_P\rangle o_1 \vdash @_{m_2}\langle\alpha_P\rangle{\downarrow}o_2\ R(o_1,o_2)}{\text{BS};\ \frac{R(i_1,i_2), @_{i_1}\langle\alpha_{NLBP}\rangle o_1 \vdash @_{i_2}\langle\alpha_{NLBP}\rangle{\downarrow}o_2\ R(o_1,o_2)}{\text{BS}^*\ \frac{R(i_1,i_2), @_{i_1}\langle\alpha^*_{NLBP}\rangle o_1 \vdash @_{i_2}\langle\alpha^*_{NLBP}\rangle{\downarrow}o_2\ R(o_1,o_2)}{}}}}}$$

Next we prove the lemma $\mathcal{D}_c$ for controller safety, which itself uses a lemma $\mathcal{D}_i$ for controller inversion. The lemma application immediately splits us into four branches b_1 to b_4

b1: (abbrev $\Gamma_{LB} \equiv R(i_1,i_2), @_{i_1}\langle\alpha_{LB}\rangle m_1, @_{m_1}(gr = 0 \wedge gm = 0)$)

$$\mathbb{R}\ \frac{*}{R(i_1,i_2), @_{i_1}\langle\alpha_N\rangle n_1, @_{m_1}(gr = 0 \wedge gm = 0) \vdash (@_{i_2}\max(0,p_1) - (p_1 + \max(0,-p_1)) = \max(0,p_2) - (p_2 + \max(0,-p_2)) = 0}$$

$$\langle:=\rangle\ \frac{R(i_1,i_2), @_{i_1}\langle\alpha_N\rangle n_1, @_{m_1}(gr = 0 \wedge gm = 0) \vdash @_{i_2}\langle n_i := \max(0,p_i) - (p_i + \max(0,-p_i))\rangle n_1 = n_2 = 0}{\langle:=*\rangle, \text{NTV}\ \frac{R(i_1,i_2), @_{i_1}\langle\alpha_N\rangle n_1, @_{m_1}(gr = 0 \wedge gm = 0) \vdash @_{i_2}\langle\alpha_N\rangle{\downarrow}n_2\ (n_1 = n_2 = 0 \wedge R(n_1,n_2))}{\text{BS};\ \frac{R(i_1,i_2), @_{i_1}\langle\alpha_N\rangle n_1, @_{m_1}(gr = 0 \wedge gm = 0) \vdash @_{i_2}\langle\alpha_N\rangle\langle\alpha_{LB}\rangle{\downarrow}m_2\ R(m_1,m_2)}{\langle;\rangle\ \frac{R(i_1,i_2), @_{i_1}\langle\alpha_{NLB}\rangle m_1, @_{m_1}(gr = 0 \wedge gm = 0) \vdash @_{i_2}\langle\alpha_N\rangle\langle\alpha_{LB}\rangle{\downarrow}m_2\ R(m_1,m_2)}{}}}$$

$$\frac{\mathcal{D}_{lb1}}{\Gamma_{LB}, @_{n_2}(n_1 = n_2 = 0) \vdash @_{n_2}\langle\alpha_{LB}\rangle{\downarrow}m_2\ R(m_1,m_2)}$$

$\mathcal{D}_{lb1}$: Below we abbreviate $\Gamma \equiv R(n_1,n_2), @_{n_1}\langle\alpha_{LB}\rangle m_1, @_{m_1}(gr = 0 \wedge gm = 0), @_{n_2}n_1 = n_2 = 0$:

$$\mathbb{R}\ \frac{*}{\Gamma \vdash @_{n_2}\neg((n_1 \geq thresh \wedge n_2 < 0) \vee (n_2 \geq thresh \wedge n_1 < 0))}$$

$$\langle\cup\rangle,\langle?\rangle\ \frac{\Gamma \vdash @_{n_2}\neg((n_1 \geq thresh \wedge n_2 < 0) \vee (n_2 \geq thresh \wedge n_1 < 0))}{\Gamma \vdash @_{n_2}\langle\alpha_L\rangle\langle\alpha_B\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}$$

$$\text{NTV}\ \frac{*}{\Gamma \vdash @_{n_1}t = @_{m_1}t}$$

$$\mathbb{R},\text{assumption}\ R(n_1,n_2)\ \frac{\Gamma \vdash @_{n_1}t = @_{m_1}t}{\Gamma \vdash @_{n_2}t = @_{m_1}t}$$

$$\mathbb{R},\text{assumption}\ \frac{\Gamma \vdash @_{n_2}t = @_{m_1}t}{\Gamma \vdash @_{n_2}(0,0,t) = @_{m_1}(gr,gm,t)}$$

$$\langle:=\rangle\ \frac{\Gamma \vdash @_{n_2}(0,0,t) = @_{m_1}(gr,gm,t)}{\langle\cup\rangle,\langle;\rangle\ \frac{\Gamma \vdash @_{n_2}\langle b_i, bm_i, gr, gm := 0,0,0,0\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}{\langle:=\rangle\ \frac{\Gamma \vdash @_{n_2}\langle\alpha_B{}^0_m\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}{\Gamma \vdash @_{n_2}\langle m := 0\rangle\langle\alpha_B\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}}$$

$$\langle;\rangle\ \frac{\Gamma \vdash @_{n_2}\langle\alpha_L\rangle\langle\alpha_B\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}{@\text{hom},{\downarrow}\ \frac{\Gamma \vdash @_{n_2}\langle\alpha_{LB}\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}{\Gamma \vdash @_{n_2}\langle\alpha_{LB}\rangle{\downarrow}m_2\ R(m_1,m_2)}}$$

b2 Below we abbreviate $\Gamma \equiv R(i_1,i_2), @_{i_1}\langle\alpha_N\rangle n_1, @_{m_1}(gr = n_1 \wedge gm = m)$
$\Gamma_2 \equiv R(n_1,n_2), @_{n_1}\langle\alpha_{LB}\rangle m_1, @_{m_1}(gr = n_1 \wedge gm = m), @_{n_2}n_1 = @_{n_1}n_1 \wedge n_2 = 0$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\Gamma \vdash (@_{i_2}(\max(0, @_{i2}p_1 + @_{m_2}n_1) - (p_1 + \max(0, -@_{i2}p_1 - @_{m_2}n_1)) = @_{m_2}n_1) \wedge (\max(0,p_2) - (p_2 + \max(0,-p_2)) = 0)}\,\mathbb{R}}{\Gamma \vdash @_{i_2}\langle n_1, n_2 := \max(0, @_{i2}p_1 + @_{m_2}n_1) - (p_1 + \max(0, -@_{i2}p_1 - @_{m_2}n_1)), \max(0,p_2) - (p_2 + \max(0,-p_2))\rangle(n_1 = @_{n_1}n_1) \wedge (n_2 = 0)}\,\langle:=\rangle}{\Gamma \vdash @_{i_2}\langle\alpha_N\rangle\downarrow_{n_2} @_{n_2}(n_1 = @_{n_1}n_1 \wedge n_2 = 0) \wedge R(n_1,n_2)}\,\langle:=*\rangle, \text{NTV} \qquad \dfrac{\mathcal{D}_{lb2}}{\Gamma_2 \vdash @_{n_2}\langle\alpha_{LB}\rangle\downarrow_{m_2} R(m_1,m_2)}}{R(i_1,i_2), @_{i_1}\langle\alpha_{NLB}\rangle m_1, @_{m_1}(gr = n_1 \wedge gm = m) \vdash @_{i_2}\langle\alpha_N\rangle\langle\alpha_{LB}\rangle\downarrow_{m_2} R(m_1,m_2)}\,\text{BS};}{R(i_1,i_2), @_{i_1}\langle\alpha_{NLB}\rangle o_1, (gr = n_1 \wedge gm = m) \vdash @_{i_2}\langle\alpha_{NLB}\rangle\downarrow_{o_2} R(o_1,o_2)}\,\langle;\rangle$$

$\mathcal{D}_{lb2}$ Below we abbreviate $\Gamma \equiv R(n_1,n_2), @_{n_1}\langle\alpha_{LB}\rangle m_1, @_{m_1}(gr = n_1 \wedge gm = m), @_{n_2}n_1 = n_2 = 0$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\Gamma \vdash @_{n_2}\neg((n_1 \geq thresh \wedge n_2 < 0) \vee (n_2 \geq thresh \wedge n_1 < 0))}\,\mathbb{R}}{\Gamma \vdash @_{n_2}\langle\alpha_L\rangle\langle\alpha_B\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}\,\langle\cup\rangle,\langle?\rangle \qquad \dfrac{\dfrac{\dfrac{\dfrac{*}{\Gamma \vdash (@_{n_1}t) = (@_{m_1}t)}\,\text{NTV}}{\Gamma \vdash @_{n_2}(t = @_{m_1}t)}\,\mathbb{R},\text{assumption}\ R(n_1,n_2)}{\Gamma \vdash @_{n_2}(n_1 + n_2, m \cdot (-1) + m \cdot (-1)^2, t) = @_{m_1}(gr,gm,t)}\,\mathbb{R},\text{assumption}}{\Gamma \vdash @_{n_2}\langle \ldots, gr, gm := \ldots, n_1 + n_2, m \cdot (-1) + m \cdot (-1)^2\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}\,\langle:=\rangle}{\Gamma \vdash @_{n_2}\langle\alpha_B{}^0_m\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}\,\langle\cup\rangle,\langle;\rangle}{\Gamma \vdash @_{n_2}\langle m := 0\rangle\langle\alpha_B\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}\,\langle:=\rangle}{\Gamma \vdash @_{n_2}\langle\alpha_L\rangle\langle\alpha_B\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}\,\langle;\rangle$$
$$\dfrac{\Gamma \vdash @_{n_2}\langle\alpha_L\rangle\langle\alpha_B\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}{\Gamma \vdash @_{n_2}\langle\alpha_{LB}\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}\,@\text{hom},\downarrow$$
$$\Gamma \vdash @_{n_2}\langle\alpha_{LB}\rangle\downarrow_{m_2} R(m_1,m_2)$$

b3 This case is symmetric with b2. We include it in full for the sake of completeness. Below we abbreviate $\Gamma \equiv R(i_1,i_2), @_{i_1}\langle\alpha_N\rangle n_1, @_{m_1}(gr = n_2 \wedge gm = -m)$
$\Gamma_2 \equiv R(n_1,n_2), @_{n_1}\langle\alpha_{LB}\rangle m_1, @_{m_1}(gr = n_2 \wedge gm = -m), @_{n_2}(n_2 = @_{n_1}n_2 \wedge n_1 = 0)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\Gamma \vdash (@_{i_2}\max(0, (@_{i2}p_2 + @_{m_2}n_2) - (p_2 + \max(0, -@_{i2}p_2 - @_{m_2}n_2)) = (@_{m_2}n_2)) \wedge (\max(0,p_1) - (p_1 + \max(0,-p_1)) = 0))}\,\mathbb{R}}{\Gamma \vdash @_{i_2}\langle n_1, n_2 := \max(0,p_1) - (p_1 + \max(0,-p_1)), \max(0, @_{i2}p_2 + @_{m_2}n_2) - (p_2 + \max(0, -@_{i2}p_2 - @_{m_2}n_2))\rangle n_2 = @_{n_1}n_2 \wedge n_1 = 0}\,\langle:=\rangle}{\Gamma \vdash @_{i_2}\langle\alpha_N\rangle\downarrow_{n_2} @_{n_2}n_1 = @_{n_1}n_1 \wedge n_2 = 0 \wedge R(n_1,n_2)}\,\langle:=*\rangle, \text{NTV} \qquad \dfrac{\mathcal{D}_{lb3}}{\Gamma_2 \vdash @_{n_2}\langle\alpha_{LB}\rangle\downarrow_{m_2} R(m_1,m_2)}}{R(i_1,i_2), @_{i_1}\langle\alpha_{NLB}\rangle m_1, @_{m_1}(gr = n_2 \wedge gm = -m) \vdash @_{i_2}\langle\alpha_N\rangle\langle\alpha_{LB}\rangle\downarrow_{m_2} R(m_1,m_2)}\,\text{BS};}{R(i_1,i_2), @_{i_1}\langle\alpha_{NLB}\rangle o_1, (gr = n_2 \wedge gm = -m) \vdash @_{i_2}\langle\alpha_{NLB}\rangle\downarrow_{o_2} R(o_1,o_2)}\,\langle;\rangle$$

$\mathcal{D}_{lb3}$ Below we abbreviate $\Gamma \equiv R(n_1,n_2), @_{n_1}\langle\alpha_{LB}\rangle m_1, @_{m_1}(gr = n_2 \wedge gm = -m), @_{n_2}(n_2 = @_{n_1}n_2 \wedge n_1 = 0)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\Gamma \vdash @_{n_2}\neg((n_1 \geq thresh \wedge n_2 < 0) \vee (n_2 \geq thresh \wedge n_1 < 0))}\,\mathbb{R}}{\Gamma \vdash @_{n_2}\langle\alpha_L\rangle\langle\alpha_B\rangle(gr,gm,t) = (@_{m_1}gr,gm,t)}\,\langle\cup\rangle,\langle?\rangle \qquad \dfrac{\dfrac{\dfrac{\dfrac{*}{\Gamma \vdash (@_{n_1}t) = (@_{m_1}t)}\,\text{NTV}}{\Gamma \vdash @_{n_2}(t = @_{m_1}t)}\,\mathbb{R},\text{assumption}\ R(n_1,n_2)}{\Gamma \vdash @_{n_2}(n_1 + n_2, m \cdot (-1) + m \cdot (-1)^2, t) = @_{m_1}(gr,gm,t)}\,\mathbb{R},\text{assumption}}{\Gamma \vdash @_{n_2}\langle \ldots, gr, gm := \ldots, n_1 + n_2, m \cdot (-1) + m \cdot (-1)^2\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}\,\langle:=\rangle}{\Gamma \vdash @_{n_2}\langle\alpha_B{}^0_m\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}\,\langle\cup\rangle,\langle;\rangle}{\Gamma \vdash @_{n_2}\langle m := 0\rangle\langle\alpha_B\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}\,\langle:=\rangle}{\Gamma \vdash @_{n_2}\langle\alpha_L\rangle\langle\alpha_B\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}\,\langle;\rangle$$
$$\dfrac{\Gamma \vdash @_{n_2}\langle\alpha_L\rangle\langle\alpha_B\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}{\Gamma \vdash @_{n_2}\langle\alpha_{LB}\rangle(gr,gm,t) = @_{m_1}(gr,gm,t)}\,@\text{hom},\downarrow$$
$$\Gamma \vdash @_{n_2}\langle\alpha_{LB}\rangle\downarrow_{m_2} R(m_1,m_2)$$

b4 First define $\theta_1 = \max(0, @_{i_2} p_1 + @_{m_2} n_1) - (p_1 + \max(0, -@_{i_2} p_1 - @_{m_2} n_1))$ and $\theta_2 = \max(0, @_{i_2} p_2 + @_{m_2} n_2) - (p_2 + \max(0, -@_{i_2} p_2 - @_{m_2} n_2))$

$$\langle := * \rangle, \text{NTV}\ \frac{\langle := \rangle\ \dfrac{\mathbb{R}\ \dfrac{*}{\Gamma \vdash) @_{i_2} \theta_1 = @_{m_2} n_1 \wedge \theta_2 = @_{m_2} n_2}}{\Gamma \vdash @_{i_2} \langle n_1, n_2 := \theta_1, \theta_2 \rangle n_2 = @_{n_1} n_2 \wedge n_1 = @_{n_1} n_1}}{\Gamma \vdash @_{i_2} \langle \alpha_N \rangle \downarrow n_2\ @_{n_2} n_1 = @_{n_1} n_1 \wedge n_2 = @_{n_1} n_2 \wedge R(n_1, n_2)} \quad \frac{\mathcal{D}_{lb4}}{\Gamma_2 \vdash @_{n_2} \langle \alpha_{LB} \rangle \downarrow m_2\ R(m_1, m_2)}$$
$$\text{BS}; \frac{\langle ; \rangle\ \dfrac{R(i_1, i_2), @_{i_1} \langle \alpha_{NLB} \rangle m_1, @_{m_1} (gr = n_1 + n_2 \wedge gm = 0) \vdash @_{i_2} \langle \alpha_N \rangle \langle \alpha_{LB} \rangle \downarrow m_2\ R(m_1, m_2)}{R(i_1, i_2), @_{i_1} \langle \alpha_{NLB} \rangle o_1, (gr = n_1 + n_2 \wedge gm = 0) \vdash @_{i_2} \langle \alpha_{NLB} \rangle \downarrow o_2\ R(o_1, o_2)}}{}$$

$\mathcal{D}_{lb4}$

$$\text{NTV}\ \frac{*}{\Gamma \vdash @_{n_2} \langle \alpha_L \rangle \downarrow l_2\ (R(m_1, m_2) \wedge (@_{l_2} n_i) = (@_{n_2} n_i))} \qquad \frac{\langle \cup \rangle, \langle ; \rangle\ \dfrac{\langle := \rangle\ \dfrac{\mathbb{R}, \text{assumption}\ \dfrac{\mathbb{R}, \text{assumption}\ R(l_1, l_2)\ \dfrac{\text{NTV}\ \dfrac{*}{\Gamma \vdash @_{l_1} t = @_{m_1} t}}{\Gamma \vdash @_{l_2} t = @_{m_1} t}}{\Gamma \vdash @_{l_2} n_1 + n_2, m \cdot (-1) + m \cdot (-1)^2, t = @_{m_1} gr, gm, t}}{\Gamma \vdash @_{l_2} \langle \ldots, gr, gm := \ldots, n_1 + n_2, m \cdot (-1) + m \cdot (-1)^2 \rangle gr, gm, t = @_{m_1} gr, gm, t}}{@\text{hom}, \downarrow\ \dfrac{\Gamma \vdash @_{l_2} \langle \alpha_B \rangle gr, gm, t = @_{m_1} gr, gm, t}{\Gamma \vdash @_{l_2} \langle \alpha_{LB} \rangle \downarrow m_2\ R(m_1, m_2)}}$$
$$\text{BS}; \frac{}{\Gamma \vdash @_{n_2} \langle \alpha_{LB} \rangle \downarrow m_2\ R(m_1, m_2)}$$

The controller inversion lemma: Here we abbreviate

$$\phi_1 \equiv (gr = 0 \wedge gm = 0)$$
$$\phi_2 \equiv (gr = n_1 \wedge gm = m)$$
$$\phi_3 \equiv (gr = n_2 \wedge gm = -m)$$
$$\phi_4 \equiv (gr = n_1 + n_2 \wedge gm = 0)$$

for formulas and

$$\gamma = gr := 0;\ bm_i := 0;\ gm := 0$$
$$\beta_{1a} \equiv ?(B_1 < B_{max}) \vee (n_1 > 0 \wedge B_1 > 0); b_1 := -n_1;\ bm_1 := bm_1 + m \cdot (-1)^2$$
$$\beta_{1b} \equiv b_1 := 0;\ gr := gr + n_1;\ gm := gm + m \cdot (-1)^2$$
$$\beta_{2a} \equiv ?(B_2 < B_{max}) \vee (n_2 > 0 \wedge B_2 > 0); b_2 := -n_2;\ bm_2 := bm_2 + m \cdot (-1)^3$$
$$\beta_{2b} \equiv b_2 := 0;\ gr := gr + n_2;\ gm := gm + m \cdot (-1)^3$$

Then the lemma is: $@_{i_1} \langle \alpha_{NLB} \rangle m_1 \rightarrow @_{m_1} (\phi_1 \vee \phi_2 \vee \phi_3 \vee \phi_4)$

Here $b_1, \ldots, b_4$ are branches, proved after the main lemma.

$$\langle\cup\rangle, \wedge\mathrm{R}\ \frac{b_1 \quad b_2 \quad b_2 \quad b_3}{[\gamma][\beta_{1a}\cup\beta_{1b}][\beta_{2a}\cup\beta_{2b}](\phi_1\vee\phi_2\vee\phi_3\vee\phi_4)}$$
$$\langle;\rangle\ \frac{[\gamma][\beta_{1a}\cup\beta_{1b}][\beta_{2a}\cup\beta_{2b}](\phi_1\vee\phi_2\vee\phi_3\vee\phi_4)}{[\alpha_B](\phi_1\vee\phi_2\vee\phi_3\vee\phi_4)}$$
$$\mathrm{G}_@\ \frac{[\alpha_B](\phi_1\vee\phi_2\vee\phi_3\vee\phi_4)}{@_{l_1}[\alpha_B](\phi_1\vee\phi_2\vee\phi_3\vee\phi_4)}$$
$$\langle\overline{n}\rangle\ \frac{@_{l_1}\langle\alpha_B\rangle m_1 \to @_{m_1}(\phi_1\vee\phi_2\vee\phi_3\vee\phi_4)}{@_{i_1}\langle\alpha_{NLB}\rangle m_1 \to @_{m_1}(\phi_1\vee\phi_2\vee\phi_3\vee\phi_4)}$$
$$\langle;\rangle,\exists W,@\mathrm{I}\ \frac{@_{l_1}\langle\alpha_B\rangle m_1 \to @_{m_1}(\phi_1\vee\phi_2\vee\phi_3\vee\phi_4)}{@_{i_1}\langle\alpha_{NLB}\rangle m_1 \to @_{m_1}(\phi_1\vee\phi_2\vee\phi_3\vee\phi_4)}$$

We also define $\sigma = \{gr \mapsto 0, bm_i \mapsto 0, gm \mapsto 0\}$
b_1

$$\mathbb{R}\ \frac{*}{0=0\wedge 0=0}$$
$$\text{by defn}\ \frac{}{\sigma(\phi_1)}$$
$$\langle\cup\rangle, \langle:=\rangle, \langle;\rangle\ \frac{}{[\beta_{1a}][\beta_{2a}]\sigma(\phi_1)}$$
$$\mathrm{M}[\cdot]\ \frac{[\beta_{1a}][\beta_{2a}]\sigma(\phi_1)}{[\beta_{1a}][\beta_{2a}]\sigma(\phi_1\vee\phi_2\vee\phi_3\vee\phi_4)}$$
$$\langle:=\rangle\ \frac{[\beta_{1a}][\beta_{2a}]\sigma(\phi_1\vee\phi_2\vee\phi_3\vee\phi_4)}{[\gamma][\beta_{1a}][\beta_{2a}](\phi_1\vee\phi_2\vee\phi_3\vee\phi_4)}$$

b_2 Define $\alpha \equiv gr := 0;\ bm_i := 0;\ gm := 0$ then

$$\mathbb{R}\ \frac{*}{(0+n_2=n_2\wedge m\cdot(-1)^3=-m)}$$
$$\langle;\rangle, \langle:=\rangle, \langle?\rangle\ \frac{}{[\alpha][?(B_1<B_{max})\vee(n_1>0\wedge B_1>0);b_1:=-n_1;\ bm_1:=bm_1+m\cdot(-1)^2](gr+n_2=n_2\wedge m\cdot(-1)^3=-m)}$$
$$\langle;\rangle, \langle:=\rangle\ \frac{}{[\alpha][?(B_1<B_{max})\vee(n_1>0\wedge B_1>0);b_1:=-n_1;\ bm_1:=bm_1+m\cdot(-1)^2][b_2:=0;\ gr:=gr+n_2;\ gm:=m\cdot(-1)^3](gr=n_2\wedge gm=-m)}$$
$$\text{by defn}\ \frac{}{[\gamma][\beta_{1a}][\beta_{2b}]\phi_3}$$
$$\mathrm{M}[\cdot]\ \frac{[\gamma][\beta_{1a}][\beta_{2b}]\phi_3}{[\gamma][\beta_{1a}][\beta_{2b}](\phi_1\vee\phi_2\vee\phi_3\vee\phi_4)}$$

b_3 Define $\alpha \equiv gr := 0;\ bm_i := 0;\ gm := 0$ then

$$\mathbb{R}\ \frac{*}{(0+n_1=n_1\wedge m\cdot(-1)^2=m)}$$
$$\langle;\rangle, \langle:=\rangle\ \frac{}{[\alpha](gr+n_1=n_1\wedge m\cdot(-1)^2=m)}$$
$$\langle;\rangle, \langle:=\rangle\ \frac{}{[\alpha][b_1:=0;\ gr:=gr+n_1;\ gm:=gm+m\cdot(-1)^2](gr=n_1\wedge m\cdot(-1)^2=m)}$$
$$\langle;\rangle, \langle:=\rangle\ \frac{}{[\alpha][b_1:=0;\ gr:=gr+n_1;\ gm:=gm+m\cdot(-1)^2](gr=n_1\wedge gm=m)}$$
$$\langle?\rangle, \langle;\rangle, \langle:=\rangle\ \frac{}{[\alpha][b_1:=0;\ gr:=gr+n_1;\ gm:=gm+m\cdot(-1)^2][?(B_2<B_{max})\vee(n_2>0\wedge B_2>0);b_2:=-n_2;\ bm_2:=bm_2+m\cdot(-1)^3](gr=n_1\wedge gm=m)}$$
$$\text{by defn}\ \frac{}{[\gamma][\beta_{1b}][\beta_{2a}]\phi_2}$$
$$\mathrm{M}[\cdot]\ \frac{[\gamma][\beta_{1b}][\beta_{2a}]\phi_2}{[\gamma][\beta_{1b}][\beta_{2a}](\phi_1\vee\phi_2\vee\phi_3\vee\phi_4)}$$

b_4 Define $\alpha \equiv gr := 0;\ bm_i := 0;\ gm := 0$ then

$$\langle;\rangle,\langle:=\rangle\ \frac{\langle;\rangle,\langle:=\rangle\ \frac{\langle;\rangle,\langle:=\rangle\ \frac{\mathbb{R}\ \frac{*}{(0+n_1+n_2=n_1+n_2\land 0+m\cdot(-1)^2+m\cdot(-1)^3=0)}}{[\alpha](gr+n_1+n_2=n_1+n_2\land gm+m\cdot(-1)^2+m\cdot(-1)^3=0)}}{[\alpha][b_1:=0;\ gr:=gr+n_1;\ gm:=gm+m\cdot(-1)^2](gr+n_2=n_1+n_2\land gm+m\cdot(-1)^3=0)}}{\text{by defn}\ \dfrac{[\alpha][b_1:=0;\ gr:=gr+n_1;\ gm:=gm+m\cdot(-1)^2][b_2:=0;\ gr:=gr+n_2;\ gm:=gm+m\cdot(-1)^3](gr=n_1+n_2\land gm=0)}{\mathrm{M}[\cdot]\ \dfrac{[\gamma][\beta_{1b}][\beta_{2b}]\phi_4}{[\gamma][\beta_{1b}][\beta_{2b}](\phi_1\lor\phi_2\lor\phi_3\lor\phi_4)}}}$$

□

F Full proof of insecure smart grid model

See note at the start of Appendix E on extra notations used in the appendices.

Lemma 33. Formula $\exists i_1 : \mathcal{W}\ @_i\langle B_i := B_{max}; t := 0; gr := 0\rangle i_1$ is valid.

Proof.
$$\exists W,\mathrm{M}\ \frac{\mathrm{BW}\ \dfrac{\mathrm{G}_@\ \dfrac{\langle;\rangle,\langle:=\rangle\ \dfrac{\mathbb{R}\ \dfrac{*}{\top}}{\langle B_i := B_{max}; t := 0; gr := 0\rangle\top}}{@_i\langle B_i := B_{max}; t := 0; gr := 0\rangle\top}}{@_i\langle B_i := B_{max}; t := 0; gr := 0\rangle\exists i_1 : \mathcal{W}\ i_1}}{\exists i_1 : \mathcal{W}\ @_i\langle B_i := B_{max}; t := 0; gr := 0\rangle i_1}$$

□

Lemma 34. Formula $@_i\langle B_i := B_{max}; t := 0; gr := 0\rangle i_1 \to @_{i_1} B_i = B_{max} \land t = 0 \land gr = 0$ is valid.

Proof.
$$\to\mathrm{R}\ \frac{\langle\overline{n}\rangle\ \dfrac{\mathrm{G}_@\ \dfrac{[:=],[;]\ \dfrac{\mathbb{R}\ \dfrac{*}{\vdash B_{max}=B_{max}\land 0=0\land 0=0}}{\vdash [B_i := B_{max}; t := 0; gr := 0](B_i = B_{max}\land t = 0\land gr = 0)}}{\vdash @_i[B_i := B_{max}; t := 0; gr := 0](B_i = B_{max}\land t = 0\land gr = 0)}}{@_i\langle B_i := B_{max}; t := 0; gr := 0\rangle i_1 \vdash @_{i_1}(B_i = B_{max}\land t = 0\land gr = 0)}}{@_i\langle B_i := B_{max}; t := 0; gr := 0\rangle i_1 \to @_{i_1}(B_i = B_{max}\land t = 0\land gr = 0)}$$

□

Lemma 35. Formula $\exists i_2 : \mathcal{W}\ @_i\langle B_i := \frac{1}{2}B_{max}; t := 0; gr := 0\rangle i_2$ is valid.

Proof.
$$\exists W,\mathrm{M}\ \frac{\mathrm{BW}\ \dfrac{\mathrm{G}_@\ \dfrac{\langle;\rangle,\langle:=\rangle\ \dfrac{\mathbb{R}\ \dfrac{*}{\top}}{\langle B_i := \frac{1}{2}B_{max}; t := 0; gr := 0\rangle\top}}{@_i\langle B_i := \frac{1}{2}B_{max}; t := 0; gr := 0\rangle\top}}{@_i\langle B_i := \frac{1}{2}B_{max}; t := 0; gr := 0\rangle\exists i_1 : \mathcal{W}\ i_1}}{\exists i_1 : \mathcal{W}\ @_i\langle B_i := \frac{1}{2}B_{max}; t := 0; gr := 0\rangle i_1}$$

□

Lemma 36. Formula $@_i\langle B_i := \frac{1}{2}B_{max}; t := 0; gr := 0\rangle i_2 \rightarrow @_{i_2} B_i = \frac{1}{2}B_{max} \land t = 0 \land gr = 0$ is valid.

Proof.

$$
{\scriptstyle\rightarrow\mathrm{R}}\;\dfrac{{\scriptstyle\langle\overline{n}\rangle}\;\dfrac{{\scriptstyle\mathrm{G}_{@}}\;\dfrac{{\scriptstyle[:=],[;]}\;\dfrac{{\scriptstyle\mathbb{R}}\;\dfrac{*}{\vdash \frac{1}{2}B_{max} = \frac{1}{2}B_{max} \land 0 = 0 \land 0 = 0}}{\vdash [B_i := \frac{1}{2}B_{max}; t := 0; gr := 0](B_i = \frac{1}{2}B_{max} \land t = 0 \land gr = 0)}}{\vdash @_i[B_i := \frac{1}{2}B_{max}; t := 0; gr := 0](B_i = \frac{1}{2}B_{max} \land t = 0 \land gr = 0)}}{@_i\langle B_i := \frac{1}{2}B_{max}; t := 0; gr := 0\rangle i_2 \vdash @_{i_2}(B_i = \frac{1}{2}B_{max} \land t = 0 \land gr = 0)}}{@_i\langle B_i := \frac{1}{2}B_{max}; t := 0; gr := 0\rangle i_2 \rightarrow @_{i_2}(B_i = \frac{1}{2}B_{max} \land t = 0 \land gr = 0)}
$$

□

Lemma 37. Formula $@_{i_1}(B_i = B_{max} \land gr = 0 \land t = 0) \rightarrow \exists o_2 : \mathcal{W}\left((@_{i_2}\langle\alpha_F\rangle o_2) \land @_{o_2}(t = 0 \land gr > 0)\right)$ is valid.

Proof. Everywhere in this proof, $@_{i_1}(B_i = B_{max} \land gr = 0 \land t = 0)$ is implicitly in the context. Define also $\phi_{QE} \equiv \neg\left((n_i \leq 0 \land B_i < B_{max}) \lor (n_i > 0 \land B_i > 0)\right)$.

$$
{\scriptstyle @\mathrm{I}}\;\dfrac{{\scriptstyle\mathrm{BW}}\;\dfrac{{\scriptstyle\exists\mathcal{W},\mathrm{M}}\;\dfrac{{\scriptstyle\langle *\rangle}\;\dfrac{{\scriptstyle\langle:=\rangle}\;\dfrac{{\scriptstyle\langle:=*\rangle,\langle ?\rangle,\mathbb{R}}\;\dfrac{{\scriptstyle\mathbb{R},\mathrm{CE}}\;\dfrac{{\scriptstyle\langle:=\rangle}\;\dfrac{{\scriptstyle\langle;\rangle,\langle if\rangle 2}\;\dfrac{{\scriptstyle\mathbb{R}}\;\dfrac{*}{(@_{i_2}\neg(n_i \geq thresh \land N_{\bar{i}} < 0))^{1}_{n_i}} \qquad {\scriptstyle\langle:=\rangle}\;\dfrac{{\scriptstyle\langle;\rangle}\;\dfrac{{\scriptstyle\langle'\rangle}\;\dfrac{{\scriptstyle\langle if\rangle 2,\langle;\rangle,\mathrm{CE}}\;\dfrac{{\scriptstyle\mathbb{R}}\;\dfrac{*}{\phi_{QE}} \qquad {\scriptstyle\langle:=\rangle}\;\dfrac{{\scriptstyle\langle:=\rangle}\;\dfrac{{\scriptstyle\langle:=\rangle}\;\dfrac{{\scriptstyle\mathrm{subst}}\;\dfrac{{\scriptstyle\langle:=\rangle}\;\dfrac{{\scriptstyle\mathbb{R},\,\mathrm{assump}\,R}\;\dfrac{*}{@_{i_2}(t = 0 \land 0 + 1 + 1 > 0)}}{@_{i_2}\langle gr := 0 + 1 + 1; gm := 0 + 0((-1)^2 + (-1)^3)\rangle(t = 0 \land gr > 0)}}{@_{i_2}\langle(gr := 0 + n_1 + n_2; gm := 0 + m(-1^2 + -1^3))^{1,0}_{n_i,m}\rangle(t = 0 \land gr > 0)}}{@_{i_2}\langle(b_i := 0; gr := 0 + n_1 + n_2; gm := 0 + m(-1^2 + -1^3))^{1,0}_{n_i,m}\rangle(t = 0 \land gr > 0)}}{@_{i_2}\langle(gm := 0; b_i := 0; gr := 0 + n_1 + n_2; gm := gm + m(-1^2 + -1^3))^{1,0}_{n_i,m}\rangle(t = 0 \land gr > 0)}}{@_{i_2}\langle(gr := 0; bm_i := 0; gm := 0; b_i := 0; gr := gr + n_1 + n_2; gm := gm + m(-1^2 + -1^3))^{1,0}_{n_i,m}\rangle(t = 0 \land gr > 0)}}{@_{i_2}\langle(\alpha_B)^{1,0}_{n_i,m}\rangle(t = 0 \land gr > 0)}}{@_{i_2}\langle(\alpha_B)^{1,0}_{n_i,m}\rangle\langle(\alpha_P)^{1,0}_{n_i,m}\rangle(t = 0 \land gr > 0)}}{@_{i_2}\langle(\alpha_{BP})^{1,0}_{n_i,m}\rangle(t = 0 \land gr > 0)}}{@_{i_2}\langle m := 0\rangle\langle(\alpha_{BP})^{1}_{n_i}\rangle(t = 0 \land gr > 0)}}{@_{i_2}\langle(\alpha_{LBP})^{1}_{n_i}\rangle(t = 0 \land gr > 0)}}{@_{i_2}\langle n_i := 1\rangle\langle\alpha_{LBP}\rangle(t = 0 \land gr > 0)}}{@_{i_2}\langle n_i := \max(0, p_i) + 1 - (p_i + \max(0, -p_i))\rangle\langle\alpha_{LBP}\rangle(t = 0 \land gr > 0)}}{@_{i_2}\langle d_i := *\rangle\langle ?d_i \geq 0\rangle\langle r_i := *\rangle\langle ?r_i \geq 0\rangle\langle n_i := \cdots\rangle\langle\alpha_{LBP}\rangle(t = 0 \land gr > 0)}}{@_{i_2}\langle\alpha_{NLBP}\rangle(t = 0 \land gr > 0)}}{@_{i_2}\langle\alpha_F\rangle(t = 0 \land gr > 0)}}{@_{i_2}\langle\alpha_F\rangle\exists o_2 : \mathcal{W}(o_2 \land @_{o_2}(t = 0 \land gr > 0))}}{\exists o_2 @_{i_2}\langle\alpha_F\rangle(o_2 \land @_{o_2}(t = 0 \land gr > 0))}}{\exists o_2(@_{i_2}\langle\alpha_F\rangle o_2) \land @_{o_2}(t = 0 \land gr > 0)}
$$

□

Lemma 38. Formula $@_{i_2}[\alpha_F](t \geq 0 \land (t = 0 \rightarrow (gr = 0 \land B_1 = B_2 = B_{max})))$ is valid.

Proof. Define $J \equiv t \geq 0 \wedge (t = 0 \rightarrow (gr = 0 \wedge B_1 = B_2 = B_{max}))$ Define $cond \equiv (n_i \leq 0 \wedge B_i < B_{max}) \vee (n_i > 0 \wedge B_i > 0)$. Define branches $\gamma_1 \equiv b_i := -n_i;\ bm_i := bm_i + m \cdot (-1)^{i+1}$ and $\gamma_2 \equiv \{\ b_i := 0;\ gr := gr + n_i;\ gm := gm + m \cdot (-1)^{i+1}\ \}$.

$$
\dfrac{\mathbb{R}\ \dfrac{*}{@_{i_2}J} \qquad \mathrm{M}[\cdot]\ \dfrac{\mathrm{V}\ \dfrac{*}{Ja \vdash [\alpha_{NL}]J}}{J \vdash [\alpha_{NLBP}]J}}{\mathrm{I}\ \dfrac{}{@_{i_2}[\alpha_F]J}}
\qquad
\dfrac{\mathbb{R}\ \dfrac{*}{J, \neg cond \vdash \bot}}{\mathrm{hide}\ \dfrac{}{J, \neg cond \vdash \cdots}}
$$

$$
[if]\ \dfrac{\dfrac{\mathbb{R}\ \dfrac{*}{J \vdash t' \geq 0 \rightarrow (t + t' \geq 0 \wedge (t + t' = 0 \rightarrow y_G(t', 0) = 0 \wedge B_i = \frac{1}{2}B_{max}))}}{\forall \mathrm{R}\ \dfrac{}{J \vdash \forall t' \geq 0 : \mathbb{R}\ (t + t' \geq 0 \wedge (t + t' = 0 \rightarrow y_G(t', 0) = 0 \wedge B_i = \frac{1}{2}B_{max}))}}{[:=]\ \dfrac{}{J \vdash [gr := 0]\forall t' \geq 0 : \mathbb{R}\ (t + t' \geq 0 \wedge (t + t' = 0 \rightarrow y_G(t', gr) = 0 \wedge B_i = \frac{1}{2}B_{max}))}}{[']\ \dfrac{}{J \vdash [gr := 0][\alpha_P{}^{n_i,0}_{b_i,gm}]J}}{[:=]\ \dfrac{}{J \vdash [gr := 0; bm_i := 0; gm := 0; \gamma_1; \alpha_P]J}}
\qquad
[;]\ \dfrac{J \vdash [\alpha_B][\alpha_P]J}{J \vdash [\alpha_{BP}]J}
$$

□

Theorem 39 (Insecurity). Formula $\exists i_1, i_2, o_1 : \mathcal{W}\ R(i_1, i_2) \wedge @_{i_1}\langle\alpha_F\rangle o_1 \wedge @_{i_2}[\alpha_f]{\downarrow}o_2\ \neg R(o_1, o_2)$ is valid.

Proof. We abbreviate:

$$
\begin{aligned}
\phi_1 &\equiv @_i\langle B_i := B_{max}; t := 0; gr := 0\rangle i_1 \\
\phi_2 &\equiv @_i\langle B_i := \frac{1}{2}B_{max}; t := 0; gr := 0\rangle i_2 \\
\phi_o &\equiv @_{i_2}\langle\alpha_F\rangle o_2 \wedge @_{o_2} t = 0 \wedge gr > 0 \\
p_1(i_1) &\equiv @_{i_1} B_i = B_{max} \wedge t = 0 \wedge gr = 0 \\
p_2(i_2) &\equiv @_{i_2} B_i = \frac{1}{2}B_{max} \wedge t = 0 \wedge gr = 0
\end{aligned}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathbb{R}\ \dfrac{*}{p_1(i_1), p_2(i_2) \rightarrow R(i_1, i_2)} \qquad \mathrm{id}\ \dfrac{*}{@_{i_1}\langle\alpha_F\rangle o_1} \qquad \mathbb{R}, {\downarrow}\ \dfrac{*}{{\downarrow}o_2\ \neg R(o_1, o_2) \vdash J} \qquad \mathrm{L38}\ \dfrac{*}{@_{i_2}[\alpha_f]J}}{@_{i_2}[\alpha_f]{\downarrow}o_2\ \neg R(o_1, o_2)}}{\phi_1, \phi_2, \phi_o, p_1(i_1), p_2(i_2), i \rightarrow R(i_1, i_2) \wedge @_{i_1}\langle\alpha_F\rangle o_1 \wedge @_{i_2}[\alpha_f]{\downarrow}o_2\ \neg R(o_1, o_2)}}{\exists \mathrm{R}\ \dfrac{}{\phi_1, \phi_2, \phi_o, p_1(i_1), p_2(i_2), i \rightarrow \exists i_1, i_2, o_1 : \mathcal{W}\ (R(i_1, i_2) \wedge @_{i_1}\langle\alpha_F\rangle o_1 \wedge @_{i_2}[\alpha_f]{\downarrow}o_2\ \neg R(o_1, o_2))}}{\mathrm{L34,L36}\ \exists \mathrm{L}\ \dfrac{}{\phi_1, \phi_2, \phi_o, i \rightarrow \exists i_1, i_2, o_1 : \mathcal{W}\ (R(i_1, i_2) \wedge @_{i_1}\langle\alpha_F\rangle o_1 \wedge @_{i_2}[\alpha_f]{\downarrow}o_2\ \neg R(o_1, o_2))}}{\mathrm{L37},\ \exists \mathrm{L}\ \dfrac{}{\phi_1, \phi_2, i \rightarrow \exists i_1, i_2, o_1 : \mathcal{W}\ (R(i_1, i_2) \wedge @_{i_1}\langle\alpha_F\rangle o_1 \wedge @_{i_2}[\alpha_f]{\downarrow}o_2\ \neg R(o_1, o_2))}}{\mathrm{L33, L35},\ \exists \mathrm{L}\ \dfrac{}{i \rightarrow \exists i_1, i_2, o_1 : \mathcal{W}\ (R(i_1, i_2) \wedge @_{i_1}\langle\alpha_F\rangle o_1 \wedge @_{i_2}[\alpha_f]{\downarrow}o_2\ \neg R(o_1, o_2))}}{\exists W, \exists \mathrm{L}\ \dfrac{}{\exists i_1, i_2, o_1 : \mathcal{W}\ (R(i_1, i_2) \wedge @_{i_1}\langle\alpha_F\rangle o_1 \wedge @_{i_2}[\alpha_f]{\downarrow}o_2\ \neg R(o_1, o_2))}}
$$

□