

Inductive Logical Relations

Rui Li

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Computer Science Department
School of Computer Science
Carnegie Mellon University
Pittsburgh, PA 15213

Thesis Committee:

Stephanie Balzer, Chair
Robert Harper

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Abstract

Logical relations are a powerful proof technique for establishing properties of programming languages. Traditionally, they are defined by induction on the structure of types, with each case specifying the meaning of computability at the corresponding type. This enables a robust methodology for program verification, as one proves that programs satisfy the computability associated with their types. In this thesis, we present an alternative, purely inductive formulation of logical relations using inference rules. To illustrate the broad applicability of the inductive formulation, we apply it to two calculi: the simply-typed lambda calculus and a process calculus based on intuitionistic linear logic session types. For the former, we contribute an inductive logical relation to prove normalizability of all well-typed open terms. For the latter, we contribute another inductive logical relation to prove termination of all closed, well-typed session programs. Together, these results demonstrate that inductive logical relations offer a promising alternative to traditional formulations.

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Contents

Acknowledgements	v
1 Introduction	1
I Normalization of Lambda Calculus	4
2 Simply-Typed Lambda Calculus	5
2.1 Syntax	5
2.2 β -reduction	8
3 Logical Relations	14
3.1 Inductive Logical Relation	15
3.2 Recursive Counterpart	18
4 Properties of Logical Relations	20
4.1 Proofs of Lemmas	24
4.2 Proofs of Theorems	44
5 Confluence	66
5.1 Proofs	69
6 Normalization	103
6.1 Proofs	107
II Termination of Process Calculus	137
7 Process Calculus	138
7.1 Syntax	140
7.2 Process Semantics	142
8 Logical Relations	144
8.1 Configuration	144
8.2 Inductive Logical Relation	147
8.3 Recursive Counterpart	148
9 Properties of Logical Relations	150
9.1 Semantic Fundamental Theorem	153
9.2 Syntactic Fundamental Theorem	162
9.3 Corollaries	170

10 Related Work	173
11 Conclusion	175
Bibliography	177

Chapter 1

Introduction

Our exploration of logical relations will be unconventional, in that we will define the main logical relation inductively as inference rules. Traditionally, logical relations are defined by induction on the structure of types, to account for the negative occurrence of the universal quantification of a computable argument occurring in the arrow-type case of the definition. For example, consider the following typical definition of the arrow case of a computability predicate for closed terms:

$$H_{A_L \rightarrow A_R}(M) \iff \forall M_L. \boxed{H_{A_L}(M_L)} \implies H_{A_R}(M M_L)$$

Here, $H_{A_L}(M_L)$ occurs on the left of the implication and hence amounts to a negative occurrence. This means that the above case cannot be defined as a rule. Extending the above logical relation with worlds Δ ranging over contexts to account for free variables that may occur in a term M , we still face the same problem in the typical definition of the arrow case of a Kripke logical relation [5, 6]:

$$H_{A_L \rightarrow A_R}^\Delta(M) \iff \forall \Delta'. \Delta' \leq \Delta \implies \forall M_L. \boxed{H_{A_L}^{\Delta'}(M_L)} \implies H_{A_R}^{\Delta'}(M M_L)$$

However, we observe that now that we have Δ , which contains at least all the free variables of a term M , we may simply apply the term which inhabits the arrow type to a fresh variable:

$$H_{A_L \rightarrow A_R}^\Delta(M) \iff H_{A_R}^{\Delta, x:A_L}(M x)$$

That is, we claim that to verify if a term M is a computable function, we simply verify if the resulting application of M to a fresh variable remains computable at the resulting type A_R . This adjustment also removes the earlier negative occurrence of a universally quantified computable argument, which suggests we can make this new definition an inference rule, as follows, with a slight change to sequent-calculus-style notation:

$$\frac{\begin{array}{c} \text{[H-ArrowT]} \\ x \notin \Delta \quad \Delta, x : A_L \implies M \quad x \in A_R \end{array}}{\Delta \implies M \in A_L \rightarrow A_R}$$

Here, we are not supplying all possible fresh variables x to M . Rather, the above rule specifies that it suffices to apply M to a single fresh x . We further observe that the typical definition of the other cases

of a Kripke logical relation can be easily turned into inference rules, which allows us to define a fully inductive logical relation, purely by rules.

This positive phrasing is not confined to the lambda calculus. For example, we can define the function case of a process calculus analogously:

[V-LolliT-R]

$$\frac{\Omega \xrightarrow{?a} \Omega' \quad \Delta, a : A_L \Rightarrow \Omega' \in A_R}{\Delta \Rightarrow \Omega \in A_L \multimap A_R}$$

Here, Ω is a collection of processes, each of which is a program executing concurrently. This rule is similar to the rule for the function case of the lambda calculus: the receipt of an address a , which corresponds to the action $?a$, is analogous to the application of a term M to a fresh variable x . The address a is also free here and is then added to the context Δ .

To verify whether a term M in the lambda calculus is computable, it suffices to check that the application of M to a fresh variable x remains computable. Similarly, this rule states that, to verify whether a configuration is computable at a linear function type $A_L \multimap A_R$, it suffices to check that the configuration remains computable after receiving an arbitrary address. This rule also makes it possible to define a fully inductive logical relation for a process calculus based on intuitionistic linear logic session types [2, 8].

This insight is further developed in this thesis, in which I develop two inductive logical relations for two calculi, the simply-typed lambda calculus and a process calculus based on intuitionistic linear logic session types, as follows:

Part I (Normalization of Lambda Calculus) focuses on designing and utilizing an inductive logical relation to prove normalization of all open well-typed lambda terms:

- **Chapter 2 (Simply-Typed Lambda Calculus)** specifies the syntax and β -reduction of a standard simply-typed lambda calculus. It also gives several examples that use natural number recursion to implement the mathematical operations of addition, multiplication, and exponentiation.
- **Chapter 3 (Logical Relations)** motivates and defines an inductive logical relation for the simply-typed lambda calculus. It then shows how a proof of the fundamental theorem fails when only the definition of the logical relation is available. This motivates proving that the inductive logical relation is equivalent to a recursive logical relation. Through this equivalence, the compositionality of the inductive logical relation can be restored.
- **Chapter 4 (Properties of Logical Relations)** proves various properties of the inductive and recursive logical relations. By the definition of the inductive logical relation, the normalizability of closed terms with respect to β -reduction is established.
- **Chapter 5 (Confluence)** identifies the importance of proving forward closure of normalizability and then dedicates the whole chapter to proving confluence, which implies this closure.
- **Chapter 6 (Normalization)** utilizes the inductive logical relation to prove that any term inhabiting the inductive logical relation is normalizable. Combined with the fundamental theorem, this shows that any well-typed open term is normalizable.

Part II (Termination of Process Calculus) likewise focuses on designing and utilizing an inductive logical relation to prove termination of all closed session-typed programs:

- [Chapter 7 \(Process Calculus\)](#) defines a process calculus based on intuitionistic linear logic session types. It also defines two kinds of actions for processes. Right actions are not annotated with a destination address, and this distinction more closely matches the provider behaviors of nameless configurations.
- [Chapter 8 \(Logical Relations\)](#) defines configurations as collections of processes. It then defines an inductive logical relation applicable to these configurations.
- [Chapter 9 \(Properties of Logical Relations\)](#) proves various properties of the logical relations. It ends by showing that any closed session-typed configuration at the unit type 1 terminates.

Finally, this thesis summarizes related work and concludes both parts:

- [Chapter 10 \(Related Work\)](#) summarizes the main works upon which this thesis builds and highlights how the thesis differs from them.
- [Chapter 11 \(Conclusion\)](#) summarizes the main pipelines established in the two parts, which show that all well-typed open programs in the lambda calculus normalize and that all well-typed closed programs in the process calculus terminate.

Both [Part I \(Normalization of Lambda Calculus\)](#) and [Part II \(Termination of Process Calculus\)](#) are formalized on paper in this thesis document. In addition, [Part I \(Normalization of Lambda Calculus\)](#) is fully mechanized in the proof assistant Istari [3], including all the chapters mentioned above.

Next, we give a few examples illustrating the style of proof trees used throughout this thesis document. First, we use a single line to define inference rules:

$$\frac{J_1 \quad J_2 \quad J_3}{J}$$

Next, we use double-line right-pointing arrows for statements of theorems, lemmas, and corollaries:

$$\frac{J_1 \quad J_2 \quad J_3}{J} \Rightarrow$$

The above tree states that J_1 , J_2 , and J_3 together imply J via a proof. The implication is therefore represented by the long double-line right-pointing arrow $\Rightarrow$. Sometimes, the premises in an inference rule or in a lemma statement take up too much space on a single line, so we stack them:

$$\frac{\begin{array}{l} \text{(very long) } J_1 \\ \text{~~~~~} \\ \text{(very long) } J_2 \\ \text{~~~~~} \\ \text{(very long) } J_3 \end{array}}{J} \quad \frac{\begin{array}{l} \text{(very long) } J_1 \\ \text{~~~~~} \\ \text{(very long) } J_2 \\ \text{~~~~~} \\ \text{(very long) } J_3 \end{array}}{J} \Rightarrow$$

Here, the stacked premises are separated by wiggly lines, and the trees above are equivalent to the first two examples. Finally, during a proof, we use a dashed line to separate the available information from the conclusion to be established:

$$\frac{J_1 \quad J_2 \quad J_3}{J} \text{ --- }$$

Part I

Normalization of Lambda Calculus

Chapter 2

Simply-Typed Lambda Calculus

2.1 Syntax

We work with the simply-typed lambda calculus with constructions for units, products, arrows, sums, and natural numbers. The language has primitive recursion over natural numbers, which, as will become clear shortly, iterates over a natural number once for each successor constructor it contains. Since this applies only to a single recursion, we may nest recursions inside recursions to, for example, implement multiplication using addition and exponentiation using multiplication.

$$\begin{array}{ll} \text{Var} & x, \dots \\ \text{Term } M & ::= x \mid \langle \rangle \mid \langle M_L, M_R \rangle \mid M \cdot L \mid M \cdot R \\ & \mid \lambda x \Rightarrow M \mid M M_L \\ & \mid L \cdot M \mid R \cdot M \mid \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \\ & \mid \text{zero} \mid \text{succ } M \mid \text{natrec } M (M_0 \mid p, r \Rightarrow M_S) \\ \text{Type } A & ::= 1 \mid A_L \times A_R \mid A_L \rightarrow A_R \mid A_L + A_R \mid \text{nat} \end{array}$$

While we do not assume a particular representation of variables, we do assume that we can decide if two variables are equal, and given a collection of variables, we can pick a fresh variable that does not already appear in the collection. Additionally, the notion of terms comes before that of types, and hence each term is not intrinsically typed but rather assigned a type later. Thus, an untyped, and more specifically, untypable term may loop forever, and one of the goals we want to achieve is to show that if a term can be typed, then it can reduce to a unique β -normal form. We will derive this ending result step by step during the coming chapters.

$$\boxed{\Gamma \vdash M : A}$$

$$\begin{array}{c} \text{Var-WT} \qquad \text{Empty-WT} \\ \hline \Gamma, x : A \vdash x : A \qquad \Gamma \vdash \langle \rangle : 1 \\ \text{Pair-WT} \\ \hline \frac{\Gamma \vdash M_L : A_L \quad \Gamma \vdash M_R : A_R}{\Gamma \vdash \langle M_L, M_R \rangle : A_L \times A_R} \end{array}$$

$$\begin{array}{c}
\text{ProjL-WT} \qquad \text{ProjR-WT} \\
\frac{\Gamma \vdash M : A_L \times A_R}{\Gamma \vdash M \cdot L : A_L} \qquad \frac{\Gamma \vdash M : A_L \times A_R}{\Gamma \vdash M \cdot R : A_R} \\
\\
\text{InL-WT} \qquad \text{InR-WT} \\
\frac{\Gamma \vdash M : A_L}{\Gamma \vdash L \cdot M : A_L + A_R} \qquad \frac{\Gamma \vdash M : A_R}{\Gamma \vdash R \cdot M : A_L + A_R} \\
\\
\text{Case-WT} \\
\frac{\Gamma \vdash M : A_L + A_R \quad \Gamma, x_L : A_L \vdash M_L : B \quad \Gamma, x_R : A_R \vdash M_R : B}{\Gamma \vdash \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) : B} \\
\\
\text{Zero-WT} \qquad \text{Succ-WT} \\
\frac{}{\Gamma \vdash \text{zero} : \text{nat}} \qquad \frac{\Gamma \vdash M : \text{nat}}{\Gamma \vdash \text{succ } M : \text{nat}} \\
\\
\text{NatRec-WT} \\
\frac{\Gamma \vdash M : \text{nat} \quad \Gamma \vdash M_0 : A \quad \Gamma, p : \text{nat}, r : A \vdash M_S : A}{\Gamma \vdash \text{natrec } M (M_0 \mid p, r \Rightarrow M_S) : A} \\
\\
\text{Lam-WT} \qquad \text{App-WT} \\
\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x \Rightarrow M : A \rightarrow B} \qquad \frac{\Gamma \vdash M_L : A \rightarrow B \quad \Gamma \vdash M_R : A}{\Gamma \vdash M_L M_R : B}
\end{array}$$

Now, let's come up with a function that implements the addition of any two natural-number terms x and y :

$$\text{add} \triangleq \lambda x \Rightarrow \lambda y \Rightarrow \text{natrec } x (y \mid p, r \Rightarrow \text{succ } r)$$

With the typing rules above, we can state and prove that it is a well-typed function of the type $\text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$:

Example-Addition-WT

$$\frac{}{\frac{}{\cdot \vdash \lambda x \Rightarrow \lambda y \Rightarrow \text{natrec } x (y \mid p, r \Rightarrow \text{succ } r) : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}} \rightarrow}$$

Proof with $\Gamma_{x,y} = x : \text{nat}, y : \text{nat} :$

$$\begin{array}{c}
\frac{\frac{\Gamma_{x,y} \vdash x : \text{nat}}{1} \quad \frac{\Gamma_{x,y} \vdash y : \text{nat}}{2} \quad \frac{\frac{\Gamma_{x,y}, p : \text{nat}, r : \text{nat} \vdash r : \text{nat}}{3} \quad \Gamma_{x,y}, p : \text{nat}, r : \text{nat} \vdash \text{succ } r : \text{nat}}{4}}{5} \\
\frac{\Gamma_{x,y} \vdash \text{natrec } x (y \mid p, r \Rightarrow \text{succ } r) : \text{nat}}{6} \\
\frac{x : \text{nat} \vdash \lambda y \Rightarrow \text{natrec } x (y \mid p, r \Rightarrow \text{succ } r) : \text{nat} \rightarrow \text{nat}}{7} \\
\cdot \vdash \lambda x \Rightarrow \lambda y \Rightarrow \text{natrec } x (y \mid p, r \Rightarrow \text{succ } r) : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}
\end{array}$$

1 Var-WT 2 Var-WT 3 Var-WT 4 Succ-WT 5 NatRec-WT 6 Lam-WT 7 Lam-WT □

As we mentioned at the beginning of this chapter, we may nest recursions inside recursions. Though `add` is defined using a single recursion, we may then wrap it in another recursion to define a multiplication function:

$$\text{mul} \triangleq \lambda x \Rightarrow \lambda y \Rightarrow \text{natrec } x (\text{zero} \mid p, r \Rightarrow \text{add } r y)$$

Predictably, this term has the type $\text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$:

Example-Mul-WT

$$\cdot \vdash \lambda x \Rightarrow \lambda y \Rightarrow \text{natrec } x (\text{zero} \mid p, r \Rightarrow \text{add } r y) : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$$

Proof with $\Gamma_{x,y} = x : \text{nat}, y : \text{nat}$ and $\Gamma_{x,y,p,r} = \Gamma_{x,y}, p : \text{nat}, r : \text{nat}$:

$$\begin{array}{c}
\frac{\frac{\frac{\Gamma_{x,y,p,r} \vdash \text{add} : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}}{1} \quad \Gamma_{x,y,p,r} \vdash r : \text{nat}}{2} \quad \Gamma_{x,y,p,r} \vdash \text{add } r : \text{nat} \rightarrow \text{nat}}{3} \quad \Gamma_{x,y,p,r} \vdash y : \text{nat}}{4} \\
\frac{\Gamma_{x,y,p,r} \vdash \text{add } r y : \text{nat}}{5}
\end{array}$$

1 weaken Example-Addition-WT 2 Var-WT 3 App-WT 4 Var-WT 5 App-WT

Therefore:

$$\begin{array}{c}
\frac{\Gamma_{x,y} \vdash x : \text{nat}}{1} \quad \frac{\Gamma_{x,y} \vdash \text{zero} : \text{nat}}{2} \quad \Gamma_{x,y,p,r} \vdash \text{add } r y : \text{nat}}{3} \\
\frac{\Gamma_{x,y} \vdash \text{natrec } x (\text{zero} \mid p, r \Rightarrow \text{add } r y) : \text{nat}}{4} \\
\frac{x : \text{nat} \vdash \lambda y \Rightarrow \text{natrec } x (\text{zero} \mid p, r \Rightarrow \text{add } r y) : \text{nat} \rightarrow \text{nat}}{5} \\
\cdot \vdash \lambda x \Rightarrow \lambda y \Rightarrow \text{natrec } x (\text{zero} \mid p, r \Rightarrow \text{add } r y) : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}
\end{array}$$

1 Var-WT 2 Zero-WT 3 NatRec-WT 4 Lam-WT 5 Lam-WT □

We can take an even further step to use `mul` to define an exponentiation function:

$$\text{exp} \triangleq \lambda x \Rightarrow \lambda y \Rightarrow \text{natrec } x (\text{succ zero} \mid p, r \Rightarrow \text{mul } y)$$

Here, the function should implement the mathematical function $x \mapsto y \mapsto y^x$. That is, it should implement the operation of raising y to the power of x . Again, this function has the type $\text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$,

but we will not repeat deriving a proof tree of this fact since it would be similar to the proof of [Example-Mul-WT].

We have been mentioning repeatedly that the functions we defined, `add`, `mul`, and `exp`, should implement some corresponding mathematical functions. But what does this mean precisely? One possible meaning is that once we define a notion of reduction that applies to the terms of our language, we may reduce the applications of these functions to some natural-number terms, and their results should match the applications of mathematical addition, multiplication, and exponentiation to the same natural numbers.

More precisely, let's define the numeral $\bar{n}$ of any mathematical natural number $n \in \mathbb{N}$, where $\mathbb{N}$ is the mathematical notion of all natural numbers. Let's postulate that $\mathbb{N}$ consists only of the natural numbers $0, 1, 2, \dots$ and that it supports induction on $n \in \mathbb{N}$. Then define $\bar{n}$ by induction on n :

Definition 2.1.1 (Numerals)

$$\begin{aligned}\bar{0} &= \text{zero} \\ \overline{n+1} &= \text{succ } \bar{n}\end{aligned}$$

Hence, the numeral $\bar{n}$ consists of n applications of the successor constructor to `zero`. We can now state more precisely the meaning of having a lambda term $M : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$ implementing a mathematical function $f \in \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}$.

Definition 2.1.2 (Implementing a Mathematical Function)

We say that a term $M : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$ implements a mathematical function $f \in \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}$ if and only if:

$$\forall n_1, n_2 \in \mathbb{N} . M \bar{n}_1 \bar{n}_2 \rightarrow_{\beta}^* \overline{f(n_1)(n_2)}$$

That is, given two mathematical natural numbers $n_1, n_2 \in \mathbb{N}$, whenever we supply their numerals to the lambda term M , the application $M \bar{n}_1 \bar{n}_2$ should compute towards the result of taking the numeral of the mathematical application of f to n_1 and then to n_2 . Here, the notion of computation is phrased in terms of β -reduction, which we define below.

2.2 β -reduction

Next, we define β -reduction, which is applicable to all open terms:

$$\boxed{M \rightarrow_{\beta} M'}$$

Pair-Under-L

$$\frac{M_L \rightarrow_{\beta} M'_L}{\langle M_L, M_R \rangle \rightarrow_{\beta} \langle M'_L, M_R \rangle}$$

Pair-Under-R

$$\frac{M_R \rightarrow_{\beta} M'_R}{\langle M_L, M_R \rangle \rightarrow_{\beta} \langle M_L, M'_R \rangle}$$

ProjL-Under

$$\frac{M \rightarrow_{\beta} M'}{M \cdot L \rightarrow_{\beta} M' \cdot L}$$

ProjR-Under

$$\frac{M \rightarrow_{\beta} M'}{M \cdot R \rightarrow_{\beta} M' \cdot R}$$

ProjL-beta

$$\overline{\langle M_L, M_R \rangle \cdot L \rightarrow_{\beta} M_L}$$

ProjR-beta

$$\overline{\langle M_L, M_R \rangle \cdot R \rightarrow_{\beta} M_R}$$

Lam-Under

$$\frac{M \rightarrow_{\beta} M'}{\lambda x \Rightarrow M \rightarrow_{\beta} \lambda x \Rightarrow M'}$$

App-Under-Fun

$$\frac{M \rightarrow_{\beta} M'}{M M_L \rightarrow_{\beta} M' M_L}$$

App-Under-Arg

$$\frac{M_L \rightarrow_{\beta} M'_L}{M M_L \rightarrow_{\beta} M M'_L}$$

App-beta

$$\overline{(\lambda x \Rightarrow M) N \rightarrow_{\beta} [N/x](M)}$$

InL-Under

$$\frac{M \rightarrow_{\beta} M'}{L \cdot M \rightarrow_{\beta} L \cdot M'}$$

InR-Under

$$\frac{M \rightarrow_{\beta} M'}{R \cdot M \rightarrow_{\beta} R \cdot M'}$$

Case-Under-Main

$$\frac{M \rightarrow_{\beta} M'}{\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta} \text{case } M' (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)}$$

Case-Under-L

$$\frac{M_L \rightarrow_{\beta} M'_L}{\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta} \text{case } M (x_L \Rightarrow M'_L \mid x_R \Rightarrow M_R)}$$

Case-Under-R

$$\frac{M_R \rightarrow_{\beta} M'_R}{\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta} \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M'_R)}$$

Case-InL-beta

$$\frac{}{\text{case } L \cdot M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_\beta [M/x_L](M_L)}$$

Case-InR-beta

$$\frac{}{\text{case } R \cdot M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_\beta [M/x_R](M_R)}$$

Succ-Under

$$\frac{M \rightarrow_\beta M'}{\text{succ } M \rightarrow_\beta \text{succ } M'}$$

NatRec-Under-Main

$$\frac{M \rightarrow_\beta M'}{\text{natrec } M \ (M_0 \mid x, y \Rightarrow M_S) \rightarrow_\beta \text{natrec } M' \ (M_0 \mid x, y \Rightarrow M_S)}$$

NatRec-Under-Zero

$$\frac{M_0 \rightarrow_\beta M'_0}{\text{natrec } M \ (M_0 \mid x, y \Rightarrow M_S) \rightarrow_\beta \text{natrec } M \ (M'_0 \mid x, y \Rightarrow M_S)}$$

NatRec-Under-Succ

$$\frac{M_S \rightarrow_\beta M'_S}{\text{natrec } M \ (M_0 \mid x, y \Rightarrow M_S) \rightarrow_\beta \text{natrec } M \ (M_0 \mid x, y \Rightarrow M'_S)}$$

NatRec-Zero-beta

$$\frac{}{\text{natrec zero } (M_0 \mid x, y \Rightarrow M_S) \rightarrow_\beta M_0}$$

NatRec-Succ-beta

$$\frac{}{\text{natrec succ } M \ (M_0 \mid x, y \Rightarrow M_S) \rightarrow_\beta [M/x, \text{natrec } M \ (M_0 \mid x, y \Rightarrow M_S)/y](M_S)}$$

We would like to emphasize that β -reduction applies to all terms, not just closed terms. There are two kinds of rules defining β -reduction, the first of which consists of compatibility rules that state that a term reduces if one of its immediate subterms reduces, such as [Succ-Under]. The second kind consists of β -rules where an elimination form is applied to an introduction form, such as the rule [NatRec-Succ-beta], where we are eliminating on a successor of a natural number.

Next, we want to define the notion of taking zero or more β -reductions, which we call multi-step β -reduction:

$$\boxed{M \rightarrow_{\beta}^* M'}$$

Beta-Multi-Refl

$$\frac{}{M \rightarrow_{\beta}^* M}$$

Beta-Multi-Cons

$$\frac{M \rightarrow_{\beta} M' \quad M' \rightarrow_{\beta}^* M''}{M \rightarrow_{\beta}^* M''}$$

Notably, multi-step β -reduction enjoys the following non-trivial properties:

Subject-Reduction

$$\frac{\Gamma \vdash M : A \quad M \rightarrow_{\beta}^* M'}{\Gamma \vdash M' : A}$$

Hat-Subst-Preserves-Beta-Multi

$$\frac{M \rightarrow_{\beta}^* M'}{\widehat{\gamma}(M) \rightarrow_{\beta}^* \widehat{\gamma}(M')}$$

Hat-Re-Preserves-Beta-Multi

$$\frac{M \rightarrow_{\beta}^* M'}{\widehat{\rho}(M) \rightarrow_{\beta}^* \widehat{\rho}(M')}$$

[Subject-Reduction] states that whenever we start reducing a well-typed term $M : A$ relative to a context Γ , whatever term M' we end up with after the multi-step reduction remains a well-typed term of the same type A relative to the same context Γ . Both [Hat-Subst-Preserves-Beta-Multi] and [Hat-Re-Preserves-Beta-Multi] state that whenever we already have a reduction sequence starting from M and ending at M' , we may apply some substitution γ or renaming ρ to both terms, M and M' , to obtain a reduction sequence starting from $\widehat{\gamma}(M)$ or $\widehat{\rho}(M)$ and ending at $\widehat{\gamma}(M')$ or $\widehat{\rho}(M')$.

Following are two lemmas which are helpful for proving other lemmas and theorems:

Beta-Multi-if-Single

$$\frac{M \rightarrow_{\beta} M'}{M \rightarrow_{\beta}^* M'}$$

Beta-Multi-Trans

$$\frac{M \rightarrow_{\beta}^* M' \quad M' \rightarrow_{\beta}^* M''}{M \rightarrow_{\beta}^* M''}$$

With all these tools at hand, we can now try to prove that `add`, `mul`, and `exp` implement mathematical addition, multiplication and exponentiation. Let's start proving this for `add`, and before that let's pin down a slightly more precise mathematical addition. Define the mathematical addition $n_1 + n_2$ by induction on $n_1 \in \mathbb{N}$:

Definition 2.2.1 (Mathematical Addition of Natural Numbers)

$$\begin{aligned} 0 + n_2 &= n_2 \\ (n_1 + 1) + n_2 &= (n_1 + n_2) + 1 \end{aligned}$$

Now, the statement that `add` implements the mathematical addition:

add-Implements-Addition

$$\overline{\overline{\text{add } \overline{n_1} \ \overline{n_2} \rightarrow_{\beta}^* \overline{n_1 + n_2}}}$$

Proof with $\mathcal{L}_{n_1} \triangleq \lambda y \Rightarrow \text{natrec } \overline{n_1} (y \mid p, r \Rightarrow \text{succ } r)$ & $\mathcal{R}_{n_1, n_2} \triangleq \text{natrec } \overline{n_1} (\overline{n_2} \mid p, r \Rightarrow \text{succ } r)$:

It suffices to show the following claim by induction on $n_1 \in \mathbb{N}$.

$$\overline{\overline{\mathcal{R}_{n_1, n_2} \rightarrow_{\beta}^* \overline{n_1 + n_2}}}$$

CASE $\overline{\overline{\mathcal{R}_{0, n_2} \rightarrow_{\beta}^* \overline{0 + n_2}}}$

$$\begin{array}{c} \overline{\overline{\mathcal{R}_{0, n_2} \rightarrow_{\beta} \overline{n_2}}} \quad 1 \\ \overline{\overline{\mathcal{R}_{0, n_2} \rightarrow_{\beta}^* \overline{n_2}}} \quad 2 \\ \overline{\overline{\mathcal{R}_{0, n_2} \rightarrow_{\beta}^* \overline{0 + n_2}}} \quad 3 \end{array}$$

1 NatRec-Zero-beta 2 Beta-Multi-if-Single 3 definition of +

CASE $\overline{\overline{\mathcal{R}_{n_1+1, n_2} \rightarrow_{\beta}^* \overline{(n_1 + 1) + n_2}}}$

$$\begin{array}{c} \overline{\overline{\mathcal{R}_{n_1, n_2} \rightarrow_{\beta}^* \overline{n_1 + n_2}}} \quad 2 \\ \overline{\overline{\text{succ } \mathcal{R}_{n_1, n_2} \rightarrow_{\beta}^* \text{succ } \overline{n_1 + n_2}}} \quad 3 \\ \overline{\overline{\text{succ } \mathcal{R}_{n_1, n_2} \rightarrow_{\beta}^* \overline{(n_1 + 1) + n_2}}} \quad 4 \\ \overline{\overline{\mathcal{R}_{n_1+1, n_2} \rightarrow_{\beta} \text{succ } \mathcal{R}_{n_1, n_2}}} \quad 1 \quad \overline{\overline{\mathcal{R}_{n_1+1, n_2} \rightarrow_{\beta}^* \overline{(n_1 + 1) + n_2}}} \quad 5 \end{array}$$

1 NatRec-Succ-beta 2 IH 3 β^* -under 4 definitions of + and numerals 5 Beta-Multi-Cons

Shown both CASES and hence the claim.

Then:

$$\begin{array}{c} \overline{\overline{\text{add } \overline{n_1} \rightarrow_{\beta} \mathcal{L}_{n_1}}} \quad 1 \quad \overline{\overline{\mathcal{L}_{n_1} \ \overline{n_2} \rightarrow_{\beta} \mathcal{R}_{n_1, n_2}}} \quad 3 \\ \overline{\overline{\text{add } \overline{n_1} \ \overline{n_2} \rightarrow_{\beta} \mathcal{L}_{n_1} \ \overline{n_2}}} \quad 2 \quad \overline{\overline{\mathcal{L}_{n_1} \ \overline{n_2} \rightarrow_{\beta}^* \mathcal{R}_{n_1, n_2}}} \quad 4 \\ \overline{\overline{\text{add } \overline{n_1} \ \overline{n_2} \rightarrow_{\beta}^* \mathcal{R}_{n_1, n_2}}} \quad 5 \end{array}$$

1 App-beta 2 App-Under-Fun 3 App-beta 4 Beta-Multi-if-Single 5 Beta-Multi-Cons

Finally:

$$\frac{\frac{\text{add } \overline{n_1} \ \overline{n_2} \rightarrow_{\beta}^* \mathcal{R}_{n_1, n_2} \quad \frac{\mathcal{R}_{n_1, n_2} \rightarrow_{\beta}^* \overline{n_1 + n_2}}{1}}{2}}{\text{add } \overline{n_1} \ \overline{n_2} \rightarrow_{\beta}^* \overline{n_1 + n_2}}$$

1 claim 2 Beta-Multi-Trans

□

Likewise, we may pin down more precise notions of multiplication and exponentiation:

Definition 2.2.2 (Mathematical Multiplication and Exponentiation)

We define multiplication by induction on $n_1 \in \mathbb{N}$:

$$\begin{aligned} 0 \times n_2 &= 0 \\ (n_1 + 1) \times n_2 &= n_1 \times n_2 + n_2 \end{aligned}$$

We define exponentiation by induction on $n_1 \in \mathbb{N}$:

$$\begin{aligned} n_2^0 &= 1 \\ n_2^{n_1+1} &= n_2^{n_1} \times n_2 \end{aligned}$$

Now, the pair of statements that `mul` and `exp` implement these two mathematical functions:

mul-Implements-Multiplication

exp-Implements-Exponentiation

$$\frac{\text{mul } \overline{n_1} \ \overline{n_2} \rightarrow_{\beta}^* \overline{n_1 \times n_2}}{}$$

$$\frac{\text{exp } \overline{n_1} \ \overline{n_2} \rightarrow_{\beta}^* \overline{n_2^{n_1}}}{}$$

Proof by induction on $n_1 \in \mathbb{N}$, similar to the proof of [add-Implements-Addition], where in the proof of [mul-Implements-Multiplication], we make use of [add-Implements-Addition], and after this and in the proof of [exp-Implements-Exponentiation], we make use of [mul-Implements-Multiplication]. □

These examples illustrate that β -reduction, as it is applicable to all terms, assigns meaning to closed terms in particular, such as applications of `add`, `mul`, and `exp` to pairs of numerals, and that we will be able to follow the rules of β -reduction to find out the ending numerals that these applications intend to compute.

We will also find out that we can derive a stronger theorem: when the terms under reduction are open, in the sense of having zero or more free variables, we can follow the rules of β -reduction to derive β -normal forms of these terms if they are computable, where computability is a notion to be pinned down by logical relations we are going to define in [Chapter 3 \(Logical Relations\)](#). We will generalize our study to open terms in the next chapters.

Chapter 3

Logical Relations

In the last chapter, we reasoned about how we can define two-place lambda functions on natural-number terms to implement mathematical addition, multiplication, and exponentiation. Now, we would like to extend our study to open terms, or in other words, all terms, meaning now these terms could contain free variables that are not bound anywhere. Our goal would be to prove that there is a collection of well-behaved terms, which we will call computable, that will always normalize, i.e., reduce to their unique normal forms. To do so, we make use of logical relations.

Our exploration of logical relations will be unconventional, in that we will define the main logical relation inductively as inference rules. Traditionally, logical relations are defined by induction on the structure of types, to account for the negative occurrence of the universal quantification of a computable argument occurring in the arrow-type case of the definition. For example, consider the following typical definition of the arrow case of a computability predicate for closed terms:

$$H_{A_L \rightarrow A_R}(M) \iff \forall M_L. \boxed{H_{A_L}(M_L)} \implies H_{A_R}(M M_L)$$

Here, $H_{A_L}(M_L)$ is to the left of the implication and hence is a negative occurrence. This means that the above case cannot be defined as a rule. Extending the above logical relation with worlds Δ ranging over contexts to account for free variables that may occur in a term M , we still face the same problem in the typical definition of the arrow case of a Kripke logical relation [5, 6]:

$$H_{A_L \rightarrow A_R}^\Delta(M) \iff \forall \Delta'. \Delta' \leq \Delta \implies \forall M_L. \boxed{H_{A_L}^{\Delta'}(M_L)} \implies H_{A_R}^{\Delta'}(M M_L)$$

However, we observe that now that we have Δ , which contains at least all the free variables of a term M , we may simply apply the term which inhabits the arrow type to a fresh variable:

$$H_{A_L \rightarrow A_R}^\Delta(M) \iff H_{A_R}^{\Delta, x:A_L}(M x)$$

That is, we claim that to verify if a term M is a computable function, we simply verify if the resulting application of M to a fresh variable remains computable at the resulting type A_R . This adjustment also removes the earlier negative occurrence of a universally quantified computable argument, which suggests we can make this new definition an inference rule, as follows, with a slight change to sequent-calculus-style notation:

[H-ArrowT]

$$\frac{x \notin \Delta \quad \Delta, x : A_L \Rightarrow M \quad x \in A_R}{\Delta \Rightarrow M \in A_L \rightarrow A_R}$$

Here, we are not supplying all possible fresh variables x to M . Rather, the above rule specifies that it suffices to apply M to a single fresh x . We further observe that the typical definition of the other cases of a Kripke logical relation can be easily turned into inference rules, which allows us to define a fully inductive logical relation, purely by rules. Since the terms M under consideration by a Kripke computability predicate range over open terms, we also need to factor in the possibility that a term M reduces to a stuck spine, hence neutral, and the Kripke logical relation therefore demands an additional neutral computability judgement.

3.1 Inductive Logical Relation

These considerations motivate us to come up with the following definition, where the judgments $\Delta \xRightarrow{\mathcal{H}} M \in A$ and $\Delta \xRightarrow{\mathcal{N}} M \in A$ are defined simultaneously:

$\Delta \xRightarrow{\mathcal{H}} M \in A$	$\Delta \xRightarrow{\mathcal{N}} M \in A$
H-EmptyM $\frac{M \rightarrow_{\beta}^* \langle \rangle}{\Delta \xRightarrow{\mathcal{H}} M \in 1}$ H-Neutral-1T $\frac{M \rightarrow_{\beta}^* M' \quad \Delta \xRightarrow{\mathcal{N}} M' \in 1}{\Delta \xRightarrow{\mathcal{H}} M \in 1}$	
H-TimesT $\frac{\Delta \xRightarrow{\mathcal{H}} M \cdot L \in A_L \quad \Delta \xRightarrow{\mathcal{H}} M \cdot R \in A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L \times A_R}$	
H-ArrowT $\frac{x \notin \Delta \quad \Delta, x : A \xRightarrow{\mathcal{H}} M \quad x \in B}{\Delta \xRightarrow{\mathcal{H}} M \in A \rightarrow B}$	
H-InLM $\frac{M \rightarrow_{\beta}^* L \cdot M_L \quad \Delta \xRightarrow{\mathcal{H}} M_L \in A_L}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R}$ H-InRM $\frac{M \rightarrow_{\beta}^* R \cdot M_R \quad \Delta \xRightarrow{\mathcal{H}} M_R \in A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R}$	
H-Neutral-SumT $\frac{M \rightarrow_{\beta}^* M' \quad \Delta \xRightarrow{\mathcal{N}} M' \in A_L + A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R}$	

$$\begin{array}{c}
\text{H-ZeroM} \qquad \text{H-SuccM} \qquad \text{H-Neutral-NatT} \\
\frac{M \rightarrow_{\beta}^* \text{zero}}{\Delta \xRightarrow{\mathcal{H}} M \in \text{nat}} \quad \frac{M \rightarrow_{\beta}^* \text{succ } M' \quad \Delta \xRightarrow{\mathcal{H}} M' \in \text{nat}}{\Delta \xRightarrow{\mathcal{H}} M \in \text{nat}} \quad \frac{M \rightarrow_{\beta}^* M' \quad \Delta \xRightarrow{\mathcal{N}} M' \in \text{nat}}{\Delta \xRightarrow{\mathcal{H}} M \in \text{nat}} \\
\text{N-VarM} \\
\frac{}{\Delta, x : A \xRightarrow{\mathcal{N}} x \in A} \\
\text{N-projLM} \qquad \text{N-projRM} \\
\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R}{\Delta \xRightarrow{\mathcal{N}} M \cdot L \in A_L} \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R}{\Delta \xRightarrow{\mathcal{N}} M \cdot R \in A_R} \\
\text{N-AppM} \\
\frac{\Delta \xRightarrow{\mathcal{N}} M \in A \rightarrow B \quad \Delta \xRightarrow{\mathcal{H}} N \in A}{\Delta \xRightarrow{\mathcal{N}} M N \in B} \\
\text{N-CaseM} \\
\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L + A_R \quad \Delta, x_L : A_L \xRightarrow{\mathcal{H}} M_L \in B \quad \Delta, x_R : A_R \xRightarrow{\mathcal{H}} M_R \in B}{\Delta \xRightarrow{\mathcal{N}} \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B} \\
\text{N-NatRecM} \\
\frac{\Delta \xRightarrow{\mathcal{N}} M \in \text{nat} \quad \Delta \xRightarrow{\mathcal{H}} M_0 \in A \quad \Delta, x : \text{nat}, y : A \xRightarrow{\mathcal{H}} M_S \in A}{\Delta \xRightarrow{\mathcal{N}} \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \in A}
\end{array}$$

Ideally, this logical relation should admit the fundamental theorem which states that any well-typed term is computable:

$$\begin{array}{c}
\text{[FTLR]} \\
\frac{\Gamma \vdash M : A}{\Gamma \xRightarrow{\mathcal{H}} M \in A}
\end{array}$$

This statement of the fundamental theorem is slightly unconventional in that we do not supply a computable substitution to $\Gamma \vdash M : A$. Nonetheless, we will be able to prove this theorem as stated.

For now, however, without knowing any properties of the inductive logical relation, a proof by induction on $\Gamma \vdash M : A$ in which we only have access to all the primitive inference rules will get stuck in at least the application case:

$$\boxed{\text{CASE}} \quad \frac{\Gamma \vdash M_L : A \rightarrow B \quad \Gamma \vdash M_R : A}{\Gamma \vdash M_L M_R : B} [\text{App-WT}]$$

$$\Gamma \xRightarrow{\mathcal{H}} M_L M_R \in B$$

By IH on $\Gamma \vdash M_L : A \rightarrow B$, we have $\Gamma \xRightarrow{\mathcal{H}} M_L \in A \rightarrow B$.

By IH on $\Gamma \vdash M_R : A$, we have $\Gamma \xRightarrow{\mathcal{H}} M_R \in A$.

However, it is now not clear how to proceed, as the definition of $\Gamma \xRightarrow{\mathcal{H}} M_L \in A \rightarrow B$ does not tell us anything about what happens when we apply this supposedly computable function to a computable argument such as $\Gamma \xRightarrow{\mathcal{H}} M_R \in A$. We would therefore like to derive the following as a property of the inductive logical relation, with which we can immediately finish the proof of this case by applying it to $\Gamma \xRightarrow{\mathcal{H}} M_L \in A \rightarrow B$, $\Gamma \leq \Gamma$, and $\Gamma \xRightarrow{\mathcal{H}} M_R \in A$:

$$\boxed{\text{[H-App-Compositionality]}}$$

$$\frac{\Delta \xRightarrow{\mathcal{H}} M \in A \rightarrow B \quad \Delta' \leq \Delta \quad \Delta' \xRightarrow{\mathcal{H}} N \in A}{\Delta' \xRightarrow{\mathcal{H}} M N \in B}$$

The only other move we have here is to apply inversion to $\Gamma \xRightarrow{\mathcal{H}} M_L \in A \rightarrow B$, which tells us that $\Gamma, x : A \xRightarrow{\mathcal{H}} M_L x \in B$. But this alone does not give us anything useful. We would naturally want some substitution lemma that lets us replace this fresh x with the computable argument we already have in our context $\Gamma \xRightarrow{\mathcal{H}} M_R \in A$. Thus, if we have the following also as a property of the inductive logical relation, we can immediately close this case by applying it to $\Gamma \xRightarrow{\mathcal{H}} M_R \in A$ and $\Gamma, x : A \xRightarrow{\mathcal{H}} M_L x \in B$:

$$\boxed{\text{[Semantic-Cut]}}$$

$$\frac{\Delta \xRightarrow{\mathcal{H}} M \in A \quad \Delta, x : A \xRightarrow{\mathcal{H}} N \in B}{\Delta \xRightarrow{\mathcal{H}} [M/x]N \in B}$$

To unblock this case of the fundamental theorem, we need [H-App-Compositionality] or [Semantic-Cut], which we will derive as properties of the inductive logical relation. Hence, we cannot attack the fundamental theorem directly, as a proof without knowing any properties of the inductive logical relation will definitely get stuck as described above.

How can we prove these two theorems? One natural move is to define a recursive logical relation as a counterpart of the inductive logical relation, where in the arrow case we have the compositionality built in. If we are able to prove that the inductive logical relation holds if and only if the recursive logical relation holds, then we may simply transport the compositionality in the arrow case of the recursive logical relation to apply to the inductive logical relation.

3.2 Recursive Counterpart

$$\boxed{h_A^\Delta(M)}$$

$$\begin{aligned}
h_1^\Delta(M) &\iff M \rightarrow_\beta^* \langle \rangle \vee \exists M' . M \rightarrow_\beta^* M' \wedge \Delta \xRightarrow{\textcolor{red}{N}} M' \in 1 \\
h_{A_L \times A_R}^\Delta(M) &\iff h_{A_L}^\Delta(M \cdot L) \wedge h_{A_R}^\Delta(M \cdot R) \\
h_{A_L \rightarrow A_R}^\Delta(M) &\iff \boxed{\forall \Delta' \leq \Delta, M_L . h_{A_L}^{\Delta'}(M_L) \implies h_{A_R}^{\Delta'}(M M_L)} \\
h_{A_L + A_R}^\Delta(M) &\iff \begin{aligned} &\exists N . M \rightarrow_\beta^* L \cdot N \wedge h_{A_L}^\Delta(N) \\ &\vee \exists N . M \rightarrow_\beta^* R \cdot N \wedge h_{A_R}^\Delta(N) \\ &\vee \exists N . M \rightarrow_\beta^* N \wedge \Delta \xRightarrow{\textcolor{red}{N}} N \in A_L + A_R \end{aligned} \\
h_{\text{nat}}^\Delta(M) &\iff \Delta \xRightarrow{\textcolor{blue}{H}} M \in \text{nat}
\end{aligned}$$

We have boxed the desired compositionality we would like to recover for the inductive logical relation, once we can prove the above recursive logical relation is equivalent to the inductive logical relation. In the definition of the arrow case, $\Delta' \leq \Delta$ means that every typing assumption valid in Δ is also valid in Δ' :

$$\Delta' \leq \Delta \triangleq \Delta \vdash x : A \implies \Delta' \vdash x : A$$

In addition, observe that in the unit-type case, we appeal to $\Delta \xRightarrow{\textcolor{red}{N}} M' \in 1$ which is already defined when we are defining $h_1^\Delta(M)$.

Now, a more precise statement of the equivalence between the two logical relations we would like to prove:

$$\Delta \xRightarrow{\textcolor{blue}{H}} M \in A \iff h_A^\Delta(M) \quad (\text{equivalence})$$

Note that, once this equivalence is established, we can prove [H-App-Compositionality] easily, so that the fundamental theorem will be unblocked:

$$\begin{array}{c}
\frac{\Delta \xRightarrow{\textcolor{blue}{H}} M \in A \rightarrow B}{h_{A \rightarrow B}^\Delta(M)} \quad 1 \qquad \Delta' \leq \Delta \qquad \frac{\Delta' \xRightarrow{\textcolor{blue}{H}} N \in A}{h_A^{\Delta'}(N)} \quad 2 \\
\hline
\frac{h_{A \rightarrow B}^\Delta(M) \quad h_A^{\Delta'}(N)}{h_B^{\Delta'}(M N)} \quad 3 \\
\hline
\Delta' \xRightarrow{\textcolor{blue}{H}} M N \in B \quad 4
\end{array}$$

1 $\rightarrow$ equivalence 2 $\rightarrow$ equivalence 3 unfold h at $A \rightarrow B$ and $\forall / \implies$ -E 4 $\leftarrow$ equivalence

Observe how we annotate the proofs, where $\rightarrow$ equivalence stands for applying the left-to-right direction of the stated equivalence, and likewise for $\leftarrow$ equivalence. The proof establishes that if $\Delta \xRightarrow{\textcolor{blue}{H}} M \in A \rightarrow B$, $\Delta' \leq \Delta$, and $\Delta' \xRightarrow{\textcolor{blue}{H}} N \in A$, then we have $\Delta' \xRightarrow{\textcolor{blue}{H}} M N \in B$. Hence, with equivalence, we can transport the compositionality which applies to the recursive logical relation by definition to also apply to the inductive logical relation, by the above proof.

Now, with this backward reasoning, we find out that it remains to actually show the equivalence, which we can decompose into the following soundness and completeness theorems. Since our focus is on the inductive logical relation, [h-if-H] corresponds to soundness and [H-if-h] corresponds to completeness.

We will spend the entire next [Chapter 4 \(Properties of Logical Relations\)](#) proving this pair of theorems and various helper lemmas for them.

$[\text{h-if-H}]$ $\Delta \xRightarrow{\textcolor{teal}{H}} M \in A$ $\xRightarrow{\textcolor{teal}{h}_A^\Delta(M)} \rightarrow$	$[\text{H-if-h}]$ $\xRightarrow{\textcolor{teal}{h}_A^\Delta(M)} \rightarrow$ $\Delta \xRightarrow{\textcolor{teal}{H}} M \in A$
--	--

Chapter 4

Properties of Logical Relations

In the last chapter, we motivated why we need to establish the equivalence between the inductive and recursive logical relations. In this chapter, we move on to prove various helper lemmas and then prove the equivalence. First and foremost, we would like to show expansion holds for both the inductive and recursive logical relations:

[H-Expansion]	[h-Expansion]
$\frac{M \rightarrow_{\beta}^* M' \quad \Delta \xRightarrow{\mathcal{H}} M' \in A}{\Delta \xRightarrow{\mathcal{H}} M \in A}$	$\frac{M \rightarrow_{\beta}^* M' \quad h_A^{\Delta}(M')}{h_A^{\Delta}(M)}$

These two lemmas are helpful, for example, when we want to show $\Delta \xRightarrow{\mathcal{H}} \langle M_L, M_R \rangle \cdot L \in A$ or $h_A^{\Delta}(\langle M_L, M_R \rangle \cdot L)$. By expansion, it then suffices to show $\Delta \xRightarrow{\mathcal{H}} M_L \in A$ or $h_A^{\Delta}(M_L)$, which will match with the hypotheses we have in our proof context better.

Next, we want to show that the logical relations are stable under extension of variables to the indexing worlds Δ . Since the arrow case of the inductive logical relation is defined by saying it suffices to pick an arbitrary fresh variable, we need to show first that the inductive logical relations are stable under renamings to later derive that they are stable under extension to Δ by picking identity renamings:

[H-Rename]	[N-Rename]
$\frac{\Gamma \vdash \rho : \Delta \quad \Delta \xRightarrow{\mathcal{H}} M \in A}{\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in A}$	$\frac{\Gamma \vdash \rho : \Delta \quad \Delta \xRightarrow{\mathcal{N}} M \in A}{\Gamma \xRightarrow{\mathcal{N}} \hat{\rho}(M) \in A}$

$\begin{array}{c} \text{[H-Anti-Mono]} \\ \Delta' \leq \Delta \quad \Delta \xRightarrow{\mathcal{H}} M \in A \\ \hline \Delta' \xRightarrow{\mathcal{H}} M \in A \end{array}$	$\begin{array}{c} \text{[N-Anti-Mono]} \\ \Delta' \leq \Delta \quad \Delta \xRightarrow{\mathcal{N}} M \in A \\ \hline \Delta' \xRightarrow{\mathcal{N}} M \in A \end{array}$	$\begin{array}{c} \text{[h-Anti-Mono]} \\ \Delta' \leq \Delta \quad h_A^\Delta(M) \\ \hline h_A^{\Delta'}(M) \end{array}$
---	---	---

Next, we can establish the various relationships, except the soundness of the inductive logical relation with respect to the recursive logical relation, between the different judgments of the logical relations:

$\begin{array}{c} \text{[H-if-N]} \\ \Delta \xRightarrow{\mathcal{N}} M \in A \\ \hline \Delta \xRightarrow{\mathcal{H}} M \in A \end{array}$	$\begin{array}{c} \text{[h-if-N]} \\ \Delta \xRightarrow{\mathcal{N}} M \in A \\ \hline h_A^\Delta(M) \end{array}$	$\begin{array}{c} \text{[H-if-h]} \\ h_A^\Delta(M) \\ \hline \Delta \xRightarrow{\mathcal{H}} M \in A \end{array}$
---	--	--

Both [H-if-N] and [h-if-N] allow us to consider variables, which are neutral by definition in any world that contains them, to be computable. [H-if-h] has already recovered the completeness direction of the equivalence we wish to show.

Define computable $\mathcal{H}$ substitutions for $\gamma : \mathbb{V}\text{ar} \rightarrow \mathbb{T}\text{erm}$:

$$h_\Delta^\Gamma(\gamma) \triangleq \Delta \vdash x : A \Rightarrow h_A^\Gamma(\gamma(x))$$

With this definition of computable substitutions and the lemmas, we can prove the most important pair of theorems, by simultaneous induction on $\Delta \xRightarrow{\mathcal{H}} M \in A$ and $\Delta \xRightarrow{\mathcal{N}} M \in A$.

$\begin{array}{c} \text{[h-Cut-H]} \\ h_\Delta^\Gamma(\gamma) \quad \Delta \xRightarrow{\mathcal{H}} M \in A \\ \hline h_A^\Gamma(\hat{\gamma}(M)) \end{array}$	$\begin{array}{c} \text{[h-Cut-N]} \\ h_\Delta^\Gamma(\gamma) \quad \Delta \xRightarrow{\mathcal{N}} M \in A \\ \hline h_A^\Gamma(\hat{\gamma}(M)) \end{array}$
---	---

Observe these two theorems are similar to the traditional fundamental theorem, where we would have swapped both $\Delta \xRightarrow{\mathcal{H}} M \in A$ and $\Delta \xRightarrow{\mathcal{N}} M \in A$ with $\Delta \vdash M : A$:

$\begin{array}{c} \text{Traditional-FTLR} \\ h_\Delta^\Gamma(\gamma) \quad \Delta \vdash M : A \\ \hline h_A^\Gamma(\hat{\gamma}(M)) \end{array}$

But we will not prove such an FTLR. We will prove a simpler one presented later.

Now, with [h-Cut-H], we may show the soundness of the inductive logical relation with respect to the recursive logical relation:

[h-if-H]

$$\frac{\Delta \xRightarrow{\mathcal{H}} M \in A}{h_A^\Delta(M)}$$

Proof:

$$\frac{\frac{\frac{x \in \Delta}{1} \quad \Delta \xRightarrow{\mathcal{N}} x \in \Delta(x)}{2} \quad \frac{h_{\Delta(x)}^\Delta(x)}{3} \quad \frac{\forall x \in \Delta . h_{\Delta(x)}^\Delta(\iota(x))}{4}}{\frac{h_\Delta^\Delta(\iota)}{5} \quad \Delta \xRightarrow{\mathcal{H}} M \in A} \quad \frac{h_A^\Delta(\iota(M))}{6} \quad h_A^\Delta(M)$$

1 N-VarM 2 h-if-N 3 $\forall$ -I($x \in \Delta$) & fold $\iota(x)$ 4 fold $h_\Delta^\Delta(\iota)$ 5 h-Cut-H 6 reduce □

Note that, here, showing $h_\Delta^\Delta(\iota)$ requires showing that every variable in Δ is computable. This follows simply from [h-if-N] and the definition of neutrality, which states that any variable is neutrally computable in a world that already contains it.

Next, we are able to show a nice characterization of the notion of compositionality of the inductive logical relation, as a corollary of [h-Cut-H]:

[Semantic-Cut]

$$\frac{\Delta \xRightarrow{\mathcal{H}} M \in A \quad \Delta, x : A \xRightarrow{\mathcal{H}} N \in B}{\Delta \xRightarrow{\mathcal{H}} [M/x]N \in B}$$

Proof:

$$\frac{\frac{\frac{\Delta \xRightarrow{\mathcal{H}} M \in A}{1} \quad h_A^\Delta(M)}{2} \quad \frac{h_{\Delta, x:A}^\Delta([M/x]) \quad \Delta, x : A \xRightarrow{\mathcal{H}} N \in B}{3}}{\frac{h_B^\Delta([M/x]N)}{4} \quad \Delta \xRightarrow{\mathcal{H}} [M/x]N \in B}$$

1 h-if-H 2 fold $h_{\Delta, x:A}^\Delta([M/x])$ 3 h-Cut-H 4 H-if-h □

We prove three specialized compositionality results for elimination forms, which are useful for proving the fundamental theorem:

[H-App-Compositionality]

$$\frac{\Delta \xRightarrow{\mathcal{H}} M \in A \rightarrow B \quad \Delta' \leq \Delta \quad \Delta' \xRightarrow{\mathcal{H}} N \in A}{\Delta' \xRightarrow{\mathcal{H}} M N \in B}$$

[H-Case-Compositionality]

$$\frac{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R \quad \Delta, x_L : A_L \xRightarrow{\mathcal{H}} M_L \in B \quad \Delta, x_R : A_R \xRightarrow{\mathcal{H}} M_R \in B}{\Delta \xRightarrow{\mathcal{H}} \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B}$$

[H-NatRec-Compositionality]

$$\frac{\Delta \xRightarrow{\mathcal{H}} M \in \text{nat} \quad \Delta \xRightarrow{\mathcal{H}} M_0 \in A \quad \Delta, x : \text{nat}, y : A \xRightarrow{\mathcal{H}} M_S \in A}{\Delta \xRightarrow{\mathcal{H}} \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \in A}$$

Then, we are prepared to prove the fundamental theorem, which is simpler than a traditional fundamental theorem, since it does not require supplying computable substitutions:

[FTLR]

$$\frac{\Delta \vdash M : A}{\Delta \xRightarrow{\mathcal{H}} M \in A}$$

With [FTLR], we can derive various adequacy theorems that state well-typed closed terms reduce to some canonical forms. [Impossible-Neutrality] is for discharging some impossible cases as a neutral term must contain at least one free variable, by definition.

[Impossible-Neutrality]

$$\frac{\cdot \xRightarrow{\mathcal{N}} M \in A}{\perp}$$

[Step-To-Null-if-H-at-1]

$$\frac{\cdot \xRightarrow{\mathcal{H}} M \in 1}{M \rightarrow_{\beta}^* \langle \rangle}$$

[Step-To-Null-if-WT-at-1]

$$\frac{\cdot \vdash M : 1}{M \rightarrow_{\beta}^* \langle \rangle}$$

[Step-To-Inj-if-H-at-1+1]

$$\frac{\cdot \xRightarrow{\mathcal{H}} M \in 1 + 1}{M \rightarrow_{\beta}^* L \cdot \langle \rangle \vee M \rightarrow_{\beta}^* R \cdot \langle \rangle}$$

[Step-To-Inj-if-WT-at-1+1]

$$\frac{\cdot \vdash M : 1 + 1}{M \rightarrow_{\beta}^* L \cdot \langle \rangle \vee M \rightarrow_{\beta}^* R \cdot \langle \rangle}$$

[Step-To-Succ-if-H-at-NatT]

$$\frac{\cdot \xRightarrow{\mathcal{H}} M \in \text{nat}}{\exists n . M \rightarrow_{\beta}^* \text{succ}^n \text{zero}}$$

[Step-To-Succ-if-WT-at-NatT]

$$\frac{\cdot \vdash M : \text{nat}}{\exists n . M \rightarrow_{\beta}^* \text{succ}^n \text{zero}}$$

The next two sections of this chapter, [Section 4.1 \(Proofs of Lemmas\)](#) and [Section 4.2 \(Proofs of Theorems\)](#), contain the detailed proofs of the lemmas and theorems we discussed above.

4.1 Proofs of Lemmas

H-Expansion

$$\frac{M \rightarrow_{\beta}^* M' \quad \Delta \xRightarrow{\mathcal{H}} M' \in A}{\Delta \xRightarrow{\mathcal{H}} M \in A}$$

Proof by induction on the derivation of $\Delta \xRightarrow{\mathcal{H}} M' \in A$:

CASE
$$\frac{M \rightarrow_{\beta}^* M' \quad \frac{M' \rightarrow_{\beta}^* \langle \rangle}{\Delta \xRightarrow{\mathcal{H}} M' \in 1} \text{ [H-EmptyM]}}{\Delta \xRightarrow{\mathcal{H}} M \in 1}$$

$$\frac{\frac{M \rightarrow_{\beta}^* M' \quad M' \rightarrow_{\beta}^* \langle \rangle}{M \rightarrow_{\beta}^* \langle \rangle} 1}{\Delta \xRightarrow{\mathcal{H}} M \in 1} 2$$

1 Beta-Multi-Trans 2 H-EmptyM

$$\begin{array}{c}
\boxed{\text{CASE}} \quad M \rightarrow_{\beta}^* M' \quad \frac{M' \rightarrow_{\beta}^* M'' \quad \Delta \Rightarrow M'' \in 1}{\Delta \Rightarrow M' \in 1} \text{ [H-Neutral-1T]} \\
\hline
\Delta \Rightarrow M \in 1
\end{array}$$

$$\begin{array}{c}
\frac{M \rightarrow_{\beta}^* M' \quad M' \rightarrow_{\beta}^* M''}{M \rightarrow_{\beta}^* M''} \xrightarrow{1} \quad \Delta \Rightarrow M'' \in 1 \\
\hline
\Delta \Rightarrow M \in 1 \quad 2
\end{array}$$

1 Beta-Multi-Trans 2 H-Neutral-1T

$$\begin{array}{c}
\boxed{\text{CASE}} \quad M \rightarrow_{\beta}^* M' \quad \frac{\Delta \Rightarrow M' \cdot L \in A_L \quad \Delta \Rightarrow M' \cdot R \in A_R}{\Delta \Rightarrow M' \in A_L \times A_R} \text{ [H-TimesT]} \\
\hline
\Delta \Rightarrow M \in A_L \times A_R
\end{array}$$

$$\begin{array}{c}
\frac{M \rightarrow_{\beta}^* M'}{M \cdot L \rightarrow_{\beta}^* M' \cdot L} \xrightarrow{1} \quad \Delta \Rightarrow M' \cdot L \in A_L \quad \vdots \text{ (likewise)} \\
\hline
\Delta \Rightarrow M \cdot L \in A_L \quad \Delta \Rightarrow M \cdot R \in A_R \quad 3 \\
\hline
\Delta \Rightarrow M \in A_L \times A_R
\end{array}$$

1 β^* -under 2 IH 3 H-TimesT

$$\begin{array}{c}
\boxed{\text{CASE}} \quad M \rightarrow_{\beta}^* M' \quad \frac{x \notin \Delta \quad \Delta, x : A \Rightarrow M' x \in B}{\Delta \Rightarrow M' \in A \rightarrow B} \text{ [H-ArrowT]} \\
\hline
\Delta \Rightarrow M \in A \rightarrow B
\end{array}$$

$$\begin{array}{c}
\frac{M \rightarrow_{\beta}^* M'}{M x \rightarrow_{\beta}^* M' x} \xrightarrow{1} \quad \Delta, x : A \Rightarrow M' x \in B \\
\hline
\Delta, x : A \Rightarrow M x \in B \quad 3 \\
\hline
\Delta \Rightarrow M \in A \rightarrow B
\end{array}$$

1 β^* -under 2 IH 3 H-ArrowT

$$\begin{array}{c}
\text{CASE} \quad \frac{M \rightarrow_{\beta}^* M' \quad \frac{M' \rightarrow_{\beta}^* L \cdot M_L \quad \Delta \xRightarrow{\mathcal{H}} M_L \in A_L}{\Delta \xRightarrow{\mathcal{H}} M' \in A_L + A_R} [\text{H-InLM}]}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R} \\
\\
\frac{\frac{M \rightarrow_{\beta}^* M' \quad M' \rightarrow_{\beta}^* L \cdot M_L}{M \rightarrow_{\beta}^* L \cdot M_L} \xrightarrow{1} \quad \Delta \xRightarrow{\mathcal{H}} M_L \in A_L}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R} 2
\end{array}$$

1 Beta-Multi-Trans 2 H-InLM

$$\begin{array}{c}
\text{CASE} \quad \frac{M \rightarrow_{\beta}^* M' \quad \frac{M' \rightarrow_{\beta}^* R \cdot M_R \quad \Delta \xRightarrow{\mathcal{H}} M_R \in A_R}{\Delta \xRightarrow{\mathcal{H}} M' \in A_L + A_R} [\text{H-InRM}]}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R}
\end{array}$$

Entirely analogous to the previous case.

$$\begin{array}{c}
\text{CASE} \quad \frac{M \rightarrow_{\beta}^* M' \quad \frac{M' \rightarrow_{\beta}^* M'' \quad \Delta \xRightarrow{\mathcal{N}} M'' \in A_L + A_R}{\Delta \xRightarrow{\mathcal{H}} M' \in A_L + A_R} [\text{H-Neutral-SumT}]}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R} \\
\\
\frac{\frac{M \rightarrow_{\beta}^* M' \quad M' \rightarrow_{\beta}^* M''}{M \rightarrow_{\beta}^* M''} \xrightarrow{1} \quad \Delta \xRightarrow{\mathcal{N}} M'' \in A_L + A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R} 2
\end{array}$$

1 Beta-Multi-Trans 2 H-Neutral-SumT

$$\begin{array}{c}
\text{CASE} \quad \frac{M \rightarrow_{\beta}^* M' \quad \frac{M' \rightarrow_{\beta}^* \text{zero}}{\Delta \xRightarrow{\mathcal{H}} M' \in \text{nat}} [\text{H-ZeroM}]}{\Delta \xRightarrow{\mathcal{H}} M \in \text{nat}}
\end{array}$$

$$\begin{array}{c}
\frac{M \rightarrow_{\beta}^* M' \quad M' \rightarrow_{\beta}^* \text{zero}}{M \rightarrow_{\beta}^* \text{zero}} \xrightarrow{1} \\
\frac{M \rightarrow_{\beta}^* \text{zero}}{\Delta \xRightarrow{\mathcal{H}} M \in \text{nat}} 2
\end{array}$$

1 Beta-Multi-Trans 2 H-ZeroM

$$\begin{array}{c}
 \boxed{\text{CASE}} \quad M \rightarrow_{\beta}^* M' \quad \frac{M' \rightarrow_{\beta}^* \text{succ } M'' \quad \Delta \xRightarrow{\mathcal{H}} M'' \in \text{nat}}{\Delta \xRightarrow{\mathcal{H}} M' \in \text{nat}} \text{ [H-SuccM]} \\
 \hline
 \Delta \xRightarrow{\mathcal{H}} M \in \text{nat} \\
 \\
 \frac{M \rightarrow_{\beta}^* M' \quad M' \rightarrow_{\beta}^* \text{succ } M''}{M \rightarrow_{\beta}^* \text{succ } M''} \xrightarrow{1} \Delta \xRightarrow{\mathcal{H}} M'' \in \text{nat} \quad 2 \\
 \hline
 \Delta \xRightarrow{\mathcal{H}} M \in \text{nat}
 \end{array}$$

1 Beta-Multi-Trans 2 H-SuccM

$$\begin{array}{c}
 \boxed{\text{CASE}} \quad M \rightarrow_{\beta}^* M' \quad \frac{M' \rightarrow_{\beta}^* M'' \quad \Delta \xRightarrow{\mathcal{N}} M'' \in \text{nat}}{\Delta \xRightarrow{\mathcal{H}} M' \in \text{nat}} \text{ [H-Neutral-NatT]} \\
 \hline
 \Delta \xRightarrow{\mathcal{H}} M \in \text{nat} \\
 \\
 \frac{M \rightarrow_{\beta}^* M' \quad M' \rightarrow_{\beta}^* M''}{M \rightarrow_{\beta}^* M''} \xrightarrow{1} \Delta \xRightarrow{\mathcal{N}} M'' \in \text{nat} \quad 2 \\
 \hline
 \Delta \xRightarrow{\mathcal{H}} M \in \text{nat}
 \end{array}$$

1 Beta-Multi-Trans 2 H-Neutral-NatT

□

h-Expansion

$$\frac{M \rightarrow_{\beta}^* M' \quad h_A^{\Delta}(M')}{h_A^{\Delta}(M)}$$

Proof by induction on the structure of type A :

$$\boxed{\text{CASE}} \quad \frac{M \rightarrow_{\beta}^* M' \quad h_1^{\Delta}(M')}{h_1^{\Delta}(M)}$$

Case on $h_1^{\Delta}(M')$:

$$\boxed{\text{SUBCASE}} \quad M' \rightarrow_{\beta}^* \langle \rangle :$$

$$\frac{\frac{M \rightarrow_{\beta}^* M' \quad M' \rightarrow_{\beta}^* \langle \rangle}{M \rightarrow_{\beta}^* \langle \rangle} 1}{h_1^{\Delta}(M)} 2$$

1 Beta-Multi-Trans 2 fold h

SUBCASE $M' \rightarrow_{\beta}^* M'' \wedge \Delta \stackrel{\mathcal{N}}{\Rightarrow} M'' \in 1 :$

$$\frac{\frac{M \rightarrow_{\beta}^* M' \quad M' \rightarrow_{\beta}^* M''}{M \rightarrow_{\beta}^* M''} 1 \quad \Delta \stackrel{\mathcal{N}}{\Rightarrow} M'' \in 1}{h_1^{\Delta}(M)} 2$$

1 Beta-Multi-Trans 2 fold h

CASE $\frac{M \rightarrow_{\beta}^* M' \quad h_{A_L \times A_R}^{\Delta}(M')}{h_{A_L \times A_R}^{\Delta}(M)}$

$$\frac{\frac{M \rightarrow_{\beta}^* M'}{M \cdot L \rightarrow_{\beta}^* M' \cdot L} 1 \quad \frac{h_{A_L \times A_R}^{\Delta}(M')}{h_{A_L}^{\Delta}(M' \cdot L)} 2}{h_{A_L}^{\Delta}(M \cdot L)} 3 \quad \vdots \text{ (likewise)} \quad h_{A_R}^{\Delta}(M \cdot R) 4$$

$$\frac{h_{A_L}^{\Delta}(M \cdot L) \quad h_{A_R}^{\Delta}(M \cdot R)}{h_{A_L \times A_R}^{\Delta}(M)} 4$$

1 β^* -under 2 unfold h 3 IH 4 fold h

CASE $\frac{M \rightarrow_{\beta}^* M' \quad h_{A_L \rightarrow A_R}^{\Delta}(M')}{h_{A_L \rightarrow A_R}^{\Delta}(M)}$

Fix arbitrary $\Delta' \leq \Delta$ and N such that $h_{A_L}^{\Delta'}(N)$.

$$\frac{\frac{M \rightarrow_{\beta}^* M'}{M N \rightarrow_{\beta}^* M' N} 1 \quad \frac{h_{A_L \rightarrow A_R}^{\Delta}(M') \quad \Delta' \leq \Delta \quad h_{A_L}^{\Delta'}(N)}{h_{A_R}^{\Delta'}(M' N)} 2}{h_{A_R}^{\Delta'}(M N)} 3$$

$$\frac{h_{A_R}^{\Delta'}(M N)}{h_{A_L \rightarrow A_R}^{\Delta}(M)} 4$$

1 β^* -under 2 unfold h at $A_L \rightarrow A_R$ & $\forall / \Rightarrow$ -E 3 IH 4 fold h

$$\boxed{\text{CASE}} \quad \frac{M \rightarrow_{\beta}^* M' \quad h_{A_L+A_R}^{\Delta}(M')}{h_{A_L+A_R}^{\Delta}(M)}$$

Case on $h_{A_L+A_R}^{\Delta}(M')$:

$$\boxed{\text{SUBCASE}} \quad M' \rightarrow_{\beta}^* L \cdot M_L \wedge h_{A_L}^{\Delta}(M_L) :$$

$$\frac{\frac{M \rightarrow_{\beta}^* M' \quad M' \rightarrow_{\beta}^* L \cdot M_L}{M \rightarrow_{\beta}^* L \cdot M_L} \quad 1 \quad h_{A_L}^{\Delta}(M_L)}{h_{A_L+A_R}^{\Delta}(M)} \quad 2$$

1 Beta-Multi-Trans 2 fold h

$$\boxed{\text{SUBCASE}} \quad M' \rightarrow_{\beta}^* R \cdot M_R \wedge h_{A_R}^{\Delta}(M_R) :$$

Entirely analogous to the previous subcase.

$$\boxed{\text{SUBCASE}} \quad M' \rightarrow_{\beta}^* M'' \wedge \Delta \xRightarrow{\mathcal{N}} M'' \in A_L + A_R :$$

$$\frac{\frac{M \rightarrow_{\beta}^* M' \quad M' \rightarrow_{\beta}^* M''}{M \rightarrow_{\beta}^* M''} \quad 1 \quad \Delta \xRightarrow{\mathcal{N}} M'' \in A_L + A_R}{h_{A_L+A_R}^{\Delta}(M)} \quad 2$$

1 Beta-Multi-Trans 2 fold h

$$\boxed{\text{CASE}} \quad \frac{M \rightarrow_{\beta}^* M' \quad h_{\text{nat}}^{\Delta}(M')}{h_{\text{nat}}^{\Delta}(M)}$$

$$\frac{\frac{M \rightarrow_{\beta}^* M' \quad \frac{h_{\text{nat}}^{\Delta}(M')}{\Delta \xRightarrow{\mathcal{H}} M' \in \text{nat}} \quad 1}{\Delta \xRightarrow{\mathcal{H}} M \in \text{nat}} \quad 2}{h_{\text{nat}}^{\Delta}(M)} \quad 3$$

1 unfold h 2 H-Expansion 3 fold h

□

H-Rename	N-Rename
$\frac{\Gamma \vdash \rho : \Delta \quad \Delta \xRightarrow{\mathcal{H}} M \in A}{\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in A}$	$\frac{\Gamma \vdash \rho : \Delta \quad \Delta \xRightarrow{\mathcal{N}} M \in A}{\Gamma \xRightarrow{\mathcal{N}} \hat{\rho}(M) \in A}$

Proof by simultaneous induction on $\Delta \xRightarrow{\mathcal{H}} M \in A$ and $\Delta \xRightarrow{\mathcal{N}} M \in A$:

CASE $\Gamma \vdash \rho : \Delta \quad \frac{M \rightarrow_{\beta}^* \text{zero}}{\Delta \xRightarrow{\mathcal{H}} M \in \text{nat}} \text{ [H-ZeroM]}$

$\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in \text{nat}$

$$\frac{\frac{\frac{M \rightarrow_{\beta}^* \text{zero}}{\hat{\rho}(M) \rightarrow_{\beta}^* \hat{\rho}(\text{zero})} \xrightarrow{1}}{\hat{\rho}(M) \rightarrow_{\beta}^* \text{zero}} \xrightarrow{2}}{\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in \text{nat}} \xrightarrow{3}$$

1 Hat-Re-Preserves-Beta-Multi 2 reduce 3 H-ZeroM

CASE $\Gamma \vdash \rho : \Delta \quad \frac{M \rightarrow_{\beta}^* \text{succ } M' \quad \Delta \xRightarrow{\mathcal{H}} M' \in \text{nat}}{\Delta \xRightarrow{\mathcal{H}} M \in \text{nat}} \text{ [H-SuccM]}$

$\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in \text{nat}$

$$\frac{\frac{\frac{M \rightarrow_{\beta}^* \text{succ } M'}{\hat{\rho}(M) \rightarrow_{\beta}^* \hat{\rho}(\text{succ } M')} \xrightarrow{1}}{\hat{\rho}(M) \rightarrow_{\beta}^* \text{succ } \hat{\rho}(M')} \xrightarrow{2} \quad \frac{\Gamma \vdash \rho : \Delta \quad \Delta \xRightarrow{\mathcal{H}} M' \in \text{nat}}{\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M') \in \text{nat}} \xrightarrow{3}}{\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in \text{nat}} \xrightarrow{4}$$

1 Hat-Re-Preserves-Beta-Multi 2 reduce 3 IH 4 H-SuccM

CASE $\Gamma \vdash \rho : \Delta \quad \frac{M \rightarrow_{\beta}^* \langle \rangle}{\Delta \xRightarrow{\mathcal{H}} M \in 1} \text{ [H-EmptyM]}$

$\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in 1$

$$\begin{array}{c}
M \rightarrow_{\beta}^* \langle \rangle \\
\hline
\hat{\rho}(M) \rightarrow_{\beta}^* \hat{\rho}(\langle \rangle) \quad 1 \\
\hline
\hat{\rho}(M) \rightarrow_{\beta}^* \langle \rangle \quad 2 \\
\hline
\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in 1 \quad 3
\end{array}$$

1 Hat-Re-Preserves-Beta-Multi 2 reduce 3 H-EmptyM

$$\boxed{\text{CASE}} \quad \frac{\Gamma \vdash \rho : \Delta \quad \frac{M \rightarrow_{\beta}^* M' \quad \Delta \xRightarrow{\mathcal{N}} M' \in 1}{\Delta \xRightarrow{\mathcal{H}} M \in 1} [\text{H-Neutral-1T}]}{\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in 1}$$

$$\begin{array}{c}
M \rightarrow_{\beta}^* M' \quad \Gamma \vdash \rho : \Delta \quad \Delta \xRightarrow{\mathcal{N}} M' \in 1 \\
\hline
\hat{\rho}(M) \rightarrow_{\beta}^* \hat{\rho}(M') \quad \Gamma \xRightarrow{\mathcal{N}} \hat{\rho}(M') \in 1 \quad 2 \\
\hline
\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in 1 \quad 3
\end{array}$$

1 Hat-Re-Preserves-Beta-Multi 2 IH 3 H-Neutral-1T

$$\boxed{\text{CASE}} \quad \frac{\Gamma \vdash \rho : \Delta \quad \frac{\Delta \xRightarrow{\mathcal{H}} M \cdot L \in A_L \quad \Delta \xRightarrow{\mathcal{H}} M \cdot R \in A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L \times A_R} [\text{H-TimesT}]}{\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in A_L \times A_R}$$

$$\begin{array}{c}
\Gamma \vdash \rho : \Delta \quad \Delta \xRightarrow{\mathcal{H}} M \cdot L \in A_L \quad 1 \\
\hline
\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M \cdot L) \in A_L \quad 2 \quad \vdots \text{ (likewise)} \\
\hline
\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \cdot L \in A_L \quad \Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \cdot R \in A_R \quad 3 \\
\hline
\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in A_L \times A_R
\end{array}$$

1 IH 2 reduce 3 H-TimesT

$$\boxed{\text{CASE}} \quad \frac{\Gamma \vdash \rho : \Delta \quad \frac{x \notin \Delta \quad \Delta, x : A \xRightarrow{\mathcal{H}} M \quad x \in B}{\Delta \xRightarrow{\mathcal{H}} M \in A \rightarrow B} [\text{H-ArrowT}]}{\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in A \rightarrow B}$$

Pick a fresh $y \notin \Gamma$. Then:

$$\begin{array}{c}
\frac{y \notin \Gamma \quad \Gamma \vdash \rho : \Delta \quad x \notin \Delta}{\Gamma, y : A \vdash \rho, x \mapsto y : \Delta, x : A} \xrightarrow{1} \quad \Delta, x : A \xRightarrow{\mathcal{H}} M \ x \in B \\
\hline
\frac{\Gamma, y : A \xRightarrow{\mathcal{H}} (\rho, \widehat{x \mapsto y})(M \ x) \in B}{\Gamma, y : A \xRightarrow{\mathcal{H}} \hat{\rho}(M) \ y \in B} \xrightarrow{2} \quad 3 \\
\hline
\frac{y \notin \Gamma \quad \Gamma, y : A \xRightarrow{\mathcal{H}} \hat{\rho}(M) \ y \in B}{\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in A \rightarrow B} \xrightarrow{4}
\end{array}$$

1 definition of renaming well-formedness 2 IH 3 reduce 4 H-ArrowT

$$\begin{array}{c}
\text{CASE} \quad \Gamma \vdash \rho : \Delta \quad \frac{M \rightarrow_{\beta}^* L \cdot M_L \quad \Delta \xRightarrow{\mathcal{H}} M_L \in A_L}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R} \text{ [H-InLM]} \\
\hline
\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in A_L + A_R \\
\\
\frac{\frac{\frac{M \rightarrow_{\beta}^* L \cdot M_L}{\hat{\rho}(M) \rightarrow_{\beta}^* \hat{\rho}(L \cdot M_L)} \xrightarrow{1} \quad \Gamma \vdash \rho : \Delta \quad \Delta \xRightarrow{\mathcal{H}} M_L \in A_L}{\hat{\rho}(M) \rightarrow_{\beta}^* L \cdot \hat{\rho}(M_L)} \xrightarrow{2} \quad \Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M_L) \in A_L}{\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in A_L + A_R} \xrightarrow{3} \quad 4
\end{array}$$

1 Hat-Re-Preserves-Beta-Multi 2 reduce 3 IH 4 H-InLM

$$\begin{array}{c}
\text{CASE} \quad \Gamma \vdash \rho : \Delta \quad \frac{M \rightarrow_{\beta}^* R \cdot M_R \quad \Delta \xRightarrow{\mathcal{H}} M_R \in A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R} \text{ [H-InRM]} \\
\hline
\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in A_L + A_R
\end{array}$$

Entirely analogous to the previous case.

$$\begin{array}{c}
\text{CASE} \quad \Gamma \vdash \rho : \Delta \quad \frac{M \rightarrow_{\beta}^* M' \quad \Delta \xRightarrow{\mathcal{N}} M' \in A_L + A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R} \text{ [H-Neutral-SumT]} \\
\hline
\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in A_L + A_R \\
\\
\frac{\frac{\frac{M \rightarrow_{\beta}^* M'}{\hat{\rho}(M) \rightarrow_{\beta}^* \hat{\rho}(M')} \xrightarrow{1} \quad \Gamma \vdash \rho : \Delta \quad \Delta \xRightarrow{\mathcal{N}} M' \in A_L + A_R}{\hat{\rho}(M) \rightarrow_{\beta}^* \hat{\rho}(M')} \xrightarrow{2} \quad \Gamma \xRightarrow{\mathcal{N}} \hat{\rho}(M') \in A_L + A_R}{\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M) \in A_L + A_R} \xrightarrow{3}
\end{array}$$

1 Hat-Re-Preserves-Beta-Multi 2 IH 3 H-Neutral-SumT

$$\begin{array}{c}
 \text{CASE} \quad \frac{\Gamma \vdash \rho : \Delta \quad \frac{M \rightarrow_{\beta}^* M' \quad \Delta \Rightarrow^{\mathcal{N}} M' \in \text{nat}}{\Delta \Rightarrow^{\mathcal{H}} M \in \text{nat}} \text{ [H-Neutral-NatT]}}{\Gamma \Rightarrow^{\mathcal{H}} \hat{\rho}(M) \in \text{nat}}
 \end{array}$$

$$\frac{\frac{M \rightarrow_{\beta}^* M'}{\hat{\rho}(M) \rightarrow_{\beta}^* \hat{\rho}(M')} \xrightarrow{1} \quad \frac{\Gamma \vdash \rho : \Delta \quad \Delta \Rightarrow^{\mathcal{N}} M' \in \text{nat}}{\Gamma \Rightarrow^{\mathcal{N}} \hat{\rho}(M') \in \text{nat}} \xrightarrow{2}}{\Gamma \Rightarrow^{\mathcal{H}} \hat{\rho}(M) \in \text{nat}} \quad 3$$

1 Hat-Re-Preserves-Beta-Multi 2 IH 3 H-Neutral-NatT

$$\begin{array}{c}
 \text{CASE} \quad \frac{\Gamma \vdash \rho : \Delta, x : A \quad \frac{}{\Delta, x : A \Rightarrow^{\mathcal{N}} x \in A} \text{ [N-VarM]}}{\Gamma \Rightarrow^{\mathcal{N}} \hat{\rho}(x) \in A}
 \end{array}$$

$$\frac{\frac{\Gamma \vdash \rho : \Delta, x : A \quad \frac{}{\Delta, x : A \vdash x : A} \xrightarrow{1}}{\Gamma \vdash \hat{\rho}(x) : A} \xrightarrow{2}}{\Gamma \Rightarrow^{\mathcal{N}} \hat{\rho}(x) \in A} \quad 3$$

1 Var-WT 2 unfold $\Gamma \vdash \rho : \Delta, x : A$ and $\forall / \Rightarrow$ -E 3 inversion and N-VarM

$$\begin{array}{c}
 \text{CASE} \quad \frac{\Gamma \vdash \rho : \Delta \quad \frac{\Delta \Rightarrow^{\mathcal{N}} M \in A_L \times A_R}{\Delta \Rightarrow^{\mathcal{N}} M \cdot L \in A_L} \text{ [N-projLM]}}{\Gamma \Rightarrow^{\mathcal{N}} \hat{\rho}(M \cdot L) \in A_L}
 \end{array}$$

$$\frac{\frac{\Gamma \vdash \rho : \Delta \quad \Delta \Rightarrow^{\mathcal{N}} M \in A_L \times A_R}{\Gamma \Rightarrow^{\mathcal{N}} \hat{\rho}(M) \in A_L \times A_R} \xrightarrow{1} \quad \frac{\Gamma \Rightarrow^{\mathcal{N}} \hat{\rho}(M) \in A_L \times A_R}{\Gamma \Rightarrow^{\mathcal{N}} \hat{\rho}(M) \cdot L \in A_L} \xrightarrow{2}}{\Gamma \Rightarrow^{\mathcal{N}} \hat{\rho}(M \cdot L) \in A_L} \quad 3$$

1 IH 2 N-projLM 3 fold $\hat{\rho}(M \cdot L)$

$$\begin{array}{c}
\text{CASE} \quad \frac{\Gamma \vdash \rho : \Delta \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R}{\Delta \xRightarrow{\mathcal{N}} M \cdot R \in A_R} [\text{N-projRM}]}{\Gamma \xRightarrow{\mathcal{N}} \hat{\rho}(M \cdot R) \in A_R}
\end{array}$$

Entirely analogous to the previous case.

$$\begin{array}{c}
\text{CASE} \quad \frac{\Gamma \vdash \rho : \Delta \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A \rightarrow B \quad \Delta \xRightarrow{\mathcal{H}} N \in A}{\Delta \xRightarrow{\mathcal{N}} M N \in B} [\text{N-AppM}]}{\Gamma \xRightarrow{\mathcal{N}} \hat{\rho}(M N) \in B}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma \vdash \rho : \Delta \quad \Delta \xRightarrow{\mathcal{N}} M \in A \rightarrow B}{\Gamma \xRightarrow{\mathcal{N}} \hat{\rho}(M) \in A \rightarrow B} \quad \frac{\Gamma \vdash \rho : \Delta \quad \Delta \xRightarrow{\mathcal{H}} N \in A}{\Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(N) \in A} \quad \frac{\Gamma \xRightarrow{\mathcal{N}} \hat{\rho}(M) \in A \rightarrow B \quad \Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(N) \in A}{\Gamma \xRightarrow{\mathcal{N}} \hat{\rho}(M) \hat{\rho}(N) \in B} \quad \frac{\Gamma \xRightarrow{\mathcal{N}} \hat{\rho}(M) \hat{\rho}(N) \in B}{\Gamma \xRightarrow{\mathcal{N}} \hat{\rho}(M N) \in B}
\end{array}$$

1 IH 2 IH 3 N-AppM 4 fold $\hat{\rho}(M N)$

$$\begin{array}{c}
\text{CASE} \quad \frac{\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L + A_R \quad \Delta, x_L : A_L \xRightarrow{\mathcal{H}} M_L \in B \quad \Delta, x_R : A_R \xRightarrow{\mathcal{H}} M_R \in B}{\Delta \xRightarrow{\mathcal{N}} \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B} [\text{N-CaseM}]}{\Gamma \vdash \rho : \Delta \quad \Delta \xRightarrow{\mathcal{N}} \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B} \\
\hline
\Gamma \xRightarrow{\mathcal{N}} \hat{\rho}(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \in B
\end{array}$$

Pick fresh $y_L, y_R \notin \Gamma$.

$$\frac{\Gamma \vdash \rho : \Delta \quad \Delta \xRightarrow{\mathcal{N}} M \in A_L + A_R}{\Gamma \xRightarrow{\mathcal{N}} \hat{\rho}(M) \in A_L + A_R} \quad 1$$

1 IH

$$\begin{array}{c}
\frac{y_L \notin \Gamma \quad \Gamma \vdash \rho : \Delta \quad x_L \notin \Delta}{\Gamma, y_L : A_L \vdash \rho, x_L \mapsto y_L : \Delta, x_L : A_L} \quad \frac{\Delta, x_L : A_L \xRightarrow{\mathcal{H}} M_L \in B}{\Gamma, y_L : A_L \xRightarrow{\mathcal{H}} (\rho, x_L \widehat{\mapsto} y_L)(M_L) \in B} \quad 1 \\
\hline
\Gamma, y_L : A_L \xRightarrow{\mathcal{H}} (\rho, x_L \widehat{\mapsto} y_L)(M_L) \in B \quad 2
\end{array}$$

1 definition of renaming well-formedness 2 IH

$$\begin{array}{c}
\frac{y_R \notin \Gamma \quad \Gamma \vdash \rho : \Delta \quad x_R \notin \Delta}{\Gamma, y_R : A_R \vdash \rho, x_R \mapsto y_R : \Delta, x_R : A_R} \textcolor{blue}{1} \\
\frac{\Delta, x_R : A_R \xRightarrow{\textcolor{blue}{H}} M_R \in B}{\Gamma, y_R : A_R \xRightarrow{\textcolor{blue}{H}} (\rho, x_R \widehat{\mapsto} y_R)(M_R) \in B} \textcolor{blue}{2}
\end{array}$$

$\textcolor{blue}{1}$ definition of renaming well-formedness $\textcolor{blue}{2}$ IH

Combining these three derivations:

$$\begin{array}{c}
\Gamma \xRightarrow{\textcolor{red}{N}} \hat{\rho}(M) \in A_L + A_R \\
\Gamma, y_L : A_L \xRightarrow{\textcolor{blue}{H}} (\rho, x_L \widehat{\mapsto} y_L)(M_L) \in B \\
\Gamma, y_R : A_R \xRightarrow{\textcolor{blue}{H}} (\rho, x_R \widehat{\mapsto} y_R)(M_R) \in B \\
\hline
\Gamma \xRightarrow{\textcolor{red}{N}} \text{case } \hat{\rho}(M) (y_L \Rightarrow (\rho, x_L \widehat{\mapsto} y_L)(M_L) \mid y_R \Rightarrow (\rho, x_R \widehat{\mapsto} y_R)(M_R)) \in B \textcolor{blue}{1} \\
\hline
\Gamma \xRightarrow{\textcolor{red}{N}} \hat{\rho}(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \in B \textcolor{blue}{2}
\end{array}$$

$\textcolor{blue}{1}$ N-CaseM $\textcolor{blue}{2}$ fold $\hat{\rho}(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R))$

$$\begin{array}{c}
\Delta \xRightarrow{\textcolor{red}{N}} M \in \text{nat} \\
\Delta \xRightarrow{\textcolor{blue}{H}} M_0 \in A \\
\Delta, x : \text{nat}, y : A \xRightarrow{\textcolor{blue}{H}} M_S \in A \\
\hline
\Gamma \vdash \rho : \Delta \quad \Delta \xRightarrow{\textcolor{red}{N}} \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \in A \quad [\text{N-NatRecM}] \\
\hline
\Gamma \xRightarrow{\textcolor{red}{N}} \hat{\rho}(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S)) \in A
\end{array}$$

Pick fresh $x' \notin \Gamma$ and $y' \notin \Gamma, x' : \text{nat}$.

$$\begin{array}{c}
\Gamma \vdash \rho : \Delta \quad \Delta \xRightarrow{\textcolor{red}{N}} M \in \text{nat} \\
\hline
\Gamma \xRightarrow{\textcolor{red}{N}} \hat{\rho}(M) \in \text{nat} \textcolor{blue}{1}
\end{array}$$

$\textcolor{blue}{1}$ IH

$$\begin{array}{c}
\Gamma \vdash \rho : \Delta \quad \Delta \xRightarrow{\textcolor{blue}{H}} M_0 \in A \\
\hline
\Gamma \xRightarrow{\textcolor{blue}{H}} \hat{\rho}(M_0) \in A \textcolor{blue}{1}
\end{array}$$

$\textcolor{blue}{1}$ IH

$$\frac{\frac{x' \notin \Gamma \quad \Gamma \vdash \rho : \Delta \quad x \notin \Delta}{\Gamma, x' : \text{nat} \vdash \rho, x \mapsto x' : \Delta, x : \text{nat}} \xrightarrow{1} \quad y \notin \Delta, x : \text{nat}}{\Gamma, x' : \text{nat}, y' : A \vdash \rho, x \mapsto x', y \mapsto y' : \Delta, x : \text{nat}, y : A} \xrightarrow{2}$$

1 definition of renaming well-formedness 2 definition of renaming well-formedness

$$\frac{\Gamma, x' : \text{nat}, y' : A \vdash \rho, x \mapsto x', y \mapsto y' : \Delta, x : \text{nat}, y : A \quad \Delta, x : \text{nat}, y : A \xRightarrow{\mathcal{H}} M_S \in A}{\Gamma, x' : \text{nat}, y' : A \xRightarrow{\mathcal{H}} (\rho, x \mapsto \widehat{x'}, y \mapsto y')(M_S) \in A} \xrightarrow{1}$$

1 IH

Combining these three derivations:

$$\frac{\frac{\Gamma \xRightarrow{\mathcal{N}} \hat{\rho}(M) \in \text{nat} \quad \Gamma \xRightarrow{\mathcal{H}} \hat{\rho}(M_0) \in A \quad \Gamma, x' : \text{nat}, y' : A \xRightarrow{\mathcal{H}} (\rho, x \mapsto \widehat{x'}, y \mapsto y')(M_S) \in A}{\Gamma \xRightarrow{\mathcal{N}} \text{natrec } \hat{\rho}(M) (\hat{\rho}(M_0) \mid x', y' \Rightarrow (\rho, x \mapsto \widehat{x'}, y \mapsto y')(M_S)) \in A} \xrightarrow{1}}{\Gamma \xRightarrow{\mathcal{N}} \hat{\rho}(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S)) \in A} \xrightarrow{2}$$

1 N-NatRecM 2 fold $\hat{\rho}(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S))$

□

H-Anti-Mono	N-Anti-Mono
$\frac{\Delta' \leq \Delta \quad \Delta \xRightarrow{\mathcal{H}} M \in A}{\Delta' \xRightarrow{\mathcal{H}} M \in A}$	$\frac{\Delta' \leq \Delta \quad \Delta \xRightarrow{\mathcal{N}} M \in A}{\Delta' \xRightarrow{\mathcal{N}} M \in A}$

Proof by appealing to [H-Rename] and [N-Rename] with the identity renaming $\rho = \text{id}_\rho$.

□

h-Anti-Mono
$\frac{\Delta' \leq \Delta \quad h_A^\Delta(M)}{h_A^{\Delta'}(M)}$

Proof by induction on the structure of type A :

CASE	$\frac{\Delta' \leq \Delta \quad h_1^\Delta(M)}{h_1^{\Delta'}(M)}$
------	--

Case on $h_1^\Delta(M)$:

SUBCASE $M \rightarrow_\beta^* \langle \rangle$:

$$\frac{M \rightarrow_\beta^* \langle \rangle}{h_1^{\Delta'}(M)} \quad 1$$

1 fold h

SUBCASE $M \rightarrow_\beta^* M' \wedge \Delta \xRightarrow{\mathcal{N}} M' \in 1$:

$$\frac{M \rightarrow_\beta^* M' \quad \frac{\Delta' \leq \Delta \quad \Delta \xRightarrow{\mathcal{N}} M' \in 1}{\Delta' \xRightarrow{\mathcal{N}} M' \in 1} \quad 1}{h_1^{\Delta'}(M)} \quad 2$$

1 N-Anti-Mono 2 fold h

CASE $\frac{\Delta' \leq \Delta \quad h_{A_L \times A_R}^\Delta(M)}{h_{A_L \times A_R}^{\Delta'}(M)}$

$$\frac{\Delta' \leq \Delta \quad \frac{h_{A_L \times A_R}^\Delta(M)}{h_{A_L}^\Delta(M \cdot L)} \quad 1 \quad \vdots \text{ (likewise)}}{h_{A_L}^{\Delta'}(M \cdot L) \quad h_{A_R}^{\Delta'}(M \cdot R)} \quad 2 \quad 3$$

$$\frac{\quad}{h_{A_L \times A_R}^{\Delta'}(M)} \quad 3$$

1 unfold h 2 IH 3 fold h

CASE $\frac{\Delta' \leq \Delta \quad h_{A_L \rightarrow A_R}^\Delta(M)}{h_{A_L \rightarrow A_R}^{\Delta'}(M)}$

Fix arbitrary $\Xi \leq \Delta'$ and N such that $h_{A_L}^\Xi(N)$.

$$\frac{h_{A_L \rightarrow A_R}^\Delta(M) \quad \frac{\Xi \leq \Delta' \quad \Delta' \leq \Delta}{\Xi \leq \Delta} \quad 1 \quad h_{A_L}^\Xi(N)}{h_{A_R}^\Xi(M N)} \quad 2$$

$$\frac{\quad}{h_{A_L \rightarrow A_R}^{\Delta'}(M)} \quad 3$$

1 $\leq$ -Transitivity 2 unfold h at $A_L \rightarrow A_R$ & $\forall / \Rightarrow$ -E 3 fold h

$$\boxed{\text{CASE}} \quad \frac{\Delta' \leq \Delta \quad h_{A_L+A_R}^\Delta(M)}{h_{A_L+A_R}^{\Delta'}(M)}$$

Case on $h_{A_L+A_R}^\Delta(M)$:

$$\boxed{\text{SUBCASE}} \quad M \rightarrow_\beta^* L \cdot M_L \wedge h_{A_L}^\Delta(M_L) :$$

$$\frac{M \rightarrow_\beta^* L \cdot M_L \quad \frac{\Delta' \leq \Delta \quad h_{A_L}^\Delta(M_L)}{h_{A_L}^{\Delta'}(M_L)} \xrightarrow{1}}{h_{A_L+A_R}^{\Delta'}(M)} \quad 2$$

1 IH 2 fold h

$$\boxed{\text{SUBCASE}} \quad M \rightarrow_\beta^* R \cdot M_R \wedge h_{A_R}^\Delta(M_R) :$$

Entirely analogous to the previous subcase.

$$\boxed{\text{SUBCASE}} \quad M \rightarrow_\beta^* M' \wedge \Delta \xRightarrow{\mathcal{N}} M' \in A_L + A_R :$$

$$\frac{M \rightarrow_\beta^* M' \quad \frac{\Delta' \leq \Delta \quad \Delta \xRightarrow{\mathcal{N}} M' \in A_L + A_R}{\Delta' \xRightarrow{\mathcal{N}} M' \in A_L + A_R} \xrightarrow{1}}{h_{A_L+A_R}^{\Delta'}(M)} \quad 2$$

1 N-Anti-Mono 2 fold h

$$\boxed{\text{CASE}} \quad \frac{\Delta' \leq \Delta \quad h_{\text{nat}}^\Delta(M)}{h_{\text{nat}}^{\Delta'}(M)}$$

$$\frac{\frac{\Delta' \leq \Delta \quad \Delta \xRightarrow{\mathcal{H}} M \in \text{nat}}{\Delta' \xRightarrow{\mathcal{H}} M \in \text{nat}} \xrightarrow{2} \quad \frac{h_{\text{nat}}^\Delta(M)}{\Delta' \xRightarrow{\mathcal{H}} M \in \text{nat}} \xrightarrow{1}}{h_{\text{nat}}^{\Delta'}(M)} \quad 3$$

1 unfold h 2 H-Anti-Mono 3 fold h

□

H-if-N

$$\frac{\Delta \xRightarrow{\mathcal{N}} M \in A}{\Delta \xRightarrow{\mathcal{H}} M \in A}$$

Proof by induction on the structure of type A :

CASE $\frac{\Delta \xRightarrow{\mathcal{N}} M \in 1}{\Delta \xRightarrow{\mathcal{H}} M \in 1}$

$$\frac{\overline{M \rightarrow_{\beta}^* M} \quad \Delta \xRightarrow{\mathcal{N}} M \in 1}{\Delta \xRightarrow{\mathcal{H}} M \in 1} \quad \begin{matrix} 1 \\ 2 \end{matrix}$$

1 Beta-Multi-Ref 2 H-Neutral-1T

CASE $\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L \times A_R}$

$$\frac{\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R}{\Delta \xRightarrow{\mathcal{N}} M \cdot L \in A_L} \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R}{\Delta \xRightarrow{\mathcal{N}} M \cdot R \in A_R} \quad \begin{matrix} 1 \\ 3 \end{matrix}}{\frac{\Delta \xRightarrow{\mathcal{H}} M \cdot L \in A_L}{\Delta \xRightarrow{\mathcal{H}} M \cdot R \in A_R} \quad \begin{matrix} 2 \\ 4 \end{matrix}} \quad \begin{matrix} 5 \\ \Delta \xRightarrow{\mathcal{H}} M \in A_L \times A_R \end{matrix}$$

1 N-projLM 2 IH 3 N-projRM 4 IH 5 H-TimesT

CASE $\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \rightarrow A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L \rightarrow A_R}$

Pick a fresh $x \notin \Delta$.

$$\begin{array}{c}
\frac{\frac{\frac{}{\Delta, x : A_L \leq \Delta} \text{1}}{\Delta, x : A_L \Rightarrow M \in A_L \rightarrow A_R} \text{2} \quad \frac{\frac{}{\Delta, x : A_L \Rightarrow x \in A_L} \text{3}}{\Delta, x : A_L \Rightarrow x \in A_L} \text{4}}{\Delta, x : A_L \Rightarrow M \in A_L \rightarrow A_R} \text{5} \\
\frac{\frac{\frac{}{\Delta, x : A_L \Rightarrow M x \in A_R} \text{6}}{\Delta, x : A_L \Rightarrow M x \in A_R} \text{7}}{x \notin \Delta} \text{7} \\
\Delta \Rightarrow M \in A_L \rightarrow A_R
\end{array}$$

1 definition of $\leq$ 2 N-Anti-Mono 3 N-VarM 4 IH 5 N-AppM 6 IH 7 H-ArrowT

CASE $\frac{\Delta \Rightarrow M \in A_L + A_R}{\Delta \Rightarrow M \in A_L + A_R}$

$$\frac{\frac{}{M \rightarrow_{\beta}^* M} \text{1} \quad \Delta \Rightarrow M \in A_L + A_R \text{2}}{\Delta \Rightarrow M \in A_L + A_R} \text{2}$$

1 Beta-Multi-Refl 2 H-Neutral-SumT

CASE $\frac{\Delta \Rightarrow M \in \text{nat}}{\Delta \Rightarrow M \in \text{nat}}$

$$\frac{\frac{}{M \rightarrow_{\beta}^* M} \text{1} \quad \Delta \Rightarrow M \in \text{nat} \text{2}}{\Delta \Rightarrow M \in \text{nat}} \text{2}$$

1 Beta-Multi-Refl 2 H-Neutral-NatT

□

$ \begin{array}{ccc} & \text{h-if-N} & \text{H-if-h} \\ & \frac{\Delta \Rightarrow M \in A}{h_A^{\Delta}(M)} & \frac{h_A^{\Delta}(M)}{\Delta \Rightarrow M \in A} \end{array} $

Proof of the two lemmas as a conjunction by induction on the structure of type A :

CASE $\frac{\Delta \Rightarrow M \in 1}{h_1^{\Delta}(M)}$

$$\frac{\overline{M \rightarrow_{\beta}^* M} \quad \Delta \xRightarrow{\mathcal{N}} M \in 1}{h_1^{\Delta}(M)} \quad \begin{matrix} 1 \\ 2 \end{matrix}$$

1 Beta-Multi-Refl 2 fold h

$$\boxed{\text{CASE}} \quad \frac{h_1^{\Delta}(M)}{\Delta \xRightarrow{\mathcal{H}} M \in 1}$$

Case on $h_1^{\Delta}(M)$:

$$\boxed{\text{SUBCASE}} \quad M \rightarrow_{\beta}^* \langle \rangle :$$

$$\frac{M \rightarrow_{\beta}^* \langle \rangle}{\Delta \xRightarrow{\mathcal{H}} M \in 1} \quad 1$$

1 H-EmptyM

$$\boxed{\text{SUBCASE}} \quad M \rightarrow_{\beta}^* M' \wedge \Delta \xRightarrow{\mathcal{N}} M' \in 1 :$$

$$\frac{M \rightarrow_{\beta}^* M' \quad \Delta \xRightarrow{\mathcal{N}} M' \in 1}{\Delta \xRightarrow{\mathcal{H}} M \in 1} \quad 1$$

1 H-Neutral-1T

$$\boxed{\text{CASE}} \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R}{h_{A_L \times A_R}^{\Delta}(M)}$$

$$\frac{\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R}{\Delta \xRightarrow{\mathcal{N}} M \cdot L \in A_L} \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R}{\Delta \xRightarrow{\mathcal{N}} M \cdot R \in A_R}}{\frac{h_{A_L}^{\Delta}(M \cdot L) \quad h_{A_R}^{\Delta}(M \cdot R)}{h_{A_L \times A_R}^{\Delta}(M)}} \quad \begin{matrix} 1 & 3 \\ 2 & 4 \\ 5 \end{matrix}$$

1 N-projLM 2 IH 3 N-projRM 4 IH 5 fold h

$$\boxed{\text{CASE}} \quad \frac{h_{A_L \times A_R}^{\Delta}(M)}{\Delta \xRightarrow{\mathcal{H}} M \in A_L \times A_R}$$

$$\begin{array}{c}
\frac{h_{A_L \times A_R}^\Delta(M)}{h_{A_L}^\Delta(M \cdot L)} \quad 1 \qquad \frac{h_{A_L \times A_R}^\Delta(M)}{h_{A_R}^\Delta(M \cdot R)} \quad 3 \\
\hline
\Delta \xRightarrow{h} M \cdot L \in A_L \quad \Delta \xRightarrow{h} M \cdot R \in A_R \quad 2 \qquad 4 \\
\hline
\Delta \xRightarrow{h} M \in A_L \times A_R \quad 5
\end{array}$$

1 unfold h 2 IH 3 unfold h 4 IH 5 H-TimesT

CASE $\Delta \xRightarrow{\mathcal{N}} M \in A_L \rightarrow A_R$
 $\frac{h_{A_L \rightarrow A_R}^\Delta(M)}{\quad}$

Fix arbitrary $\Delta' \leq \Delta$ and N such that $h_{A_L}^{\Delta'}(N)$.

$$\begin{array}{c}
\frac{\Delta' \leq \Delta \quad \Delta \xRightarrow{\mathcal{N}} M \in A_L \rightarrow A_R}{\Delta' \xRightarrow{\mathcal{N}} M \in A_L \rightarrow A_R} \quad 1 \qquad \frac{h_{A_L}^{\Delta'}(N)}{\Delta' \xRightarrow{h} N \in A_L} \quad 2 \\
\hline
\Delta' \xRightarrow{\mathcal{N}} M N \in A_R \quad 3 \\
\hline
\Delta' \xRightarrow{h} M N \in A_R \quad 4 \\
\hline
\frac{h_{A_R}^{\Delta'}(M N)}{h_{A_L \rightarrow A_R}^\Delta(M)} \quad 5
\end{array}$$

1 N-Anti-Mono 2 IH 3 N-AppM 4 IH 5 fold h

CASE $\frac{h_{A_L \rightarrow A_R}^\Delta(M)}{\Delta \xRightarrow{h} M \in A_L \rightarrow A_R}$

Pick a fresh $x \notin \Delta$.

$$\begin{array}{c}
\frac{h_{A_L \rightarrow A_R}^\Delta(M) \quad \frac{\Delta, x : A_L \leq \Delta}{\Delta, x : A_L \xRightarrow{\mathcal{N}} x \in A_L} \quad 2}{\Delta, x : A_L \leq \Delta} \quad 1 \qquad \frac{\Delta, x : A_L \xRightarrow{\mathcal{N}} x \in A_L}{h_{A_L}^{\Delta, x : A_L}(x)} \quad 3 \\
\hline
\Delta, x : A_L \xRightarrow{h} M x \in A_R \quad 4 \\
\hline
\frac{h_{A_R}^{\Delta, x : A_L}(M x)}{\Delta, x : A_L \xRightarrow{h} M x \in A_R} \quad 5 \\
\hline
\frac{x \notin \Delta \quad \Delta, x : A_L \xRightarrow{h} M x \in A_R}{\Delta \xRightarrow{h} M \in A_L \rightarrow A_R} \quad 6
\end{array}$$

1 definition of $\leq$ 2 N-VarM 3 IH 4 unfold h at $A_L \rightarrow A_R$ & $\forall / \Rightarrow$ -E 5 IH 6 H-ArrowT

$$\boxed{\text{CASE}} \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L + A_R}{h_{A_L + A_R}^\Delta(M)}$$

$$\frac{\overline{M \rightarrow_\beta^* M} \quad \Delta \xRightarrow{\mathcal{N}} M \in A_L + A_R}{h_{A_L + A_R}^\Delta(M)} \quad \begin{matrix} 1 \\ 2 \end{matrix}$$

1 Beta-Multi-Refl 2 fold h

$$\boxed{\text{CASE}} \quad \frac{h_{A_L + A_R}^\Delta(M)}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R}$$

Case on $h_{A_L + A_R}^\Delta(M)$:

$$\boxed{\text{SUBCASE}} \quad M \rightarrow_\beta^* L \cdot M_L \wedge h_{A_L}^\Delta(M_L) :$$

$$\frac{M \rightarrow_\beta^* L \cdot M_L \quad \Delta \xRightarrow{\mathcal{H}} M_L \in A_L}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R} \quad \begin{matrix} \xRightarrow{h_{A_L}^\Delta(M_L)} 1 \\ 2 \end{matrix}$$

1 IH 2 H-InLM

$$\boxed{\text{SUBCASE}} \quad M \rightarrow_\beta^* R \cdot M_R \wedge h_{A_R}^\Delta(M_R) :$$

Entirely analogous to the previous subcase.

$$\boxed{\text{SUBCASE}} \quad M \rightarrow_\beta^* M' \wedge \Delta \xRightarrow{\mathcal{N}} M' \in A_L + A_R :$$

$$\frac{M \rightarrow_\beta^* M' \quad \Delta \xRightarrow{\mathcal{N}} M' \in A_L + A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R} \quad 1$$

1 H-Neutral-SumT

$$\boxed{\text{CASE}} \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in \text{nat}}{h_{\text{nat}}^\Delta(M)}$$

$$\frac{\frac{\Delta \Rightarrow^{\mathcal{N}} M \in \text{nat}}{\Delta \Rightarrow^{\mathcal{H}} M \in \text{nat}} \quad 1}{\Delta \Rightarrow^{\mathcal{H}} M \in \text{nat}} \quad 2$$

$$\frac{\Delta \Rightarrow^{\mathcal{H}} M \in \text{nat}}{h_{\text{nat}}^{\Delta}(M)}$$

1 H-if-N 2 fold h

CASE

$$\frac{h_{\text{nat}}^{\Delta}(M)}{\Delta \Rightarrow^{\mathcal{H}} M \in \text{nat}}$$

$$\frac{h_{\text{nat}}^{\Delta}(M)}{\Delta \Rightarrow^{\mathcal{H}} M \in \text{nat}} \quad 1$$

1 unfold h

□

4.2 Proofs of Theorems

Define h substitutions for $\gamma : \text{Var} \rightarrow \text{Term}$:

$$h_{\Delta}^{\Gamma}(\gamma) \triangleq \Delta \vdash x : A \Rightarrow h_A^{\Gamma}(\gamma(x))$$

h-Cut-H	h-Cut-N
$\frac{h_{\Delta}^{\Gamma}(\gamma) \quad \Delta \Rightarrow^{\mathcal{H}} M \in A}{h_A^{\Gamma}(\widehat{\gamma}(M))}$	$\frac{h_{\Delta}^{\Gamma}(\gamma) \quad \Delta \Rightarrow^{\mathcal{N}} M \in A}{h_A^{\Gamma}(\widehat{\gamma}(M))}$

Proof by simultaneous induction on $\Delta \Rightarrow^{\mathcal{H}} M \in A$ and $\Delta \Rightarrow^{\mathcal{N}} M \in A$:

CASE

$$\frac{M \rightarrow_{\beta}^* \text{zero}}{\Delta \Rightarrow^{\mathcal{H}} M \in \text{nat}} \quad [\text{H-ZeroM}]$$

$$\frac{\Delta \Rightarrow^{\mathcal{H}} M \in \text{nat}}{h_{\text{nat}}^{\Gamma}(\widehat{\gamma}(M))}$$

$$\frac{\frac{\frac{M \rightarrow_{\beta}^* \text{zero}}{\widehat{\gamma}(M) \rightarrow_{\beta}^* \widehat{\gamma}(\text{zero})} \quad 1}{\widehat{\gamma}(M) \rightarrow_{\beta}^* \text{zero}} \quad 2}{\Gamma \Rightarrow^{\mathcal{H}} \widehat{\gamma}(M) \in \text{nat}} \quad 3$$

$$\frac{\Gamma \Rightarrow^{\mathcal{H}} \widehat{\gamma}(M) \in \text{nat}}{h_{\text{nat}}^{\Gamma}(\widehat{\gamma}(M))} \quad 4$$

1 Hat-Subst-Preserves-Beta-Multi 2 reduce 3 H-ZeroM 4 fold h

$$\begin{array}{c}
 \boxed{\text{CASE}} \quad \frac{M \rightarrow_{\beta}^* \text{succ } M' \quad \Delta \xRightarrow{h} M' \in \text{nat}}{\Delta \xRightarrow{h} M \in \text{nat}} \quad [\text{H-SuccM}] \\
 \hline
 h_{\Delta}^{\Gamma}(\gamma) \\
 \hline
 h_{\text{nat}}^{\Gamma}(\widehat{\gamma}(M))
 \end{array}$$

$$\begin{array}{c}
 \frac{M \rightarrow_{\beta}^* \text{succ } M'}{\widehat{\gamma}(M) \rightarrow_{\beta}^* \widehat{\gamma}(\text{succ } M')} \xrightarrow{1} \quad \frac{h_{\Delta}^{\Gamma}(\gamma) \quad \Delta \xRightarrow{h} M' \in \text{nat}}{h_{\text{nat}}^{\Gamma}(\widehat{\gamma}(M'))} \xrightarrow{3} \\
 \frac{\widehat{\gamma}(M) \rightarrow_{\beta}^* \widehat{\gamma}(\text{succ } M')}{\widehat{\gamma}(M) \rightarrow_{\beta}^* \text{succ } \widehat{\gamma}(M')} \xrightarrow{2} \quad \frac{h_{\text{nat}}^{\Gamma}(\widehat{\gamma}(M'))}{\Gamma \xRightarrow{h} \widehat{\gamma}(M') \in \text{nat}} \xrightarrow{4} \\
 \frac{\widehat{\gamma}(M) \rightarrow_{\beta}^* \text{succ } \widehat{\gamma}(M')}{\Gamma \xRightarrow{h} \widehat{\gamma}(M) \in \text{nat}} \xrightarrow{5} \\
 \frac{\Gamma \xRightarrow{h} \widehat{\gamma}(M) \in \text{nat}}{h_{\text{nat}}^{\Gamma}(\widehat{\gamma}(M))} \xrightarrow{6}
 \end{array}$$

1 Hat-Subst-Preserves-Beta-Multi 2 reduce 3 IH 4 unfold h 5 H-SuccM 6 fold h

$$\begin{array}{c}
 \boxed{\text{CASE}} \quad \frac{M \rightarrow_{\beta}^* \langle \rangle}{\Delta \xRightarrow{h} M \in 1} \quad [\text{H-EmptyM}] \\
 \hline
 h_{\Delta}^{\Gamma}(\gamma) \\
 \hline
 h_1^{\Gamma}(\widehat{\gamma}(M))
 \end{array}$$

$$\begin{array}{c}
 \frac{M \rightarrow_{\beta}^* \langle \rangle}{\widehat{\gamma}(M) \rightarrow_{\beta}^* \widehat{\gamma}(\langle \rangle)} \xrightarrow{1} \\
 \frac{\widehat{\gamma}(M) \rightarrow_{\beta}^* \widehat{\gamma}(\langle \rangle)}{\widehat{\gamma}(M) \rightarrow_{\beta}^* \langle \rangle} \xrightarrow{2} \\
 \frac{\widehat{\gamma}(M) \rightarrow_{\beta}^* \langle \rangle}{h_1^{\Gamma}(\widehat{\gamma}(M))} \xrightarrow{3}
 \end{array}$$

1 Hat-Subst-Preserves-Beta-Multi 2 reduce 3 fold h

$$\begin{array}{c}
 \boxed{\text{CASE}} \quad \frac{M \rightarrow_{\beta}^* M' \quad \Delta \xRightarrow{\mathcal{N}} M' \in 1}{\Delta \xRightarrow{h} M \in 1} \quad [\text{H-Neutral-1T}] \\
 \hline
 h_{\Delta}^{\Gamma}(\gamma) \\
 \hline
 h_1^{\Gamma}(\widehat{\gamma}(M))
 \end{array}$$

$$\begin{array}{c}
 \frac{M \rightarrow_{\beta}^* M'}{\widehat{\gamma}(M) \rightarrow_{\beta}^* \widehat{\gamma}(M')} \xrightarrow{1} \quad \frac{h_{\Delta}^{\Gamma}(\gamma) \quad \Delta \xRightarrow{\mathcal{N}} M' \in 1}{h_1^{\Gamma}(\widehat{\gamma}(M'))} \xrightarrow{2} \\
 \frac{\widehat{\gamma}(M) \rightarrow_{\beta}^* \widehat{\gamma}(M')}{h_1^{\Gamma}(\widehat{\gamma}(M))} \xrightarrow{3}
 \end{array}$$

1 Hat-Subst-Preserves-Beta-Multi 2 IH 3 h-Expansion

CASE

$$\begin{array}{c}
\frac{h_{\Delta}^{\Gamma}(\gamma) \quad \Delta \xRightarrow{u} M \cdot L \in A_L}{h_{A_L}^{\Gamma}(\widehat{\gamma}(M \cdot L))} \quad 1 \\
\frac{h_{A_L}^{\Gamma}(\widehat{\gamma}(M \cdot L))}{h_{A_L}^{\Gamma}(\widehat{\gamma}(M) \cdot L)} \quad 2 \qquad \vdots \text{ (likewise)} \\
\frac{h_{A_L}^{\Gamma}(\widehat{\gamma}(M) \cdot L) \quad h_{A_R}^{\Gamma}(\widehat{\gamma}(M) \cdot R)}{h_{A_L \times A_R}^{\Gamma}(\widehat{\gamma}(M))} \quad 3
\end{array}$$

1 IH 2 reduce 3 fold h

CASE

Fix arbitrary $\Xi \leq \Gamma$ and $h_A^\Xi(N)$.

$$\begin{array}{c}
\frac{\Xi \leq \Gamma \quad h_{\Delta}^{\Gamma}(\gamma)}{\frac{h_{\Delta}^{\Xi}(\gamma)}{h_{\Delta, x:A}^{\Xi}(\gamma, x \mapsto N)} \quad 1} \quad \frac{h_A^{\Xi}(N)}{h_B^{\Xi}((\gamma, \widehat{x \mapsto N})(M x))} \quad 2 \\
\frac{\Delta, x:A \Rightarrow^{\mathcal{H}} M \quad x \in B}{h_B^{\Xi}((\gamma, \widehat{x \mapsto N})(M x))} \quad 3 \\
\frac{h_B^{\Xi}(\widehat{\gamma}(M) N)}{h_{A \rightarrow B}^{\Gamma}(\widehat{\gamma}(M))} \quad 4 \\
\quad \quad \quad 5
\end{array}$$

1 h-Anti-Mono 2 fold h 3 IH 4 reduce 5 fold h

CASE

$$\begin{array}{c}
M \rightarrow_{\beta}^* L \cdot M_L \\
\hline
\widehat{\gamma}(M) \rightarrow_{\beta}^* \widehat{\gamma}(L \cdot M_L) \xrightarrow{1} \xrightarrow{2} \xrightarrow{3} \xrightarrow{4} \\
\widehat{\gamma}(M) \rightarrow_{\beta}^* L \cdot \widehat{\gamma}(M_L) \\
\hline
h_{A_L+A_R}^{\Gamma}(\widehat{\gamma}(M))
\end{array}$$

1 Hat-Subst-Preserves-Beta-Multi 2 reduce 3 IH 4 fold h

$$\begin{array}{c}
\boxed{\text{CASE}} \quad h_{\Delta}^{\Gamma}(\gamma) \quad \frac{M \rightarrow_{\beta}^* R \cdot M_R \quad \Delta \xRightarrow{\mathcal{H}} M_R \in A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R} \text{ [H-InRM]} \\
\hline
h_{A_L+A_R}^{\Gamma}(\widehat{\gamma}(M))
\end{array}$$

Entirely analogous to the previous case.

$$\begin{array}{c}
\boxed{\text{CASE}} \quad h_{\Delta}^{\Gamma}(\gamma) \quad \frac{M \rightarrow_{\beta}^* M' \quad \Delta \xRightarrow{\mathcal{N}} M' \in A_L + A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R} \text{ [H-Neutral-SumT]} \\
\hline
h_{A_L+A_R}^{\Gamma}(\widehat{\gamma}(M))
\end{array}$$

$$\begin{array}{c}
M \rightarrow_{\beta}^* M' \quad \Delta \xRightarrow{\mathcal{N}} M' \in A_L + A_R \\
\hline
\widehat{\gamma}(M) \rightarrow_{\beta}^* \widehat{\gamma}(M') \xrightarrow{1} \xrightarrow{2} \xrightarrow{3} \\
\hline
h_{A_L+A_R}^{\Gamma}(\widehat{\gamma}(M))
\end{array}$$

1 Hat-Subst-Preserves-Beta-Multi 2 IH 3 h-Expansion

$$\begin{array}{c}
\boxed{\text{CASE}} \quad h_{\Delta}^{\Gamma}(\gamma) \quad \frac{M \rightarrow_{\beta}^* M' \quad \Delta \xRightarrow{\mathcal{N}} M' \in \text{nat}}{\Delta \xRightarrow{\mathcal{H}} M \in \text{nat}} \text{ [H-Neutral-NatT]} \\
\hline
h_{\text{nat}}^{\Gamma}(\widehat{\gamma}(M))
\end{array}$$

$$\begin{array}{c}
M \rightarrow_{\beta}^* M' \quad \Delta \xRightarrow{\mathcal{N}} M' \in \text{nat} \\
\hline
\widehat{\gamma}(M) \rightarrow_{\beta}^* \widehat{\gamma}(M') \xrightarrow{1} \xrightarrow{2} \xrightarrow{3} \\
\hline
h_{\text{nat}}^{\Gamma}(\widehat{\gamma}(M))
\end{array}$$

1 Hat-Subst-Preserves-Beta-Multi 2 IH 3 h-Expansion

$$\begin{array}{c}
\boxed{\text{CASE}} \quad \frac{h_{\Delta, x:A}^\Gamma(\gamma) \quad \frac{}{\Delta, x:A \Rightarrow x \in A} [\text{N-VarM}]}{h_A^\Gamma(\widehat{\gamma}(x))} \\
\\
\frac{h_{\Delta, x:A}^\Gamma(\gamma) \quad \frac{}{\Delta, x:A \vdash x:A} \begin{array}{l} 1 \\ 2 \end{array}}{h_A^\Gamma(\widehat{\gamma}(x))} 3
\end{array}$$

1 Var-WT 2 unfold h and $\forall / \Rightarrow$ -E 3 fold $\widehat{\gamma}(x)$

$$\begin{array}{c}
\boxed{\text{CASE}} \quad \frac{h_\Delta^\Gamma(\gamma) \quad \frac{\Delta \Rightarrow M \in A_L \times A_R}{\Delta \Rightarrow M \cdot L \in A_L} [\text{N-projLM}]}{h_{A_L}^\Gamma(\widehat{\gamma}(M \cdot L))} \\
\\
\frac{h_\Delta^\Gamma(\gamma) \quad \Delta \Rightarrow M \in A_L \times A_R}{\frac{h_{A_L \times A_R}^\Gamma(\widehat{\gamma}(M))}{h_{A_L}^\Gamma(\widehat{\gamma}(M) \cdot L)} \begin{array}{l} 1 \\ 2 \end{array}} \begin{array}{l} 2 \\ 3 \end{array} \\
h_{A_L}^\Gamma(\widehat{\gamma}(M \cdot L))
\end{array}$$

1 IH 2 unfold h 3 fold $\widehat{\gamma}(M \cdot L)$

$$\boxed{\text{CASE}} \quad \frac{h_\Delta^\Gamma(\gamma) \quad \frac{\Delta \Rightarrow M \in A_L \times A_R}{\Delta \Rightarrow M \cdot R \in A_R} [\text{N-projRM}]}{h_{A_R}^\Gamma(\widehat{\gamma}(M \cdot R))}$$

Entirely analogous to the previous case.

$$\boxed{\text{CASE}} \quad \frac{h_\Delta^\Gamma(\gamma) \quad \frac{\Delta \Rightarrow M \in A \rightarrow B \quad \Delta \Rightarrow N \in A}{\Delta \Rightarrow M N \in B} [\text{N-AppM}]}{h_B^\Gamma(\widehat{\gamma}(M N))}$$

$$\begin{array}{c}
\frac{h_{\Delta}^{\Gamma}(\gamma) \quad \Delta \xRightarrow{\mathcal{N}} M \in A \rightarrow B}{h_{A \rightarrow B}^{\Gamma}(\widehat{\gamma}(M))} \xrightarrow{1} \frac{}{\Gamma \leq \bar{\Gamma}} \xrightarrow{2} \frac{h_{\Delta}^{\Gamma}(\gamma) \quad \Delta \xRightarrow{\mathcal{H}} N \in A}{h_A^{\Gamma}(\widehat{\gamma}(N))} \xrightarrow{3} \\
\frac{}{h_B^{\Gamma}(\widehat{\gamma}(M) \widehat{\gamma}(N))} \xrightarrow{4} \frac{}{h_B^{\Gamma}(\widehat{\gamma}(M N))} \xrightarrow{5}
\end{array}$$

1 IH 2 definition of $\leq$ 3 IH 4 unfold h at $A \rightarrow B$ and $\forall / \Rightarrow$ -E 5 fold $\widehat{\gamma}(M N)$

$$\begin{array}{c}
\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L + A_R}{\Delta, x_L : A_L \xRightarrow{\mathcal{H}} M_L \in B} \\
\frac{\Delta, x_L : A_L \xRightarrow{\mathcal{H}} M_L \in B}{\Delta, x_R : A_R \xRightarrow{\mathcal{H}} M_R \in B} \\
\text{CASE} \quad \frac{}{h_{\Delta}^{\Gamma}(\gamma) \quad \Delta \xRightarrow{\mathcal{N}} \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B} \text{ [N-CaseM]} \\
\frac{}{h_B^{\Gamma}(\widehat{\gamma}(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)))}
\end{array}$$

Define:

$$\begin{array}{lcl}
\widehat{\text{case}} & \triangleq & \widehat{\gamma}(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \\
\widehat{M}_L(N) & \triangleq & (\gamma, \widehat{x_L} \mapsto N)(M_L) \\
\widehat{M}_R(N) & \triangleq & (\gamma, \widehat{x_R} \mapsto N)(M_R)
\end{array}$$

Observe:

$$\frac{h_{\Delta}^{\Gamma}(\gamma) \quad \Delta \xRightarrow{\mathcal{N}} M \in A_L + A_R}{h_{A_L + A_R}^{\Gamma}(\widehat{\gamma}(M))} \xrightarrow{1}$$

1 IH

Case on $h_{A_L + A_R}^{\Gamma}(\widehat{\gamma}(M))$:

$$\text{SUBCASE} \quad \widehat{\gamma}(M) \rightarrow_{\beta}^* L \cdot N \wedge h_{A_L}^{\Gamma}(N)$$

$$\begin{array}{c}
\frac{\widehat{\gamma}(M) \rightarrow_{\beta}^* L \cdot N}{\widehat{\text{case}} \rightarrow_{\beta}^* \widehat{M}_L(N)} \xrightarrow{1} \frac{h_{\Delta}^{\Gamma}(\gamma) \quad h_{A_L}^{\Gamma}(N)}{h_{\Delta, x_L : A_L}^{\Gamma}(\gamma, x_L \mapsto N)} \xrightarrow{2} \frac{}{\Delta, x_L : A_L \xRightarrow{\mathcal{H}} M_L \in B} \xrightarrow{3} \\
\frac{}{h_B^{\Gamma}(\widehat{M}_L(N))} \xrightarrow{4} \frac{}{h_B^{\Gamma}(\widehat{\text{case}})}
\end{array}$$

1 β^* -under/ β 2 fold h 3 IH 4 h-Expansion

$$\text{SUBCASE} \quad \widehat{\gamma}(M) \rightarrow_{\beta}^* R \cdot N \wedge h_{A_R}^{\Gamma}(N)$$

Entirely analogous to the previous subcase.

SUBCASE $\widehat{\gamma}(M) \rightarrow_{\beta}^* N \wedge \Gamma \xRightarrow{\mathcal{N}} N \in A_L + A_R$

Pick fresh $y_L, y_R \notin \Gamma$.

$$\begin{array}{c}
\begin{array}{c} \xRightarrow{\quad} 1 \\ \Gamma, y_L : A_L \leq \Gamma \end{array} \xRightarrow{h_{\Delta}^{\Gamma}(\gamma)} 2 \quad \begin{array}{c} \xRightarrow{\quad} 3 \\ \Gamma, y_L : A_L \xRightarrow{\mathcal{N}} y_L \in A_L \end{array} \xRightarrow{h_{A_L}^{\Gamma, y_L : A_L}(y_L)} 4 \\
\hline
h_{\Delta}^{\Gamma, y_L : A_L}(\gamma) \quad h_{A_L}^{\Gamma, y_L : A_L}(y_L) \quad 5 \\
\hline
h_{\Delta, x_L : A_L}^{\Gamma, y_L : A_L}(\gamma, x_L \mapsto y_L) \quad \Delta, x_L : A_L \xRightarrow{\mathcal{H}} M_L \in B \quad 6 \\
\hline
h_B^{\Gamma, y_L : A_L}(\widehat{M}_L(y_L)) \quad 7 \\
\hline
\Gamma, y_L : A_L \xRightarrow{\mathcal{H}} \widehat{M}_L(y_L) \in B
\end{array}$$

1 definition of $\leq$ 2 h-Anti-Mono 3 N-VarM 4 h-if-N 5 fold h 6 IH 7 H-if-h

Analogously, $\Gamma, y_R : A_R \xRightarrow{\mathcal{H}} \widehat{M}_R(y_R) \in B$. Therefore:

$$\begin{array}{c}
\Gamma \xRightarrow{\mathcal{N}} N \in A_L + A_R \quad \Gamma, y_L : A_L \xRightarrow{\mathcal{H}} \widehat{M}_L(y_L) \in B \quad \Gamma, y_R : A_R \xRightarrow{\mathcal{H}} \widehat{M}_R(y_R) \in B \quad 1 \\
\hline
\Gamma \xRightarrow{\mathcal{N}} \text{case } N (y_L \Rightarrow \widehat{M}_L(y_L) \mid y_R \Rightarrow \widehat{M}_R(y_R)) \in B \\
\hline
h_B^{\Gamma}(\text{case } N (y_L \Rightarrow \widehat{M}_L(y_L) \mid y_R \Rightarrow \widehat{M}_R(y_R))) \quad 2
\end{array}$$

1 N-CaseM 2 h-if-N

$$\begin{array}{c}
\widehat{\gamma}(M) \rightarrow_{\beta}^* N \\
\hline
\widehat{\text{case}} \rightarrow_{\beta}^* \text{case } N (y_L \Rightarrow \widehat{M}_L(y_L) \mid y_R \Rightarrow \widehat{M}_R(y_R)) \\
\hline
h_B^{\Gamma}(\text{case } N (y_L \Rightarrow \widehat{M}_L(y_L) \mid y_R \Rightarrow \widehat{M}_R(y_R))) \quad 2 \\
\hline
h_B^{\Gamma}(\widehat{\text{case}})
\end{array}$$

1 reduce & β^* -under 2 h-Expansion

$$\begin{array}{c}
\Delta \xRightarrow{\mathcal{N}} M \in \text{nat} \\
\Delta \xRightarrow{\mathcal{H}} M_0 \in A \\
\Delta, x : \text{nat}, y : A \xRightarrow{\mathcal{H}} M_S \in A \quad [\text{N-NatRecM}] \\
\hline
h_{\Delta}^{\Gamma}(\gamma) \quad \Delta \xRightarrow{\mathcal{N}} \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \in A \\
\hline
h_A^{\Gamma}(\widehat{\gamma}(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S)))
\end{array}$$

Pick fresh $x' \notin \Gamma$ and $y' \notin \Gamma, x' : \text{nat}$.

Define:

$$\begin{aligned}\widehat{M}_0 &\triangleq \widehat{\gamma}(M_0) \\ \widehat{M}_S(N, R) &\triangleq (\gamma, x \mapsto \widehat{N}, y \mapsto R)(M_S) \\ \widehat{\text{rec}}(N) &\triangleq \text{natrec } N (\widehat{M}_0 \mid x', y' \Rightarrow \widehat{M}_S(x', y'))\end{aligned}$$

We prove the following claim by induction on the derivation of $\Gamma \xRightarrow{\mathcal{H}} N \in \text{nat}$:

$$\Gamma \xRightarrow{\mathcal{H}} N \in \text{nat} \implies h_A^\Gamma(\widehat{\text{rec}}(N))$$

CASE OF THE CLAIM

$$\frac{N \rightarrow_\beta^* \text{zero}}{\Gamma \xRightarrow{\mathcal{H}} N \in \text{nat}} \text{ [H-ZeroM]}$$

$$\frac{}{h_A^\Gamma(\widehat{\text{rec}}(N))}$$

$$\begin{array}{ccc} N \rightarrow_\beta^* \text{zero} & & h_\Delta^\Gamma(\gamma) \quad \Delta \xRightarrow{\mathcal{H}} M_0 \in A \\ \xrightarrow{\quad 1 \quad} & \xrightarrow{\quad 2 \quad} & \\ \widehat{\text{rec}}(N) \rightarrow_\beta^* \widehat{M}_0 & & h_A^\Gamma(\widehat{M}_0) \\ \xrightarrow{\quad 3 \quad} & & \\ h_A^\Gamma(\widehat{\text{rec}}(N)) & & \end{array}$$

1 β^* -under/ β 2 outer IH 3 h-Expansion

CASE OF THE CLAIM

$$\frac{N \rightarrow_\beta^* \text{succ } N' \quad \Gamma \xRightarrow{\mathcal{H}} N' \in \text{nat}}{\Gamma \xRightarrow{\mathcal{H}} N \in \text{nat}} \text{ [H-SuccM]}$$

$$\frac{}{h_A^\Gamma(\widehat{\text{rec}}(N))}$$

$$\begin{array}{ccc} \Gamma \xRightarrow{\mathcal{H}} N' \in \text{nat} & & \Gamma \xRightarrow{\mathcal{H}} N' \in \text{nat} \\ h_\Delta^\Gamma(\gamma) \quad h_{\text{nat}}^\Gamma(N') & \xrightarrow{\quad 1 \quad} & h_A^\Gamma(\widehat{\text{rec}}(N')) \\ \xrightarrow{\quad 3 \quad} & & \\ h_{\Delta, x: \text{nat}, y: A}^\Gamma(\gamma, x \mapsto N', y \mapsto \widehat{\text{rec}}(N')) & & \Delta, x: \text{nat}, y: A \xRightarrow{\mathcal{H}} M_S \in A \\ \xrightarrow{\quad 4 \quad} & & \\ h_A^\Gamma(\widehat{M}_S(N', \widehat{\text{rec}}(N'))) & & \end{array}$$

1 fold h 2 IH 3 fold h 4 outer IH

$$\begin{array}{ccc} N \rightarrow_\beta^* \text{succ } N' & & \\ \xrightarrow{\quad 1 \quad} & & \\ \widehat{\text{rec}}(N) \rightarrow_\beta^* \widehat{M}_S(N', \widehat{\text{rec}}(N')) & & h_A^\Gamma(\widehat{M}_S(N', \widehat{\text{rec}}(N'))) \\ \xrightarrow{\quad 2 \quad} & & \\ h_A^\Gamma(\widehat{\text{rec}}(N)) & & \end{array}$$

1 β^* -under/ β 2 h-Expansion

$$\boxed{\text{CASE OF THE CLAIM}} \quad \frac{N \rightarrow_{\beta}^* N' \quad \Gamma \xRightarrow{\mathcal{N}} N' \in \text{nat}}{\Gamma \xRightarrow{\mathcal{H}} N \in \text{nat} \quad h_A^{\Gamma}(\widehat{\text{rec}}(N))} [\text{H-Neutral-NatT}]$$

First:

$$\frac{h_{\Delta}^{\Gamma}(\gamma) \quad \Delta \xRightarrow{\mathcal{H}} M_0 \in A}{\xRightarrow{\quad} 1} \xRightarrow{h_A^{\Gamma}(\widehat{M}_0)} 2 \xRightarrow{\Gamma \xRightarrow{\mathcal{H}} \widehat{M}_0 \in A}$$

1 outer IH 2 H-if-h

Next:

$$\frac{\frac{\frac{\Gamma, x' : \text{nat}, y' : A \leq \Gamma}{\xRightarrow{\quad} 1} h_{\Delta}^{\Gamma}(\gamma)}{\xRightarrow{\quad} 2} h_{\Delta}^{\Gamma, x' : \text{nat}, y' : A}(\gamma)}{\frac{\Gamma, x' : \text{nat}, y' : A \xRightarrow{\mathcal{N}} x' \in \text{nat} \quad \Gamma, x' : \text{nat}, y' : A \xRightarrow{\mathcal{N}} y' \in A}{\xRightarrow{\quad} 4} h_{\text{nat}}^{\Gamma, x' : \text{nat}, y' : A}(x') \quad \xRightarrow{\quad} 6} h_A^{\Gamma, x' : \text{nat}, y' : A}(y')}{\xRightarrow{\quad} 7} h_{\Delta, x : \text{nat}, y : A}^{\Gamma, x' : \text{nat}, y' : A}(\gamma, x \mapsto x', y \mapsto y')$$

1 definition of $\leq$ 2 h-Anti-Mono 3 N-VarM 4 h-if-N 5 N-VarM 6 h-if-N 7 fold h

$$\frac{h_{\Delta, x : \text{nat}, y : A}^{\Gamma, x' : \text{nat}, y' : A}(\gamma, x \mapsto x', y \mapsto y') \quad \Delta, x : \text{nat}, y : A \xRightarrow{\mathcal{H}} M_S \in A}{\xRightarrow{\quad} 1} \xRightarrow{h_A^{\Gamma, x' : \text{nat}, y' : A}(\widehat{M}_S(x', y'))} 2 \xRightarrow{\Gamma, x' : \text{nat}, y' : A \xRightarrow{\mathcal{H}} \widehat{M}_S(x', y') \in A}$$

1 outer IH 2 H-if-h

Finally:

$$\begin{array}{c}
\frac{N \rightarrow_{\beta}^* N'}{\widehat{\text{rec}}(N) \rightarrow_{\beta}^* \widehat{\text{rec}}(N')} \quad 1 \qquad \frac{\Gamma \xRightarrow{\mathcal{N}} N' \in \text{nat} \quad \Gamma \xRightarrow{\mathcal{H}} \widehat{M}_0 \in A \quad \Gamma, x' : \text{nat}, y' : A \xRightarrow{\mathcal{H}} \widehat{M}_S(x', y') \in A}{\Gamma \xRightarrow{\mathcal{N}} \widehat{\text{rec}}(N') \in A} \quad 2 \\
\frac{\widehat{\text{rec}}(N) \rightarrow_{\beta}^* \widehat{\text{rec}}(N') \quad \frac{\Gamma \xRightarrow{\mathcal{N}} \widehat{\text{rec}}(N') \in A}{h_A^\Gamma(\widehat{\text{rec}}(N'))} \quad 3}{h_A^\Gamma(\widehat{\text{rec}}(N))} \quad 4
\end{array}$$

1 β^* -under 2 N-NatRecM 3 h-if-N 4 h-Expansion

This proves the claim, which then implies the proof goal:

$$\begin{array}{c}
\frac{h_\Delta^\Gamma(\gamma) \quad \Delta \xRightarrow{\mathcal{N}} M \in \text{nat}}{h_{\text{nat}}^\Gamma(\widehat{\gamma}(M))} \quad 1 \\
\frac{h_{\text{nat}}^\Gamma(\widehat{\gamma}(M))}{\Gamma \xRightarrow{\mathcal{H}} \widehat{\gamma}(M) \in \text{nat}} \quad 2 \\
\frac{\Gamma \xRightarrow{\mathcal{H}} \widehat{\gamma}(M) \in \text{nat}}{h_A^\Gamma(\widehat{\text{rec}}(\widehat{\gamma}(M)))} \quad 3 \\
\frac{h_A^\Gamma(\widehat{\text{rec}}(\widehat{\gamma}(M)))}{h_A^\Gamma(\widehat{\gamma}(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S)))} \quad 4
\end{array}$$

1 outer IH 2 H-if-h 3 claim 4 fold $\widehat{\gamma}(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S))$ □

We remark that at this point we can prove the soundness of the inductive sequent logical relation:

h-if-H

$$\frac{\Delta \xRightarrow{\mathcal{H}} M \in A}{h_A^\Delta(M)}$$

Proof:

$$\begin{array}{c}
\frac{\frac{\frac{\Delta \xRightarrow{\mathcal{H}} M \in A}{h_A^\Delta(\iota)} \quad 1}{h_A^\Delta(\iota(M))} \quad 2}{h_A^\Delta(M)}
\end{array}$$

1 h-Cut-H 2 reduce □

Due to both soundness and completeness of the inductive sequent logical relation, it is therefore equivalent to the recursive logical relation:

$$\Delta \xRightarrow{\mathcal{H}} M \in A \iff h_A^\Delta(M)$$

Proof by appealing to both [h-if-H] and [H-if-h]. $\square$

Next, we prove several compositionality theorems that will later be used to prove the fundamental theorem. These concern arrow, sum, and natural number types. We first prove a semantic cut-elimination result. That is, computability, as defined by the logical relation, is preserved under computable substitution.

Semantic-Cut

$$\frac{\Delta \xRightarrow{\mathcal{H}} M \in A \quad \Delta, x : A \xRightarrow{\mathcal{H}} N \in B}{\Delta \xRightarrow{\mathcal{H}} [M/x]N \in B}$$

Proof by showing that $[M/x]$ is a computable substitution providing $\Delta, x : A$, and then appealing to [h-Cut-H] and the equivalence between the two logical relations. $\square$

H-App-Compositionality

$$\frac{\Delta \xRightarrow{\mathcal{H}} M \in A \rightarrow B \quad \Delta' \leq \Delta \quad \Delta' \xRightarrow{\mathcal{H}} N \in A}{\Delta' \xRightarrow{\mathcal{H}} M N \in B}$$

Proof:

$$\frac{\frac{\Delta \xRightarrow{\mathcal{H}} M \in A \rightarrow B}{h_{A \rightarrow B}^{\Delta}} \quad 1 \quad \frac{\Delta' \xRightarrow{\mathcal{H}} N \in A}{h_A^{\Delta'}} \quad 2}{\frac{\Delta' \leq \Delta \quad h_A^{\Delta'}(N)}{h_B^{\Delta'}(M N)} \quad 3}{\Delta' \xRightarrow{\mathcal{H}} M N \in B} \quad 4$$

1 h-if-H 2 h-if-H 3 unfold h at $A \rightarrow B$ and $\forall / \Rightarrow$ -E 4 H-if-h $\square$

H-Case-Compositionality

$$\frac{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R \quad \Delta, x_L : A_L \xRightarrow{\mathcal{H}} M_L \in B \quad \Delta, x_R : A_R \xRightarrow{\mathcal{H}} M_R \in B}{\Delta \xRightarrow{\mathcal{H}} \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B}$$

Proof by induction on $\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R$:

$$\begin{array}{c}
\frac{M \rightarrow_{\beta}^* L \cdot N \quad \Delta \Rightarrow N \in A_L}{\Delta \Rightarrow M \in A_L + A_R} \text{ [H-InLM]} \\
\text{CASE} \\
\frac{\Delta, x_L : A_L \Rightarrow M_L \in B \quad \Delta, x_R : A_R \Rightarrow M_R \in B}{\Delta \Rightarrow \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B} \\
\\
\frac{\frac{M \rightarrow_{\beta}^* L \cdot N}{\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta}^* [N/x_L](M_L)} \xrightarrow{1} \quad \frac{\Delta \Rightarrow N \in A_L \quad \Delta, x_L : A_L \Rightarrow M_L \in B}{\Delta \Rightarrow [N/x_L](M_L) \in B} \xrightarrow{2}}{\Delta \Rightarrow \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B} \xrightarrow{3}
\end{array}$$

1 β^* -under/ β 2 Semantic-Cut 3 H-Expansion

$$\begin{array}{c}
\frac{M \rightarrow_{\beta}^* R \cdot N \quad \Delta \Rightarrow N \in A_R}{\Delta \Rightarrow M \in A_L + A_R} \text{ [H-InRM]} \\
\text{CASE} \\
\frac{\Delta, x_L : A_L \Rightarrow M_L \in B \quad \Delta, x_R : A_R \Rightarrow M_R \in B}{\Delta \Rightarrow \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B}
\end{array}$$

Entirely analogous to the previous case.

$$\begin{array}{c}
\frac{M \rightarrow_{\beta}^* N \quad \Delta \Rightarrow N \in A_L + A_R}{\Delta \Rightarrow M \in A_L + A_R} \text{ [H-Neutral-SumT]} \\
\text{CASE} \\
\frac{\Delta, x_L : A_L \Rightarrow M_L \in B \quad \Delta, x_R : A_R \Rightarrow M_R \in B}{\Delta \Rightarrow \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B} \\
\\
\frac{M \rightarrow_{\beta}^* N}{\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta}^* \text{case } N (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)} \xrightarrow{1}
\end{array}$$

1 β^* -under

$$\begin{array}{c}
\frac{\Delta \Rightarrow N \in A_L + A_R \quad \Delta, x_L : A_L \Rightarrow M_L \in B \quad \Delta, x_R : A_R \Rightarrow M_R \in B}{\Delta \Rightarrow \text{case } N (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B} \xrightarrow{1} \\
\frac{\Delta \Rightarrow \text{case } N (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B}{\Delta \Rightarrow \text{case } N (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B} \xrightarrow{2}
\end{array}$$

1 N-CaseM 2 H-if-N

$$\begin{array}{c}
 \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta}^* \text{case } N (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \\
 \hline
 \Delta \xRightarrow{\mathcal{H}} \text{case } N (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B \\
 \hline
 \Delta \xRightarrow{\mathcal{H}} \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B
 \end{array} \xrightarrow{1}$$

1 H-Expansion □

H-NatRec-Compositionality

$$\begin{array}{c}
 \Delta \xRightarrow{\mathcal{H}} M \in \text{nat} \quad \Delta \xRightarrow{\mathcal{H}} M_0 \in A \quad \Delta, x : \text{nat}, y : A \xRightarrow{\mathcal{H}} M_S \in A \\
 \hline
 \Delta \xRightarrow{\mathcal{H}} \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \in A
 \end{array}$$

Proof by induction on the derivation of $\Delta \xRightarrow{\mathcal{H}} M \in \text{nat}$:

CASE

$$\begin{array}{c}
 M \rightarrow_{\beta}^* \text{zero} \\
 \hline
 \Delta \xRightarrow{\mathcal{H}} M \in \text{nat}
 \end{array}
 \quad \text{[H-ZeroM]}$$

$$\begin{array}{c}
 \Delta \xRightarrow{\mathcal{H}} M_0 \in A \quad \Delta, x : \text{nat}, y : A \xRightarrow{\mathcal{H}} M_S \in A \\
 \hline
 \Delta \xRightarrow{\mathcal{H}} \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \in A
 \end{array}$$

$$\begin{array}{c}
 M \rightarrow_{\beta}^* \text{zero} \\
 \hline
 \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \rightarrow_{\beta}^* M_0
 \end{array}
 \xrightarrow{1}
 \Delta \xRightarrow{\mathcal{H}} M_0 \in A$$

$$\xrightarrow{2}
 \Delta \xRightarrow{\mathcal{H}} \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \in A$$

1 β^* -under/ β 2 H-Expansion

CASE

$$\begin{array}{c}
 M \rightarrow_{\beta}^* \text{succ } M' \quad \Delta \xRightarrow{\mathcal{H}} M' \in \text{nat} \\
 \hline
 \Delta \xRightarrow{\mathcal{H}} M \in \text{nat}
 \end{array}
 \quad \text{[H-SuccM]}$$

$$\begin{array}{c}
 \Delta \xRightarrow{\mathcal{H}} M_0 \in A \quad \Delta, x : \text{nat}, y : A \xRightarrow{\mathcal{H}} M_S \in A \\
 \hline
 \Delta \xRightarrow{\mathcal{H}} \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \in A
 \end{array}$$

Define $R \triangleq \text{natrec } M' (M_0 \mid x, y \Rightarrow M_S)$.

$$\begin{array}{c}
\begin{array}{c}
\Delta \xRightarrow{\mathcal{H}} M' \in \text{nat} \\
\hline
\Delta, x : \text{nat} \leq \Delta \xrightarrow{1} \Delta \xRightarrow{\mathcal{H}} M' \in \text{nat} \xrightarrow{2} \Delta \xRightarrow{\mathcal{H}} R \in A \xrightarrow{3} \Delta, x : \text{nat} \xRightarrow{\mathcal{H}} R \in A \xrightarrow{4} \Delta, x : \text{nat}, y : A \xRightarrow{\mathcal{H}} M_S \in A \xrightarrow{5} \Delta, x : \text{nat} \xRightarrow{\mathcal{H}} [R/y](M_S) \in A \xrightarrow{6} \Delta \xRightarrow{\mathcal{H}} [M'/x]([R/y](M_S)) \in A \xrightarrow{6} \Delta \xRightarrow{\mathcal{H}} [M'/x, R/y](M_S) \in A
\end{array}
\end{array}$$

1 definition of $\leq$ 2 IH 3 H-Anti-Mono 4 Semantic-Cut 5 Semantic-Cut 6 reduce

Finally:

$$\begin{array}{c}
M \rightarrow_{\beta}^* \text{succ } M' \\
\hline
\text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \rightarrow_{\beta}^* [M'/x, R/y](M_S) \xrightarrow{1} \Delta \xRightarrow{\mathcal{H}} [M'/x, R/y](M_S) \in A \xrightarrow{2} \Delta \xRightarrow{\mathcal{H}} \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \in A
\end{array}$$

1 β^* -under/ β 2 H-Expansion

$$\begin{array}{c}
M \rightarrow_{\beta}^* N \quad \Delta \xRightarrow{\mathcal{N}} N \in \text{nat} \\
\hline
\Delta \xRightarrow{\mathcal{H}} M \in \text{nat} \\
\text{CASE} \quad \Delta \xRightarrow{\mathcal{H}} M_0 \in A \quad \Delta, x : \text{nat}, y : A \xRightarrow{\mathcal{H}} M_S \in A \\
\hline
\Delta \xRightarrow{\mathcal{H}} \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \in A
\end{array}$$

[H-Neutral-NatT]

$$\begin{array}{c}
M \rightarrow_{\beta}^* N \\
\hline
\text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \rightarrow_{\beta}^* \text{natrec } N (M_0 \mid x, y \Rightarrow M_S) \xrightarrow{1}
\end{array}$$

1 β^* -under

$$\begin{array}{c}
\Delta \xRightarrow{\mathcal{N}} N \in \text{nat} \quad \Delta \xRightarrow{\mathcal{H}} M_0 \in A \quad \Delta, x : \text{nat}, y : A \xRightarrow{\mathcal{H}} M_S \in A \\
\hline
\Delta \xRightarrow{\mathcal{N}} \text{natrec } N (M_0 \mid x, y \Rightarrow M_S) \in A \xrightarrow{2} \Delta \xRightarrow{\mathcal{H}} \text{natrec } N (M_0 \mid x, y \Rightarrow M_S) \in A
\end{array}$$

1 N-NatRecM 2 H-if-N

$$\frac{\text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \rightarrow_{\beta}^* \text{natrec } N (M_0 \mid x, y \Rightarrow M_S) \quad \Delta \xRightarrow{\mathcal{H}} \text{natrec } N (M_0 \mid x, y \Rightarrow M_S) \in A}{\Delta \xRightarrow{\mathcal{H}} \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \in A} \quad 1$$

1 H-Expansion □

With these compositionality results, we can now prove the fundamental theorem of the inductive sequent logical relation:

FTLR

$$\frac{\Delta \vdash M : A}{\Delta \xRightarrow{\mathcal{H}} M \in A}$$

Proof by induction on $\Delta \vdash M : A$:

CASE $\frac{\overline{\Delta, x : A \vdash x : A} \text{ [Var-WT]}}{\Delta, x : A \xRightarrow{\mathcal{H}} x \in A}$

$$\frac{\overline{\Delta, x : A \xRightarrow{\mathcal{N}} x \in A} \quad 1}{\Delta, x : A \xRightarrow{\mathcal{H}} x \in A} \quad 2$$

1 N-VarM 2 H-if-N

CASE $\frac{\overline{\Delta \vdash \langle \rangle : 1} \text{ [Empty-WT]}}{\Delta \xRightarrow{\mathcal{H}} \langle \rangle \in 1}$

$$\frac{\overline{\langle \rangle \rightarrow_{\beta}^* \langle \rangle} \quad 1}{\Delta \xRightarrow{\mathcal{H}} \langle \rangle \in 1} \quad 2$$

1 Beta-Multi-Refl 2 H-EmptyM

CASE $\frac{\overline{\Delta \vdash M_L : A_L \quad \Delta \vdash M_R : A_R} \text{ [Pair-WT]}}{\Delta \xRightarrow{\mathcal{H}} \langle M_L, M_R \rangle \in A_L \times A_R}$

$$\begin{array}{c}
\frac{\overline{\langle M_L, M_R \rangle \cdot L \rightarrow_{\beta} M_L}}{\overline{\langle M_L, M_R \rangle \cdot L \rightarrow_{\beta}^* M_L}} \xrightarrow{1} \xrightarrow{2} \xrightarrow{4} \Delta \xRightarrow{\mathcal{H}} \langle M_L, M_R \rangle \cdot L \in A_L \\
\frac{\overline{\langle M_L, M_R \rangle \cdot R \rightarrow_{\beta} M_R}}{\overline{\langle M_L, M_R \rangle \cdot R \rightarrow_{\beta}^* M_R}} \xrightarrow{5} \xrightarrow{6} \xrightarrow{8} \Delta \xRightarrow{\mathcal{H}} \langle M_L, M_R \rangle \cdot R \in A_R \\
\frac{\Delta \xRightarrow{\mathcal{H}} \langle M_L, M_R \rangle \cdot L \in A_L \quad \Delta \xRightarrow{\mathcal{H}} \langle M_L, M_R \rangle \cdot R \in A_R}{\Delta \xRightarrow{\mathcal{H}} \langle M_L, M_R \rangle \in A_L \times A_R} 9
\end{array}$$

1 ProjL-beta 2 Beta-Multi-if-Single 3 IH 4 H-Expansion 5 ProjR-beta 6 Beta-Multi-if-Single 7 IH
8 H-Expansion 9 H-TimesT

$$\boxed{\text{CASE}} \quad \frac{\frac{\Delta \vdash M : A_L \times A_R}{\Delta \vdash M \cdot L : A_L} [\text{ProjL-WT}]}{\Delta \xRightarrow{\mathcal{H}} M \cdot L \in A_L}$$

$$\begin{array}{c}
\frac{\Delta \vdash M : A_L \times A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L \times A_R} \xrightarrow{1} \xrightarrow{2} \Delta \xRightarrow{\mathcal{H}} M \cdot L \in A_L
\end{array}$$

1 IH 2 inversion

$$\boxed{\text{CASE}} \quad \frac{\frac{\Delta \vdash M : A_L \times A_R}{\Delta \vdash M \cdot R : A_R} [\text{ProjR-WT}]}{\Delta \xRightarrow{\mathcal{H}} M \cdot R \in A_R}$$

Entirely analogous to the previous case.

$$\boxed{\text{CASE}} \quad \frac{\frac{\Delta \vdash M : A_L}{\Delta \vdash L \cdot M : A_L + A_R} [\text{InL-WT}]}{\Delta \xRightarrow{\mathcal{H}} L \cdot M \in A_L + A_R}$$

$$\frac{\overline{L \cdot M \rightarrow_{\beta}^* L \cdot M} \xrightarrow{1} \quad \frac{\Delta \vdash M : A_L}{\Delta \xRightarrow{\mathcal{H}} M \in A_L} \xrightarrow{2}}{\Delta \xRightarrow{\mathcal{H}} L \cdot M \in A_L + A_R} 3$$

1 Beta-Multi-Refl 2 IH 3 H-InLM

$$\boxed{\text{CASE}} \quad \frac{\frac{\Delta \vdash M : A_R}{\Delta \vdash R \cdot M : A_L + A_R} [\text{InR-WT}]}{\Delta \xRightarrow{\mathcal{H}} R \cdot M \in A_L + A_R}$$

Entirely analogous to the previous case.

$$\boxed{\text{CASE}} \quad \frac{\frac{\Delta \vdash M : A_L + A_R \quad \Delta, x_L : A_L \vdash M_L : B \quad \Delta, x_R : A_R \vdash M_R : B}{\Delta \vdash \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) : B} [\text{Case-WT}]}{\Delta \xRightarrow{\mathcal{H}} \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B}$$

$$\frac{\frac{\Delta \vdash M : A_L + A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R} \xrightarrow{1} \quad \frac{\Delta, x_L : A_L \vdash M_L : B}{\Delta, x_L : A_L \xRightarrow{\mathcal{H}} M_L \in B} \xrightarrow{2} \quad \frac{\Delta, x_R : A_R \vdash M_R : B}{\Delta, x_R : A_R \xRightarrow{\mathcal{H}} M_R \in B} \xrightarrow{3}}{\Delta \xRightarrow{\mathcal{H}} \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B} \xrightarrow{4}$$

1 IH 2 IH 3 IH 4 H-Case-Compositionality

$$\boxed{\text{CASE}} \quad \frac{\frac{}{\Delta \vdash \text{zero} : \text{nat}} [\text{Zero-WT}]}{\Delta \xRightarrow{\mathcal{H}} \text{zero} \in \text{nat}}$$

$$\frac{\frac{}{\text{zero} \rightarrow_{\beta}^* \text{zero}} \xrightarrow{1}}{\Delta \xRightarrow{\mathcal{H}} \text{zero} \in \text{nat}} \xrightarrow{2}$$

1 Beta-Multi-Refl 2 H-ZeroM

$$\boxed{\text{CASE}} \quad \frac{\frac{\Delta \vdash M : \text{nat}}{\Delta \vdash \text{succ } M : \text{nat}} [\text{Succ-WT}]}{\Delta \xRightarrow{\mathcal{H}} \text{succ } M \in \text{nat}}$$

$$\frac{\frac{}{\text{succ } M \rightarrow_{\beta}^* \text{succ } M} \xrightarrow{1} \quad \frac{\Delta \vdash M : \text{nat}}{\Delta \xRightarrow{\mathcal{H}} M \in \text{nat}} \xrightarrow{2}}{\Delta \xRightarrow{\mathcal{H}} \text{succ } M \in \text{nat}} \xrightarrow{3}$$

1 Beta-Multi-Refl 2 IH 3 H-SuccM

$$\boxed{\text{CASE}} \quad \frac{\frac{\Delta \vdash M : \text{nat} \quad \Delta \vdash M_0 : A \quad \Delta, x : \text{nat}, y : A \vdash M_S : A}{\Delta \vdash \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) : A} [\text{NatRec-WT}]}{\Delta \xRightarrow{\mathcal{H}} \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \in A}$$

$$\begin{array}{c}
\frac{\Delta \vdash M : \text{nat}}{\Delta \xRightarrow{\mathcal{H}} M \in \text{nat}} \xrightarrow{1} \quad \frac{\Delta \vdash M_0 : A}{\Delta \xRightarrow{\mathcal{H}} M_0 \in A} \xrightarrow{2} \quad \frac{\Delta, x : \text{nat}, y : A \vdash M_S : A}{\Delta, x : \text{nat}, y : A \xRightarrow{\mathcal{H}} M_S \in A} \xrightarrow{3} \\
\hline
\Delta \xRightarrow{\mathcal{H}} \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \in A \xrightarrow{4}
\end{array}$$

1 IH 2 IH 3 IH 4 H-NatRec-Compositionality

$$\begin{array}{c}
\frac{\Delta, x : A \vdash M : B}{\Delta \vdash \lambda x \Rightarrow M : A \rightarrow B} \text{ [Lam-WT]} \\
\hline
\Delta \xRightarrow{\mathcal{H}} \lambda x \Rightarrow M \in A \rightarrow B
\end{array}$$

CASE

$$\begin{array}{c}
\frac{\lambda x \Rightarrow M x \rightarrow_\beta [x/x](M)}{\lambda x \Rightarrow M x \rightarrow_\beta^* [x/x](M)} \xrightarrow{1} \quad \frac{\Delta, x : A \vdash M : B}{\Delta, x : A \xRightarrow{\mathcal{H}} M \in B} \xrightarrow{3} \\
\hline
\frac{\lambda x \Rightarrow M x \rightarrow_\beta^* [x/x](M)}{\Delta, x : A \xRightarrow{\mathcal{H}} [x/x](M) \in B} \xrightarrow{2} \quad \frac{\Delta, x : A \vdash M : B}{\Delta, x : A \xRightarrow{\mathcal{H}} M \in B} \xrightarrow{4} \\
\hline
\frac{x \notin \Delta \quad \Delta, x : A \xRightarrow{\mathcal{H}} \lambda x \Rightarrow M x \in B}{\Delta \xRightarrow{\mathcal{H}} \lambda x \Rightarrow M \in A \rightarrow B} \xrightarrow{5} \quad \xrightarrow{6}
\end{array}$$

1 App-beta 2 Beta-Multi-if-Single 3 IH 4 fold $[x/x](M)$ 5 H-Expansion 6 H-ArrowT

$$\begin{array}{c}
\frac{\Delta \vdash M_L : A \rightarrow B \quad \Delta \vdash M_R : A}{\Delta \vdash M_L M_R : B} \text{ [App-WT]} \\
\hline
\Delta \xRightarrow{\mathcal{H}} M_L M_R \in B
\end{array}$$

CASE

$$\begin{array}{c}
\frac{\Delta \vdash M_L : A \rightarrow B}{\Delta \xRightarrow{\mathcal{H}} M_L \in A \rightarrow B} \xrightarrow{1} \quad \frac{\Delta \vdash M_R : A}{\Delta \xRightarrow{\mathcal{H}} M_R \in A} \xrightarrow{3} \\
\hline
\Delta \xRightarrow{\mathcal{H}} M_L M_R \in B \xrightarrow{2} \quad \Delta \leq \Delta \xrightarrow{4}
\end{array}$$

1 IH 2 definition of $\leq$ 3 IH 4 H-App-Compositionality

□

Impossible-Neutrality

$$\frac{\cdot \xRightarrow{\mathcal{N}} M \in A}{\perp}$$

Proof by induction on $\cdot \xRightarrow{\mathcal{N}} M \in A$. In the base case, the rule demands that the context contain the variable being looked up, which yields $\perp$ since the context in the premise is empty. Propagate $\perp$ via the IH in the other cases. □

Step-To-Null-if-H-at-1

$$\frac{\cdot \xRightarrow{\mathcal{H}} M \in 1}{M \rightarrow_{\beta}^* \langle \rangle}$$

Proof by inversion on $\cdot \xRightarrow{\mathcal{H}} M \in 1$:

SUBCASE $\frac{M \rightarrow_{\beta}^* \langle \rangle}{\cdot \xRightarrow{\mathcal{H}} M \in 1} \text{ [H-EmptyM]}$

$$\overline{M \rightarrow_{\beta}^* \langle \rangle}$$

SUBCASE $\frac{M \rightarrow_{\beta}^* M' \quad \cdot \xRightarrow{\mathcal{N}} M' \in 1}{\cdot \xRightarrow{\mathcal{H}} M \in 1} \text{ [H-Neutral-1T]}$

$$\frac{\cdot \xRightarrow{\mathcal{N}} M' \in 1}{\perp} \xrightarrow{1} \frac{\perp}{M \rightarrow_{\beta}^* \langle \rangle} 2$$

1 Impossible-Neutrality 2 $\perp$ -E

□

Step-To-Null-if-WT-at-1

$$\frac{\cdot \vdash M : 1}{M \rightarrow_{\beta}^* \langle \rangle}$$

Proof:

$$\frac{\cdot \vdash M : 1}{\cdot \xRightarrow{\mathcal{H}} M \in 1} \xrightarrow{1} \frac{\cdot \xRightarrow{\mathcal{H}} M \in 1}{M \rightarrow_{\beta}^* \langle \rangle} 2$$

1 FTLR 2 Step-To-Null-if-H-at-1

□

Step-To-Inj-if-H-at-1+1

$$\frac{\cdot \xRightarrow{\mathcal{H}} M \in 1+1}{M \rightarrow_{\beta}^* L \cdot \langle \rangle \vee M \rightarrow_{\beta}^* R \cdot \langle \rangle}$$

Proof by inversion on $\cdot \xRightarrow{\mathcal{H}} M \in 1+1$:

SUBCASE $\frac{M \rightarrow_{\beta}^* L \cdot M_L \quad \cdot \xRightarrow{\mathcal{H}} M_L \in 1}{\cdot \xRightarrow{\mathcal{H}} M \in 1+1} [\text{H-InLM}]$

$$\frac{\cdot \xRightarrow{\mathcal{H}} M \in 1+1}{M \rightarrow_{\beta}^* L \cdot \langle \rangle \vee M \rightarrow_{\beta}^* R \cdot \langle \rangle}$$

$$\frac{\frac{\frac{\cdot \xRightarrow{\mathcal{H}} M_L \in 1}{M_L \rightarrow_{\beta}^* \langle \rangle} 1}{L \cdot M_L \rightarrow_{\beta}^* L \cdot \langle \rangle} 2}{M \rightarrow_{\beta}^* L \cdot M_L \quad L \cdot M_L \rightarrow_{\beta}^* L \cdot \langle \rangle} 3}{M \rightarrow_{\beta}^* L \cdot \langle \rangle} 4}{M \rightarrow_{\beta}^* L \cdot \langle \rangle \vee M \rightarrow_{\beta}^* R \cdot \langle \rangle}$$

1 Step-To-Null-if-H-at-1 2 β^* -under 3 Beta-Multi-Trans 4 $\vee$ -I

SUBCASE $\frac{M \rightarrow_{\beta}^* R \cdot M_R \quad \cdot \xRightarrow{\mathcal{H}} M_R \in 1}{\cdot \xRightarrow{\mathcal{H}} M \in 1+1} [\text{H-InRM}]$

$$\frac{\cdot \xRightarrow{\mathcal{H}} M \in 1+1}{M \rightarrow_{\beta}^* L \cdot \langle \rangle \vee M \rightarrow_{\beta}^* R \cdot \langle \rangle}$$

Entirely analogous to the previous case.

SUBCASE $\frac{M \rightarrow_{\beta}^* M' \quad \cdot \xRightarrow{\mathcal{N}} M' \in 1+1}{\cdot \xRightarrow{\mathcal{H}} M \in 1+1} [\text{H-Neutral-SumT}]$

$$\frac{\cdot \xRightarrow{\mathcal{H}} M \in 1+1}{M \rightarrow_{\beta}^* L \cdot \langle \rangle \vee M \rightarrow_{\beta}^* R \cdot \langle \rangle}$$

$$\frac{\frac{\cdot \xRightarrow{\mathcal{N}} M' \in 1+1}{\perp} 1}{M \rightarrow_{\beta}^* L \cdot \langle \rangle \vee M \rightarrow_{\beta}^* R \cdot \langle \rangle} 2$$

1 Impossible-Neutrality 2 $\perp$ -E

□

Step-To-Inj-if-WT-at-1+1

$$\frac{\cdot \vdash M : 1 + 1}{M \rightarrow_{\beta}^* L \cdot \langle \rangle \vee M \rightarrow_{\beta}^* R \cdot \langle \rangle}$$

Proof:

$$\frac{\frac{\cdot \vdash M : 1 + 1}{\cdot \xRightarrow{\mathcal{H}} M \in 1 + 1} \quad 1}{M \rightarrow_{\beta}^* L \cdot \langle \rangle \vee M \rightarrow_{\beta}^* R \cdot \langle \rangle} 2$$

1 FTLR 2 Step-To-Inj-if-H-at-1+1

□

Define $\text{succ}^n \text{ zero}$ to be the successor term constructor applied to the zero term constructor n times, where $n \geq 0$ is a meta-theoretic natural number.

Step-To-Succ-if-H-at-NatT

$$\frac{\cdot \xRightarrow{\mathcal{H}} M \in \text{nat}}{\exists n . M \rightarrow_{\beta}^* \text{succ}^n \text{ zero}}$$

Proof by induction on $\cdot \xRightarrow{\mathcal{H}} M \in \text{nat}$:

SUBCASE

$$\frac{\frac{M \rightarrow_{\beta}^* \text{zero}}{\cdot \xRightarrow{\mathcal{H}} M \in \text{nat}} \text{ [H-ZeroM]}}{\exists n . M \rightarrow_{\beta}^* \text{succ}^n \text{ zero}}$$

$$\frac{\frac{M \rightarrow_{\beta}^* \text{zero}}{M \rightarrow_{\beta}^* \text{succ}^0 \text{ zero}} 1}{\exists n . M \rightarrow_{\beta}^* \text{succ}^n \text{ zero}} 2$$

1 definition of $\text{succ}^0 \text{ zero}$ 2 $\exists\text{-I}(0)$

SUBCASE

$$\frac{\frac{M \rightarrow_{\beta}^* \text{succ } M' \quad \cdot \xRightarrow{\mathcal{H}} M' \in \text{nat}}{\cdot \xRightarrow{\mathcal{H}} M \in \text{nat}} \text{ [H-SuccM]}}{\exists n . M \rightarrow_{\beta}^* \text{succ}^n \text{ zero}}$$

$$\begin{array}{c}
\frac{\cdot \xRightarrow{\mathcal{H}} M' \in \text{nat}}{\exists n . M' \rightarrow_{\beta}^* \text{succ}^n \text{zero}} \xrightarrow{1} \\
\frac{\quad}{M' \rightarrow_{\beta}^* \text{succ}^n \text{zero}} \xrightarrow{2} \\
\frac{\quad}{M \rightarrow_{\beta}^* \text{succ } M'} \quad \frac{\quad}{\text{succ } M' \rightarrow_{\beta}^* \text{succ}^{n+1} \text{zero}} \xrightarrow{3} \\
\frac{\quad}{M \rightarrow_{\beta}^* \text{succ}^{n+1} \text{zero}} \xrightarrow{4} \\
\frac{\quad}{\exists k . M \rightarrow_{\beta}^* \text{succ}^k \text{zero}} \xrightarrow{5}
\end{array}$$

1 IH 2 pick n 3 β^* -under 4 Beta-Multi-Trans 5 $\exists$ -I($n + 1$)

SUBCASE

$$\frac{M \rightarrow_{\beta}^* M' \quad \cdot \xRightarrow{\mathcal{N}} M' \in \text{nat}}{\cdot \xRightarrow{\mathcal{H}} M \in \text{nat}} \quad \frac{\quad}{\exists n . M \rightarrow_{\beta}^* \text{succ}^n \text{zero}} \quad [\text{H-Neutral-NatT}]$$

$$\frac{\cdot \xRightarrow{\mathcal{N}} M' \in \text{nat}}{\perp} \xrightarrow{1} \quad \frac{\quad}{\exists n . M \rightarrow_{\beta}^* \text{succ}^n \text{zero}} \xrightarrow{2}$$

1 Impossible-Neutrality 2 $\perp$ -E

□

Step-To-Succ-if-WT-at-NatT

$$\frac{\cdot \vdash M : \text{nat}}{\exists n . M \rightarrow_{\beta}^* \text{succ}^n \text{zero}}$$

Proof:

$$\frac{\cdot \vdash M : \text{nat}}{\cdot \xRightarrow{\mathcal{H}} M \in \text{nat}} \xrightarrow{1} \quad \frac{\quad}{\exists n . M \rightarrow_{\beta}^* \text{succ}^n \text{zero}} \xrightarrow{2}$$

1 FTLR 2 Step-To-Succ-if-H-at-NatT

□

Chapter 5

Confluence

In the last chapter, we proved the following various adequacy theorems regarding closed terms:

[Step-To-Null-if-H-at-1]	[Step-To-Null-if-WT-at-1]
$\frac{\cdot \xRightarrow{H} M \in 1}{M \rightarrow_{\beta}^* \langle \rangle}$	$\frac{\cdot \vdash M : 1}{M \rightarrow_{\beta}^* \langle \rangle}$
[Step-To-Inj-if-H-at-1+1]	[Step-To-Inj-if-WT-at-1+1]
$\frac{\cdot \xRightarrow{H} M \in 1 + 1}{M \rightarrow_{\beta}^* L \cdot \langle \rangle \vee M \rightarrow_{\beta}^* R \cdot \langle \rangle}$	$\frac{\cdot \vdash M : 1 + 1}{M \rightarrow_{\beta}^* L \cdot \langle \rangle \vee M \rightarrow_{\beta}^* R \cdot \langle \rangle}$
[Step-To-Succ-if-H-at-NatT]	[Step-To-Succ-if-WT-at-NatT]
$\frac{\cdot \xRightarrow{H} M \in \text{nat}}{\exists n . M \rightarrow_{\beta}^* \text{succ}^n \text{zero}}$	$\frac{\cdot \vdash M : \text{nat}}{\exists n . M \rightarrow_{\beta}^* \text{succ}^n \text{zero}}$

We obviously want to show more. We want to derive similar results for open terms: if an open term is computable, i.e., $\Delta \xRightarrow{H} M \in A$, then it is normalizable; that is, it is able to reduce to some term M' such that M' cannot reduce further. This will be the topic of the next [Chapter 6 \(Normalization\)](#).

This [Chapter 5 \(Confluence\)](#) will prove one significantly non-trivial helper lemma for use in the next chapter: the property of being normalizable is forward closed. Because of the difficulty of proving this helper lemma, we dedicate this whole chapter to it, and the title is confluence since confluence implies that normalizability is forward closed.

Recall the relevant definitions concerning normalizability:

$$\begin{aligned} M \nrightarrow_{\beta} &\triangleq \forall M' . \neg (M \rightarrow_{\beta} M') \\ \text{norm}_{\beta}(M) &\triangleq \exists M' . M \rightarrow_{\beta}^* M' \wedge M' \nrightarrow_{\beta} \end{aligned}$$

The following are the pair of theorems we will want to derive eventually:

$$\begin{array}{cc}
 \text{[norm-if-H]} & \text{[norm-if-N]} \\
 \frac{\Delta \xRightarrow{\mathcal{H}} M \in A}{\text{norm}_\beta(M)} & \frac{\Delta \xRightarrow{\mathcal{N}} M \in A}{\text{norm}_\beta(M)}
 \end{array}$$

The proof will proceed by simultaneous induction on $\Delta \xRightarrow{\mathcal{H}} M \in A$ and $\Delta \xRightarrow{\mathcal{N}} M \in A$. The following case demands that we prove some normalization lemma regarding projections:

$$\text{CASE } \frac{\Delta \xRightarrow{\mathcal{H}} M \cdot L \in A_L \quad \Delta \xRightarrow{\mathcal{H}} M \cdot R \in A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L \times A_R} \text{ [H-TimesT]}$$

 $\text{norm}_\beta(M)$

By IH on $\Delta \xRightarrow{\mathcal{H}} M \cdot L \in A_L$, we obtain $\text{norm}_\beta(M \cdot L)$.

By IH on $\Delta \xRightarrow{\mathcal{H}} M \cdot R \in A_R$, we obtain $\text{norm}_\beta(M \cdot R)$.

If the following normalization lemma is true, then we may apply it to both $\text{norm}_\beta(M \cdot L)$ and $\text{norm}_\beta(M \cdot R)$ to derive $\text{norm}_\beta(M)$:

$$\begin{array}{c}
 \text{[WN-TimesT]} \\
 \frac{\text{norm}_\beta(M \cdot L) \quad \text{norm}_\beta(M \cdot R)}{\text{norm}_\beta(M)}
 \end{array}$$

Thus, let's try to tackle this lemma instead. We may first unfold the definition of $\text{norm}_\beta(M \cdot L)$, so that the above lemma is equivalent to:

$$\frac{M \cdot L \rightarrow_\beta^* P \quad P \not\rightarrow_\beta \quad \text{norm}_\beta(M \cdot R)}{\text{norm}_\beta(M)}$$

We may proceed by induction on $M \cdot L \rightarrow_\beta^* P$. The reflexive case is trivial, so we will skip it here (the whole proof is presented in the actual proof of [WN-TimesT] in the next chapter). The cons case yields the following incomplete proof tree:

$$\frac{\frac{M \cdot L \rightarrow_\beta N \quad N \rightarrow_\beta^* P}{M \cdot L \rightarrow_\beta^* P} \text{ [Beta-Multi-Cons]} \quad P \not\rightarrow_\beta \quad \text{norm}_\beta(M \cdot R)}{\text{norm}_\beta(M)}$$

To extract useful information on N , we may proceed by inversion on $M \cdot L \rightarrow_\beta N$, which yields the following two cases:

$$\begin{array}{c}
\text{CASE} \quad \frac{\frac{M \rightarrow_{\beta} M'}{M \cdot L \rightarrow_{\beta} M' \cdot L} \text{ [ProjL-Under]} \quad M' \cdot L \rightarrow_{\beta}^* P}{M \cdot L \rightarrow_{\beta}^* P} \quad \frac{P \not\rightarrow_{\beta} \quad \text{norm}_{\beta}(M \cdot R)}{\text{norm}_{\beta}(M)}
\end{array}$$

For this first case, $M \rightarrow_{\beta} M'$ implies $M \cdot R \rightarrow_{\beta} M' \cdot R$. If we had forward closure of $\text{norm}_{\beta}(M \cdot R)$, we could then obtain $\text{norm}_{\beta}(M' \cdot R)$, with which this case could be completed easily.

$$\begin{array}{c}
\text{CASE} \quad \frac{\frac{\langle M_L, M_R \rangle \cdot L \rightarrow_{\beta} M_L \text{ [ProjL-beta]} \quad M_L \rightarrow_{\beta}^* P}{\langle M_L, M_R \rangle \cdot L \rightarrow_{\beta}^* P} \quad P \not\rightarrow_{\beta} \quad \text{norm}_{\beta}(\langle M_L, M_R \rangle \cdot R)}{\text{norm}_{\beta}(\langle M_L, M_R \rangle)}
\end{array}$$

For this second case, if we had forward closure of $\text{norm}_{\beta}(\langle M_L, M_R \rangle \cdot R)$, we could simply step $\langle M_L, M_R \rangle \cdot R$ towards M_R so that by the forward closure we could obtain $\text{norm}_{\beta}(M_R)$, with which this case could also be completed easily.

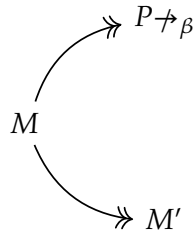
Thus, both cases demand we prove that $\text{norm}_{\beta}(-)$ is forward closed:

$$\boxed{\begin{array}{c} \text{[norm-Forward-Closure]} \\ \frac{\text{norm}_{\beta}(M) \quad M \rightarrow_{\beta}^* M'}{\text{norm}_{\beta}(M')} \end{array}}$$

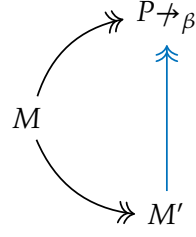
We may again unfold the definition of $\text{norm}_{\beta}(M)$ and turn the lemma statement into the following equivalent one:

$$\frac{M \rightarrow_{\beta}^* P \quad P \not\rightarrow_{\beta} \quad M \rightarrow_{\beta}^* M'}{\text{norm}_{\beta}(M')}$$

One might draw the following diagram to represent the situation, where two-headed arrows stand for multi-step β -reduction:



The forking shape starting from M naturally reminds one of confluence, which states that if a term M like the one above forks, then it will be able to join back. Since P is already β -irreducible, such a join must then force M' to multi-step β -reduce to P :



The [blue arrow](#) proves $\text{norm}_{\beta}(M')$. Hence, to prove forward closure of normalizability, it suffices to show confluence, which is what the rest of this chapter is about.

5.1 Proofs

The organization of this section follows from the standard approach to proving confluence via parallel β -reduction [7, 9].

$$\boxed{M \Rightarrow_{\beta} M'}$$

Parallel-Var

$$\overline{x \Rightarrow_{\beta} x}$$

Parallel-Empty

$$\overline{\langle \rangle \Rightarrow_{\beta} \langle \rangle}$$

Parallel-Pair

$$\frac{M_L \Rightarrow_{\beta} M'_L \quad M_R \Rightarrow_{\beta} M'_R}{\langle M_L, M_R \rangle \Rightarrow_{\beta} \langle M'_L, M'_R \rangle}$$

Parallel-ProjL-Under

$$\frac{M \Rightarrow_{\beta} M'}{M \cdot L \Rightarrow_{\beta} M' \cdot L}$$

Parallel-ProjR-Under

$$\frac{M \Rightarrow_{\beta} M'}{M \cdot R \Rightarrow_{\beta} M' \cdot R}$$

Parallel-ProjL

$$\frac{M_L \Rightarrow_{\beta} M'_L}{\langle M_L, M_R \rangle \cdot L \Rightarrow_{\beta} M'_L}$$

Parallel-ProjR

$$\frac{M_R \Rightarrow_{\beta} M'_R}{\langle M_L, M_R \rangle \cdot R \Rightarrow_{\beta} M'_R}$$

Parallel-Lam

$$\frac{M \Rightarrow_{\beta} M'}{\lambda x \Rightarrow M \Rightarrow_{\beta} \lambda x \Rightarrow M'}$$

Parallel-App-Under

$$\frac{M \Rightarrow_{\beta} M' \quad N \Rightarrow_{\beta} N'}{M N \Rightarrow_{\beta} M' N'}$$

Parallel-App

$$\frac{M \Rightarrow_{\beta} M' \quad N \Rightarrow_{\beta} N'}{\lambda x \Rightarrow M N \Rightarrow_{\beta} [N'/x](M')}$$

$$\begin{array}{c}
\text{Parallel-InL} \\
\frac{M \Rightarrow_{\beta} M'}{L \cdot M \Rightarrow_{\beta} L \cdot M'} \\
\text{Parallel-InR} \\
\frac{M \Rightarrow_{\beta} M'}{R \cdot M \Rightarrow_{\beta} R \cdot M'}
\end{array}$$

$$\begin{array}{c}
\text{Parallel-Case-Under} \\
\frac{M \Rightarrow_{\beta} M' \quad M_L \Rightarrow_{\beta} M'_L \quad M_R \Rightarrow_{\beta} M'_R}{\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} \text{case } M' (x_L \Rightarrow M'_L \mid x_R \Rightarrow M'_R)}
\end{array}$$

$$\begin{array}{c}
\text{Parallel-Case-InL} \\
\frac{M \Rightarrow_{\beta} M' \quad M_L \Rightarrow_{\beta} M'_L}{\text{case } L \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} [M'/x_L](M'_L)}
\end{array}$$

$$\begin{array}{c}
\text{Parallel-Case-InR} \\
\frac{M \Rightarrow_{\beta} M' \quad M_R \Rightarrow_{\beta} M'_R}{\text{case } R \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} [M'/x_R](M'_R)}
\end{array}$$

$$\begin{array}{c}
\text{Parallel-Zero} \qquad \text{Parallel-Succ} \\
\frac{}{\text{zero} \Rightarrow_{\beta} \text{zero}} \qquad \frac{M \Rightarrow_{\beta} M'}{\text{succ } M \Rightarrow_{\beta} \text{succ } M'}
\end{array}$$

$$\begin{array}{c}
\text{Parallel-NatRec-Under} \\
\frac{M \Rightarrow_{\beta} M' \quad M_0 \Rightarrow_{\beta} M'_0 \quad M_S \Rightarrow_{\beta} M'_S}{\text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_{\beta} \text{natrec } M' (M'_0 \mid x, y \Rightarrow M'_S)}
\end{array}$$

$$\begin{array}{c}
\text{Parallel-NatRec-Zero} \\
\frac{M_0 \Rightarrow_{\beta} M'_0}{\text{natrec zero } (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_{\beta} M'_0}
\end{array}$$

$$\begin{array}{c}
\text{Parallel-NatRec-Succ} \\
\frac{M \Rightarrow_{\beta} M' \quad M_0 \Rightarrow_{\beta} M'_0 \quad M_S \Rightarrow_{\beta} M'_S}{\text{natrec succ } M (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_{\beta} [M'/x, \text{natrec } M' (M'_0 \mid x, y \Rightarrow M'_S)/y](M'_S)}
\end{array}$$

Parallel-Refl

$$\xRightarrow{\quad} \\ M \Rightarrow_{\beta} M$$

Proof by induction on the structure of M . □

$M \Rightarrow_{\beta}^* M'$

Parallel-Multi-Refl

$$\xRightarrow{\quad} \\ M \Rightarrow_{\beta}^* M$$

Parallel-Multi-Cons

$$\frac{M \Rightarrow_{\beta} M' \quad M' \Rightarrow_{\beta}^* M''}{\xRightarrow{\quad} M \Rightarrow_{\beta}^* M''}$$

Parallel-Multi-if-Single

$$\frac{M \Rightarrow_{\beta} M'}{\xRightarrow{\quad} M \Rightarrow_{\beta}^* M'}$$

Parallel-Multi-Trans

$$\frac{M \Rightarrow_{\beta}^* M' \quad M' \Rightarrow_{\beta}^* M''}{\xRightarrow{\quad} M \Rightarrow_{\beta}^* M''}$$

Parallel-if-Usual

$$\frac{M \rightarrow_{\beta} M'}{\xRightarrow{\quad} M \Rightarrow_{\beta} M'}$$

Proof by induction on the derivation of $M \rightarrow_{\beta} M'$:

CASE

$$\frac{M_L \rightarrow_{\beta} M'_L}{\langle M_L, M_R \rangle \rightarrow_{\beta} \langle M'_L, M_R \rangle} \text{ [Pair-Under-L]}$$

$$\langle M_L, M_R \rangle \Rightarrow_{\beta} \langle M'_L, M_R \rangle$$

$$\frac{\frac{M_L \rightarrow_{\beta} M'_L}{\xRightarrow{\quad} M_L \Rightarrow_{\beta} M'_L} \text{ 1} \quad \xRightarrow{\quad} M_R \Rightarrow_{\beta} M_R \text{ 2}}{\xRightarrow{\quad} \langle M_L, M_R \rangle \Rightarrow_{\beta} \langle M'_L, M_R \rangle} \text{ 3}$$

1 IH 2 Parallel-Refl 3 Parallel-Pair

$$\begin{array}{c}
\boxed{\text{CASE}} \quad \frac{\frac{M_R \rightarrow_\beta M'_R}{\langle M_L, M_R \rangle \rightarrow_\beta \langle M_L, M'_R \rangle} \text{ [Pair-Under-R]}}{\langle M_L, M_R \rangle \Rightarrow_\beta \langle M_L, M'_R \rangle} \\
\\
\frac{\frac{\xRightarrow{\hspace{1cm}} 1 \quad \frac{M_R \rightarrow_\beta M'_R}{M_R \Rightarrow_\beta M'_R} 2}{\langle M_L, M_R \rangle \Rightarrow_\beta \langle M_L, M'_R \rangle} 3
\end{array}$$

1 Parallel-Ref 2 IH 3 Parallel-Pair

$$\begin{array}{c}
\boxed{\text{CASE}} \quad \frac{\frac{M \rightarrow_\beta M'}{M \cdot L \rightarrow_\beta M' \cdot L} \text{ [ProjL-Under]}}{M \cdot L \Rightarrow_\beta M' \cdot L} \\
\\
\frac{\frac{M \rightarrow_\beta M'}{\xRightarrow{\hspace{1cm}} 1} \quad M \Rightarrow_\beta M'}{M \cdot L \Rightarrow_\beta M' \cdot L} 2
\end{array}$$

1 IH 2 Parallel-ProjL-Under

$$\begin{array}{c}
\boxed{\text{CASE}} \quad \frac{\frac{M \rightarrow_\beta M'}{M \cdot R \rightarrow_\beta M' \cdot R} \text{ [ProjR-Under]}}{M \cdot R \Rightarrow_\beta M' \cdot R} \\
\\
\frac{\frac{M \rightarrow_\beta M'}{\xRightarrow{\hspace{1cm}} 1} \quad M \Rightarrow_\beta M'}{M \cdot R \Rightarrow_\beta M' \cdot R} 2
\end{array}$$

1 IH 2 Parallel-ProjR-Under

$$\begin{array}{c}
\boxed{\text{CASE}} \quad \frac{\frac{\langle M_L, M_R \rangle \cdot L \rightarrow_\beta M_L}{\langle M_L, M_R \rangle \cdot L \Rightarrow_\beta M_L} \text{ [ProjL-beta]}}{\langle M_L, M_R \rangle \cdot L \Rightarrow_\beta M_L} \\
\\
\frac{\xRightarrow{\hspace{1cm}} 1}{\langle M_L, M_R \rangle \cdot L \Rightarrow_\beta M_L} 2
\end{array}$$

1 Parallel-Refl 2 Parallel-ProjL

$$\boxed{\text{CASE}} \quad \frac{\overline{\langle M_L, M_R \rangle \cdot R \rightarrow_\beta M_R} \quad [\text{ProjR-beta}]}{\langle M_L, M_R \rangle \cdot R \Rightarrow_\beta M_R}$$

$$\frac{\overline{\overline{M_R \Rightarrow_\beta M_R}} \quad 1}{\langle M_L, M_R \rangle \cdot R \Rightarrow_\beta M_R} \quad 2$$

1 Parallel-Refl 2 Parallel-ProjR

$$\boxed{\text{CASE}} \quad \frac{\overline{M \rightarrow_\beta M'} \quad [\text{Lam-Under}]}{\lambda x \Rightarrow M \Rightarrow_\beta \lambda x \Rightarrow M'}$$

$$\frac{\overline{\overline{M \rightarrow_\beta M'}} \quad 1}{\lambda x \Rightarrow M \Rightarrow_\beta \lambda x \Rightarrow M'} \quad 2$$

1 IH 2 Parallel-Lam

$$\boxed{\text{CASE}} \quad \frac{\overline{M \rightarrow_\beta M'} \quad [\text{App-Under-Fun}]}{M N \rightarrow_\beta M' N}$$

$$\overline{M N \Rightarrow_\beta M' N}$$

$$\frac{\overline{\overline{M \rightarrow_\beta M'}} \quad 1 \quad \overline{\overline{N \Rightarrow_\beta N}} \quad 2}{M N \Rightarrow_\beta M' N} \quad 3$$

1 IH 2 Parallel-Refl 3 Parallel-App-Under

$$\boxed{\text{CASE}} \quad \frac{\overline{N \rightarrow_\beta N'} \quad [\text{App-Under-Arg}]}{M N \rightarrow_\beta M N'}$$

$$\overline{M N \Rightarrow_\beta M N'}$$

$$\frac{\overrightarrow{\overrightarrow{M \Rightarrow_{\beta} M}} \quad 1 \quad \overrightarrow{\overrightarrow{N \rightarrow_{\beta} N'}} \quad 2}{\overrightarrow{\overrightarrow{M N \Rightarrow_{\beta} M N'}} \quad 3}$$

1 Parallel-Refl 2 IH 3 Parallel-App-Under

$$\boxed{\text{CASE}} \quad \frac{\overrightarrow{\overrightarrow{\lambda x \Rightarrow M N \rightarrow_{\beta} [N/x](M)}} \quad [\text{App-beta}]}{\overrightarrow{\overrightarrow{\lambda x \Rightarrow M N \Rightarrow_{\beta} [N/x](M)}}$$

$$\frac{\overrightarrow{\overrightarrow{M \Rightarrow_{\beta} M}} \quad 1 \quad \overrightarrow{\overrightarrow{N \Rightarrow_{\beta} N}} \quad 2}{\overrightarrow{\overrightarrow{\lambda x \Rightarrow M N \Rightarrow_{\beta} [N/x](M)}} \quad 3$$

1 Parallel-Refl 2 Parallel-Refl 3 Parallel-App

$$\boxed{\text{CASE}} \quad \frac{\overrightarrow{\overrightarrow{M \rightarrow_{\beta} M'}} \quad [\text{InL-Under}]}{\overrightarrow{\overrightarrow{L \cdot M \rightarrow_{\beta} L \cdot M'}}} \quad \frac{\overrightarrow{\overrightarrow{L \cdot M \rightarrow_{\beta} L \cdot M'}}}{\overrightarrow{\overrightarrow{L \cdot M \Rightarrow_{\beta} L \cdot M'}}$$

$$\frac{\overrightarrow{\overrightarrow{M \rightarrow_{\beta} M'}} \quad 1}{\overrightarrow{\overrightarrow{M \Rightarrow_{\beta} M'}} \quad 2} \quad \frac{\overrightarrow{\overrightarrow{M \Rightarrow_{\beta} M'}}}{\overrightarrow{\overrightarrow{L \cdot M \Rightarrow_{\beta} L \cdot M'}}$$

1 IH 2 Parallel-InL

$$\boxed{\text{CASE}} \quad \frac{\overrightarrow{\overrightarrow{M \rightarrow_{\beta} M'}} \quad [\text{InR-Under}]}{\overrightarrow{\overrightarrow{R \cdot M \rightarrow_{\beta} R \cdot M'}}} \quad \frac{\overrightarrow{\overrightarrow{R \cdot M \rightarrow_{\beta} R \cdot M'}}}{\overrightarrow{\overrightarrow{R \cdot M \Rightarrow_{\beta} R \cdot M'}}$$

$$\frac{\overrightarrow{\overrightarrow{M \rightarrow_{\beta} M'}} \quad 1}{\overrightarrow{\overrightarrow{M \Rightarrow_{\beta} M'}} \quad 2} \quad \frac{\overrightarrow{\overrightarrow{M \Rightarrow_{\beta} M'}}}{\overrightarrow{\overrightarrow{R \cdot M \Rightarrow_{\beta} R \cdot M'}}$$

1 IH 2 Parallel-InR

$$\begin{array}{c}
\text{CASE} \quad \frac{M \rightarrow_{\beta} M'}{\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta} \text{case } M' (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)} \quad [\text{Case-Under-Main}] \\
\hline
\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} \text{case } M' (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)
\end{array}$$

$$\frac{\begin{array}{ccc} M \rightarrow_{\beta} M' & & \\ \xRightarrow{\quad} 1 & \xRightarrow{\quad} 2 & \xRightarrow{\quad} 3 \\ M \Rightarrow_{\beta} M' & M_L \Rightarrow_{\beta} M_L & M_R \Rightarrow_{\beta} M_R \end{array}}{\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} \text{case } M' (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)} \quad 4$$

1 IH 2 Parallel-Refl 3 Parallel-Refl 4 Parallel-Case-Under

$$\begin{array}{c}
\text{CASE} \quad \frac{M_L \rightarrow_{\beta} M'_L}{\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta} \text{case } M (x_L \Rightarrow M'_L \mid x_R \Rightarrow M_R)} \quad [\text{Case-Under-L}] \\
\hline
\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} \text{case } M (x_L \Rightarrow M'_L \mid x_R \Rightarrow M_R)
\end{array}$$

$$\frac{\begin{array}{ccc} \xRightarrow{\quad} 1 & M_L \rightarrow_{\beta} M'_L & \xRightarrow{\quad} 3 \\ M \Rightarrow_{\beta} M & M_L \Rightarrow_{\beta} M'_L & M_R \Rightarrow_{\beta} M_R \end{array}}{\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} \text{case } M (x_L \Rightarrow M'_L \mid x_R \Rightarrow M_R)} \quad 4$$

1 Parallel-Refl 2 IH 3 Parallel-Refl 4 Parallel-Case-Under

$$\begin{array}{c}
\text{CASE} \quad \frac{M_R \rightarrow_{\beta} M'_R}{\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta} \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M'_R)} \quad [\text{Case-Under-R}] \\
\hline
\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M'_R)
\end{array}$$

$$\frac{\begin{array}{ccc} \xRightarrow{\quad} 1 & \xRightarrow{\quad} 2 & M_R \rightarrow_{\beta} M'_R \\ M \Rightarrow_{\beta} M & M_L \Rightarrow_{\beta} M_L & M_R \Rightarrow_{\beta} M'_R \end{array}}{\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M'_R)} \quad 4$$

1 Parallel-Refl 2 Parallel-Refl 3 IH 4 Parallel-Case-Under

$$\begin{array}{c}
\text{CASE} \quad \frac{}{\text{case } L \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta} [M/x_L](M_L)} \quad [\text{Case-InL-beta}] \\
\hline
\text{case } L \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} [M/x_L](M_L)
\end{array}$$

$$\frac{\begin{array}{cc} \xRightarrow{\quad} 1 & \xRightarrow{\quad} 2 \\ M \Rightarrow_{\beta} M & M_L \Rightarrow_{\beta} M_L \end{array}}{\text{case } L \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} [M/x_L](M_L)} \quad 3$$

1 Parallel-Refl 2 Parallel-Refl 3 Parallel-Case-InL

$$\boxed{\text{CASE}} \quad \frac{\text{case } R \cdot M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_\beta [M/x_R](M_R)}{\text{case } R \cdot M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_\beta [M/x_R](M_R)} \quad [\text{Case-InR-beta}]$$

$$\frac{\frac{\overline{\overline{\overline{M} \Rightarrow_\beta M}} \quad 1 \quad \overline{\overline{\overline{M_R} \Rightarrow_\beta M_R}} \quad 2}{\text{case } R \cdot M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_\beta [M/x_R](M_R)} \quad 3$$

1 Parallel-Refl 2 Parallel-Refl 3 Parallel-Case-InR

$$\boxed{\text{CASE}} \quad \frac{\frac{M \rightarrow_\beta M'}{\text{succ } M \rightarrow_\beta \text{succ } M'} \quad [\text{Succ-Under}]}{\text{succ } M \Rightarrow_\beta \text{succ } M'}$$

$$\frac{\frac{M \rightarrow_\beta M'}{\overline{\overline{\overline{M} \Rightarrow_\beta M'}} \quad 1}}{\text{succ } M \Rightarrow_\beta \text{succ } M'} \quad 2$$

1 IH 2 Parallel-Succ

$$\boxed{\text{CASE}} \quad \frac{\frac{M \rightarrow_\beta M'}{\text{natrec } M \ (M_0 \mid x, y \Rightarrow M_S) \rightarrow_\beta \text{natrec } M' \ (M_0 \mid x, y \Rightarrow M_S)} \quad [\text{NatRec-Under-Main}]}{\text{natrec } M \ (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_\beta \text{natrec } M' \ (M_0 \mid x, y \Rightarrow M_S)}$$

$$\frac{\frac{\frac{M \rightarrow_\beta M'}{\overline{\overline{\overline{M} \Rightarrow_\beta M'}} \quad 1} \quad \overline{\overline{\overline{M_0 \Rightarrow_\beta M_0}}} \quad 2 \quad \overline{\overline{\overline{M_S \Rightarrow_\beta M_S}}} \quad 3}{\text{natrec } M \ (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_\beta \text{natrec } M' \ (M_0 \mid x, y \Rightarrow M_S)} \quad 4$$

1 IH 2 Parallel-Refl 3 Parallel-Refl 4 Parallel-NatRec-Under

$$\boxed{\text{CASE}} \quad \frac{\frac{M_0 \rightarrow_\beta M'_0}{\text{natrec } M \ (M_0 \mid x, y \Rightarrow M_S) \rightarrow_\beta \text{natrec } M \ (M'_0 \mid x, y \Rightarrow M_S)} \quad [\text{NatRec-Under-Zero}]}{\text{natrec } M \ (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_\beta \text{natrec } M \ (M'_0 \mid x, y \Rightarrow M_S)}$$

$$\frac{\begin{array}{c} \xRightarrow{\quad} 1 \\ M \Rightarrow_{\beta} M \end{array} \quad \begin{array}{c} M_0 \rightarrow_{\beta} M'_0 \\ \xRightarrow{\quad} 2 \\ M_0 \Rightarrow_{\beta} M'_0 \end{array} \quad \begin{array}{c} \xRightarrow{\quad} 3 \\ M_S \Rightarrow_{\beta} M_S \end{array}}{\text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_{\beta} \text{natrec } M (M'_0 \mid x, y \Rightarrow M_S)} 4$$

1 Parallel-Refl 2 IH 3 Parallel-Refl 4 Parallel-NatRec-Under

$$\boxed{\text{CASE}} \quad \frac{\begin{array}{c} M_S \rightarrow_{\beta} M'_S \\ \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \rightarrow_{\beta} \text{natrec } M (M_0 \mid x, y \Rightarrow M'_S) \end{array}}{\text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_{\beta} \text{natrec } M (M_0 \mid x, y \Rightarrow M'_S)} [\text{NatRec-Under-Succ}]$$

$$\frac{\begin{array}{c} \xRightarrow{\quad} 1 \\ M \Rightarrow_{\beta} M \end{array} \quad \begin{array}{c} \xRightarrow{\quad} 2 \\ M_0 \Rightarrow_{\beta} M_0 \end{array} \quad \begin{array}{c} M_S \rightarrow_{\beta} M'_S \\ \xRightarrow{\quad} 3 \\ M_S \Rightarrow_{\beta} M'_S \end{array}}{\text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_{\beta} \text{natrec } M (M_0 \mid x, y \Rightarrow M'_S)} 4$$

1 Parallel-Refl 2 Parallel-Refl 3 IH 4 Parallel-NatRec-Under

$$\boxed{\text{CASE}} \quad \frac{\begin{array}{c} \text{natrec zero } (M_0 \mid x, y \Rightarrow M_S) \rightarrow_{\beta} M_0 \end{array}}{\text{natrec zero } (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_{\beta} M_0} [\text{NatRec-Zero-beta}]$$

$$\frac{\begin{array}{c} \xRightarrow{\quad} 1 \\ M_0 \Rightarrow_{\beta} M_0 \end{array}}{\text{natrec zero } (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_{\beta} M_0} 2$$

1 Parallel-Refl 2 Parallel-NatRec-Zero

$$\boxed{\text{CASE}} \quad \frac{\begin{array}{c} \text{natrec succ } M (M_0 \mid x, y \Rightarrow M_S) \rightarrow_{\beta} [M/x, R/y](M_S) \end{array}}{\text{natrec succ } M (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_{\beta} [M/x, R/y](M_S)} [\text{NatRec-Succ-beta}]$$

Define $R \triangleq \text{natrec } M (M_0 \mid x, y \Rightarrow M_S)$.

$$\frac{\begin{array}{c} \xRightarrow{\quad} 1 \\ M \Rightarrow_{\beta} M \end{array} \quad \begin{array}{c} \xRightarrow{\quad} 2 \\ M_0 \Rightarrow_{\beta} M_0 \end{array} \quad \begin{array}{c} \xRightarrow{\quad} 3 \\ M_S \Rightarrow_{\beta} M_S \end{array}}{\text{natrec succ } M (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_{\beta} [M/x, R/y](M_S)} 4$$

1 Parallel-Refl 2 Parallel-Refl 3 Parallel-Refl 4 Parallel-NatRec-Succ

□

Parallel-Multi-if-Usual-Multi

$$\frac{M \rightarrow_{\beta}^* M'}{\overline{M \Rightarrow_{\beta}^* M'}} \Rightarrow$$

Proof by appealing to [Parallel-if-Usual]. □

Usual-Multi-if-Parallel

$$\frac{M \Rightarrow_{\beta} M'}{\overline{M \rightarrow_{\beta}^* M'}} \Rightarrow$$

Proof by induction on the derivation of $M \Rightarrow_{\beta} M'$:

CASE $\frac{\overline{x \Rightarrow_{\beta} x} \text{ [Parallel-Var]}}{\overline{x \rightarrow_{\beta}^* x}}$

$$\overline{x \rightarrow_{\beta}^* x} \quad 1$$

1 Beta-Multi-Refl

CASE $\frac{\overline{\langle \rangle \Rightarrow_{\beta} \langle \rangle} \text{ [Parallel-Empty]}}{\overline{\langle \rangle \rightarrow_{\beta}^* \langle \rangle}}$

$$\overline{\langle \rangle \rightarrow_{\beta}^* \langle \rangle} \quad 1$$

1 Beta-Multi-Refl

CASE $\frac{M_L \Rightarrow_{\beta} M'_L \quad M_R \Rightarrow_{\beta} M'_R}{\overline{\langle M_L, M_R \rangle \Rightarrow_{\beta} \langle M'_L, M'_R \rangle}} \text{ [Parallel-Pair]}$
 $\overline{\langle M_L, M_R \rangle \rightarrow_{\beta}^* \langle M'_L, M'_R \rangle}$

$$\begin{array}{c}
\begin{array}{ccc}
M_L \Rightarrow_\beta M'_L & & M_R \Rightarrow_\beta M'_R \\
\overline{\overline{\longrightarrow}} & & \overline{\overline{\longrightarrow}} \\
M_L \rightarrow_\beta^* M'_L & \text{1} & M_R \rightarrow_\beta^* M'_R \text{3}
\end{array} \\
\overline{\overline{\longrightarrow}} & \text{2} & \overline{\overline{\longrightarrow}} & \text{4} \\
\langle M_L, M_R \rangle \rightarrow_\beta^* \langle M'_L, M_R \rangle & & \langle M'_L, M_R \rangle \rightarrow_\beta^* \langle M'_L, M'_R \rangle \\
\overline{\overline{\longrightarrow}} & \text{5} & \\
\langle M_L, M_R \rangle \rightarrow_\beta^* \langle M'_L, M'_R \rangle
\end{array}$$

1 IH 2 β^* -under 3 IH 4 β^* -under 5 Beta-Multi-Trans

$$\boxed{\text{CASE}} \quad \frac{\frac{M \Rightarrow_\beta M'}{M \cdot L \Rightarrow_\beta M' \cdot L} \text{ [Parallel-ProjL-Under]}}{M \cdot L \rightarrow_\beta^* M' \cdot L}$$

$$\begin{array}{c}
M \Rightarrow_\beta M' \\
\overline{\overline{\longrightarrow}} \text{1} \\
M \rightarrow_\beta^* M' \\
\overline{\overline{\longrightarrow}} \text{2} \\
M \cdot L \rightarrow_\beta^* M' \cdot L
\end{array}$$

1 IH 2 β^* -under

$$\boxed{\text{CASE}} \quad \frac{\frac{M \Rightarrow_\beta M'}{M \cdot R \Rightarrow_\beta M' \cdot R} \text{ [Parallel-ProjR-Under]}}{M \cdot R \rightarrow_\beta^* M' \cdot R}$$

$$\begin{array}{c}
M \Rightarrow_\beta M' \\
\overline{\overline{\longrightarrow}} \text{1} \\
M \rightarrow_\beta^* M' \\
\overline{\overline{\longrightarrow}} \text{2} \\
M \cdot R \rightarrow_\beta^* M' \cdot R
\end{array}$$

1 IH 2 β^* -under

$$\boxed{\text{CASE}} \quad \frac{\frac{M_L \Rightarrow_\beta M'_L}{\langle M_L, M_R \rangle \cdot L \Rightarrow_\beta M'_L} \text{ [Parallel-ProjL]}}{\langle M_L, M_R \rangle \cdot L \rightarrow_\beta^* M'_L}$$

$$\begin{array}{c}
\frac{\langle M_L, M_R \rangle \cdot L \rightarrow_\beta M_L \text{ 1} \quad \frac{M_L \Rightarrow_\beta M'_L}{M_L \rightarrow_\beta^* M'_L} \text{ 2}}{\langle M_L, M_R \rangle \cdot L \rightarrow_\beta^* M'_L} \text{ 3}
\end{array}$$

1 ProjL-beta 2 IH 3 Beta-Multi-Cons

$$\begin{array}{c}
 \boxed{\text{CASE}} \quad \frac{M_R \Rightarrow_\beta M'_R}{\langle M_L, M_R \rangle \cdot R \Rightarrow_\beta M'_R} \text{ [Parallel-ProjR]} \\
 \hline
 \langle M_L, M_R \rangle \cdot R \rightarrow_\beta^* M'_R \\
 \\
 \frac{\frac{\langle M_L, M_R \rangle \cdot R \rightarrow_\beta M_R}{\langle M_L, M_R \rangle \cdot R \rightarrow_\beta^* M'_R} \quad \frac{M_R \Rightarrow_\beta M'_R}{M_R \rightarrow_\beta^* M'_R} \quad \begin{array}{l} 1 \quad 2 \end{array}}{\langle M_L, M_R \rangle \cdot R \rightarrow_\beta^* M'_R} 3
 \end{array}$$

1 ProjR-beta 2 IH 3 Beta-Multi-Cons

$$\begin{array}{c}
 \boxed{\text{CASE}} \quad \frac{M \Rightarrow_\beta M'}{\lambda x \Rightarrow M \Rightarrow_\beta \lambda x \Rightarrow M'} \text{ [Parallel-Lam]} \\
 \hline
 \lambda x \Rightarrow M \rightarrow_\beta^* \lambda x \Rightarrow M' \\
 \\
 \frac{\frac{M \Rightarrow_\beta M'}{M \rightarrow_\beta^* M'} \quad 1}{\lambda x \Rightarrow M \rightarrow_\beta^* \lambda x \Rightarrow M'} 2
 \end{array}$$

1 IH 2 β^* -under

$$\begin{array}{c}
 \boxed{\text{CASE}} \quad \frac{M \Rightarrow_\beta M' \quad N \Rightarrow_\beta N'}{M N \Rightarrow_\beta M' N'} \text{ [Parallel-App-Under]} \\
 \hline
 M N \rightarrow_\beta^* M' N' \\
 \\
 \frac{\frac{\frac{M \Rightarrow_\beta M'}{M \rightarrow_\beta^* M'} \quad 1}{M N \rightarrow_\beta^* M' N} \quad 2 \quad \frac{\frac{N \Rightarrow_\beta N'}{N \rightarrow_\beta^* N'} \quad 3}{M' N \rightarrow_\beta^* M' N'} 4}{M N \rightarrow_\beta^* M' N'} 5
 \end{array}$$

1 IH 2 β^* -under 3 IH 4 β^* -under 5 Beta-Multi-Trans

$$\boxed{\text{CASE}} \quad \frac{\frac{M \Rightarrow_{\beta} M' \quad N \Rightarrow_{\beta} N'}{\lambda x \Rightarrow M N \Rightarrow_{\beta} [N'/x](M')} \text{ [Parallel-App]}}{\lambda x \Rightarrow M N \rightarrow_{\beta}^* [N'/x](M')}$$

$$\frac{\frac{M \Rightarrow_{\beta} M' \quad N \Rightarrow_{\beta} N'}{\begin{array}{c} \xrightarrow{\quad} 1 \\ M \rightarrow_{\beta}^* M' \end{array}} \quad \begin{array}{c} \xrightarrow{\quad} 2 \\ N \rightarrow_{\beta}^* N' \end{array}}{\lambda x \Rightarrow M N \rightarrow_{\beta}^* [N'/x](M')} 3$$

1 IH 2 IH 3 β^* -under/ β and Beta-Multi-Trans

$$\boxed{\text{CASE}} \quad \frac{\frac{M \Rightarrow_{\beta} M'}{L \cdot M \Rightarrow_{\beta} L \cdot M'} \text{ [Parallel-InL]}}{L \cdot M \rightarrow_{\beta}^* L \cdot M'}$$

$$\frac{\frac{M \Rightarrow_{\beta} M'}{\begin{array}{c} \xrightarrow{\quad} 1 \\ M \rightarrow_{\beta}^* M' \end{array}}}{L \cdot M \rightarrow_{\beta}^* L \cdot M'} 2$$

1 IH 2 β^* -under

$$\boxed{\text{CASE}} \quad \frac{\frac{M \Rightarrow_{\beta} M'}{R \cdot M \Rightarrow_{\beta} R \cdot M'} \text{ [Parallel-InR]}}{R \cdot M \rightarrow_{\beta}^* R \cdot M'}$$

$$\frac{\frac{M \Rightarrow_{\beta} M'}{\begin{array}{c} \xrightarrow{\quad} 1 \\ M \rightarrow_{\beta}^* M' \end{array}}}{R \cdot M \rightarrow_{\beta}^* R \cdot M'} 2$$

1 IH 2 β^* -under

$$\boxed{\text{CASE}} \quad \frac{\frac{M \Rightarrow_{\beta} M' \quad M_L \Rightarrow_{\beta} M'_L \quad M_R \Rightarrow_{\beta} M'_R}{\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} \text{case } M' (x_L \Rightarrow M'_L \mid x_R \Rightarrow M'_R)} \text{ [Parallel-Case-Under]}}{\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta}^* \text{case } M' (x_L \Rightarrow M'_L \mid x_R \Rightarrow M'_R)}$$

$$\frac{\frac{M \Rightarrow_{\beta} M' \quad \frac{}{M \rightarrow_{\beta}^* M'} \quad 1}{M \Rightarrow_{\beta} M'} \quad \frac{M_L \Rightarrow_{\beta} M'_L \quad \frac{}{M_L \rightarrow_{\beta}^* M'_L} \quad 2}{M_L \Rightarrow_{\beta} M'_L} \quad \frac{M_R \Rightarrow_{\beta} M'_R \quad \frac{}{M_R \rightarrow_{\beta}^* M'_R} \quad 3}{M_R \Rightarrow_{\beta} M'_R}}{\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta}^* \text{case } M' (x_L \Rightarrow M'_L \mid x_R \Rightarrow M'_R)} \quad 4$$

1 IH 2 IH 3 IH 4 β^* -under and Beta-Multi-Trans

$$\boxed{\text{CASE}} \quad \frac{\frac{M \Rightarrow_{\beta} M' \quad M_L \Rightarrow_{\beta} M'_L}{\text{case } L \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} [M'/x_L](M'_L)} \quad [\text{Parallel-Case-InL}]}{\text{case } L \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta}^* [M'/x_L](M'_L)}$$

$$\frac{\frac{M \Rightarrow_{\beta} M' \quad \frac{}{M \rightarrow_{\beta}^* M'} \quad 1}{M \Rightarrow_{\beta} M'} \quad \frac{M_L \Rightarrow_{\beta} M'_L \quad \frac{}{M_L \rightarrow_{\beta}^* M'_L} \quad 2}{M_L \Rightarrow_{\beta} M'_L}}{\text{case } L \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta}^* [M'/x_L](M'_L)} \quad 3$$

1 IH 2 IH 3 β^* -under/ β and Beta-Multi-Trans

$$\boxed{\text{CASE}} \quad \frac{\frac{M \Rightarrow_{\beta} M' \quad M_R \Rightarrow_{\beta} M'_R}{\text{case } R \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} [M'/x_R](M'_R)} \quad [\text{Parallel-Case-InR}]}{\text{case } R \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta}^* [M'/x_R](M'_R)}$$

$$\frac{\frac{M \Rightarrow_{\beta} M' \quad \frac{}{M \rightarrow_{\beta}^* M'} \quad 1}{M \Rightarrow_{\beta} M'} \quad \frac{M_R \Rightarrow_{\beta} M'_R \quad \frac{}{M_R \rightarrow_{\beta}^* M'_R} \quad 2}{M_R \Rightarrow_{\beta} M'_R}}{\text{case } R \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta}^* [M'/x_R](M'_R)} \quad 3$$

1 IH 2 IH 3 β^* -under/ β and Beta-Multi-Trans

$$\boxed{\text{CASE}} \quad \frac{\overline{\text{zero} \Rightarrow_{\beta} \text{zero}} \quad [\text{Parallel-Zero}]}{\text{zero} \rightarrow_{\beta}^* \text{zero}}$$

$$\frac{}{\text{zero} \rightarrow_{\beta}^* \text{zero}} \quad 1$$

1 Beta-Multi-Ref

$$\boxed{\text{CASE}} \quad \frac{M \Rightarrow_{\beta} M'}{\text{succ } M \Rightarrow_{\beta} \text{succ } M'} \quad [\text{Parallel-Succ}]$$

$$\frac{}{\text{succ } M \rightarrow_{\beta}^* \text{succ } M'}$$

$$\frac{\frac{M \Rightarrow_{\beta} M'}{M \rightarrow_{\beta}^* M'} \quad 1}{\text{succ } M \rightarrow_{\beta}^* \text{succ } M'} \quad 2$$

1 IH 2 β^* -under

$$\boxed{\text{CASE}} \quad \frac{M \Rightarrow_{\beta} M' \quad M_0 \Rightarrow_{\beta} M'_0 \quad M_S \Rightarrow_{\beta} M'_S}{\text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_{\beta} \text{natrec } M' (M'_0 \mid x, y \Rightarrow M'_S)} \quad [\text{Parallel-NatRec-Under}]$$

$$\frac{}{\text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \rightarrow_{\beta}^* \text{natrec } M' (M'_0 \mid x, y \Rightarrow M'_S)}$$

$$\frac{\frac{M \Rightarrow_{\beta} M'}{M \rightarrow_{\beta}^* M'} \quad 1 \quad \frac{M_0 \Rightarrow_{\beta} M'_0}{M_0 \rightarrow_{\beta}^* M'_0} \quad 2 \quad \frac{M_S \Rightarrow_{\beta} M'_S}{M_S \rightarrow_{\beta}^* M'_S} \quad 3}{\text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \rightarrow_{\beta}^* \text{natrec } M' (M'_0 \mid x, y \Rightarrow M'_S)} \quad 4$$

1 IH 2 IH 3 IH 4 β^* -under and Beta-Multi-Trans

$$\boxed{\text{CASE}} \quad \frac{M_0 \Rightarrow_{\beta} M'_0}{\text{natrec zero } (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_{\beta} M'_0} \quad [\text{Parallel-NatRec-Zero}]$$

$$\frac{}{\text{natrec zero } (M_0 \mid x, y \Rightarrow M_S) \rightarrow_{\beta}^* M'_0}$$

$$\frac{\text{natrec zero } (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_{\beta} M'_0 \quad 1 \quad \frac{M_0 \Rightarrow_{\beta} M'_0}{M_0 \rightarrow_{\beta}^* M'_0} \quad 2}{\text{natrec zero } (M_0 \mid x, y \Rightarrow M_S) \rightarrow_{\beta}^* M'_0} \quad 3$$

1 NatRec-Zero-beta 2 IH 3 Beta-Multi-Cons

$$\boxed{\text{CASE}} \quad \frac{M \Rightarrow_{\beta} M' \quad M_0 \Rightarrow_{\beta} M'_0 \quad M_S \Rightarrow_{\beta} M'_S}{\text{natrec succ } M (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_{\beta} [M'/x, R'/y](M'_S)} \quad [\text{Parallel-NatRec-Succ}]$$

$$\frac{}{\text{natrec succ } M (M_0 \mid x, y \Rightarrow M_S) \rightarrow_{\beta}^* [M'/x, R'/y](M'_S)}$$

Define $R' \triangleq \text{natrec } M' (M'_0 \mid x, y \Rightarrow M'_S)$.

$$\frac{\frac{M \Rightarrow_{\beta} M'}{M \rightarrow_{\beta}^* M'} \quad 1 \quad \frac{M_0 \Rightarrow_{\beta} M'_0}{M_0 \rightarrow_{\beta}^* M'_0} \quad 2 \quad \frac{M_S \Rightarrow_{\beta} M'_S}{M_S \rightarrow_{\beta}^* M'_S} \quad 3}{\text{natrec succ } M (M_0 \mid x, y \Rightarrow M_S) \rightarrow_{\beta}^* [M'/x, R'/y](M'_S)} \quad 4$$

1 IH 2 IH 3 IH 4 β^* -under/ β and Beta-Multi-Trans □

Usual-Multi-if-Parallel-Multi

$$\frac{M \Rightarrow_{\beta}^* M'}{M \rightarrow_{\beta}^* M'}$$

Proof by appealing to [Usual-Multi-if-Parallel]. □

Next, define the parallel β -reduction for substitutions:

$$\gamma \Rightarrow_{\beta} \gamma' \iff \forall x. \gamma(x) \Rightarrow_{\beta} \gamma'(x)$$

Parallel-Hat-Subst

$$\frac{\gamma \Rightarrow_{\beta} \gamma' \quad M \Rightarrow_{\beta} M'}{\widehat{\gamma}(M) \Rightarrow_{\beta} \widehat{\gamma'}(M')}$$

Proof by induction on $M \Rightarrow_{\beta} M'$:

CASE $\frac{\gamma \Rightarrow_{\beta} \gamma' \quad \overline{x \Rightarrow_{\beta} x} \text{ [Parallel-Var]}}{\widehat{\gamma}(x) \Rightarrow_{\beta} \widehat{\gamma'}(x)}$

$$\frac{\frac{\gamma \Rightarrow_{\beta} \gamma'}{\gamma(x) \Rightarrow_{\beta} \gamma'(x)} \quad 1}{\widehat{\gamma}(x) \Rightarrow_{\beta} \widehat{\gamma'}(x)} \quad 2$$

1 definition of $\gamma \Rightarrow_{\beta} \gamma'$ 2 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$

CASE $\frac{\gamma \Rightarrow_{\beta} \gamma' \quad \overline{\langle \rangle \Rightarrow_{\beta} \langle \rangle} \text{ [Parallel-Empty]}}{\widehat{\gamma}(\langle \rangle) \Rightarrow_{\beta} \widehat{\gamma'}(\langle \rangle)}$

$$\frac{\overline{\langle \rangle \Rightarrow_{\beta} \langle \rangle} \quad 1}{\widehat{\gamma}(\langle \rangle) \Rightarrow_{\beta} \widehat{\gamma'}(\langle \rangle)} \quad 2$$

1 Parallel-Empty 2 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$

$$\boxed{\text{CASE}} \quad \frac{\gamma \Rightarrow_{\beta} \gamma' \quad \frac{M_L \Rightarrow_{\beta} M'_L \quad M_R \Rightarrow_{\beta} M'_R}{\langle M_L, M_R \rangle \Rightarrow_{\beta} \langle M'_L, M'_R \rangle} [\text{Parallel-Pair}]}{\widehat{\gamma}(\langle M_L, M_R \rangle) \Rightarrow_{\beta} \widehat{\gamma'}(\langle M'_L, M'_R \rangle)}$$

$$\frac{\frac{\gamma \Rightarrow_{\beta} \gamma' \quad M_L \Rightarrow_{\beta} M'_L}{\widehat{\gamma}(M_L) \Rightarrow_{\beta} \widehat{\gamma'}(M'_L)} \quad 1 \quad \frac{\gamma \Rightarrow_{\beta} \gamma' \quad M_R \Rightarrow_{\beta} M'_R}{\widehat{\gamma}(M_R) \Rightarrow_{\beta} \widehat{\gamma'}(M'_R)} \quad 2}{\frac{\langle \widehat{\gamma}(M_L), \widehat{\gamma}(M_R) \rangle \Rightarrow_{\beta} \langle \widehat{\gamma'}(M'_L), \widehat{\gamma'}(M'_R) \rangle}{\widehat{\gamma}(\langle M_L, M_R \rangle) \Rightarrow_{\beta} \widehat{\gamma'}(\langle M'_L, M'_R \rangle)} \quad 3} \quad 4$$

1 IH 2 IH 3 Parallel-Pair 4 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$

$$\boxed{\text{CASE}} \quad \frac{\gamma \Rightarrow_{\beta} \gamma' \quad \frac{M \Rightarrow_{\beta} M'}{M \cdot L \Rightarrow_{\beta} M' \cdot L} [\text{Parallel-ProjL-Under}]}{\widehat{\gamma}(M \cdot L) \Rightarrow_{\beta} \widehat{\gamma'}(M' \cdot L)}$$

$$\frac{\frac{\gamma \Rightarrow_{\beta} \gamma' \quad M \Rightarrow_{\beta} M'}{\widehat{\gamma}(M) \Rightarrow_{\beta} \widehat{\gamma'}(M')} \quad 1}{\frac{\widehat{\gamma}(M) \cdot L \Rightarrow_{\beta} \widehat{\gamma'}(M') \cdot L}{\widehat{\gamma}(M \cdot L) \Rightarrow_{\beta} \widehat{\gamma'}(M' \cdot L)} \quad 2} \quad 3$$

1 IH 2 Parallel-ProjL-Under 3 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$

$$\boxed{\text{CASE}} \quad \frac{\gamma \Rightarrow_{\beta} \gamma' \quad \frac{M \Rightarrow_{\beta} M'}{M \cdot R \Rightarrow_{\beta} M' \cdot R} [\text{Parallel-ProjR-Under}]}{\widehat{\gamma}(M \cdot R) \Rightarrow_{\beta} \widehat{\gamma'}(M' \cdot R)}$$

$$\begin{array}{c}
\frac{\gamma \Rightarrow_{\beta} \gamma' \quad M \Rightarrow_{\beta} M'}{\widehat{\gamma}(M) \Rightarrow_{\beta} \widehat{\gamma'}(M')} \quad 1 \\
\frac{\widehat{\gamma}(M) \Rightarrow_{\beta} \widehat{\gamma'}(M')}{\widehat{\gamma}(M) \cdot R \Rightarrow_{\beta} \widehat{\gamma'}(M') \cdot R} \quad 2 \\
\frac{\widehat{\gamma}(M) \cdot R \Rightarrow_{\beta} \widehat{\gamma'}(M') \cdot R}{\widehat{\gamma}(M \cdot R) \Rightarrow_{\beta} \widehat{\gamma'}(M' \cdot R)} \quad 3
\end{array}$$

1 IH 2 Parallel-ProjR-Under 3 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$

$$\boxed{\text{CASE}} \quad \frac{\gamma \Rightarrow_{\beta} \gamma' \quad \frac{M_L \Rightarrow_{\beta} M'_L}{\langle M_L, M_R \rangle \cdot L \Rightarrow_{\beta} M'_L} \text{ [Parallel-ProjL]}}{\widehat{\gamma}(\langle M_L, M_R \rangle \cdot L) \Rightarrow_{\beta} \widehat{\gamma'}(M'_L)}$$

$$\begin{array}{c}
\frac{\gamma \Rightarrow_{\beta} \gamma' \quad M_L \Rightarrow_{\beta} M'_L}{\widehat{\gamma}(M_L) \Rightarrow_{\beta} \widehat{\gamma'}(M'_L)} \quad 1 \\
\frac{\widehat{\gamma}(M_L) \Rightarrow_{\beta} \widehat{\gamma'}(M'_L)}{\langle \widehat{\gamma}(M_L), \widehat{\gamma}(M_R) \rangle \cdot L \Rightarrow_{\beta} \widehat{\gamma'}(M'_L)} \quad 2 \\
\frac{\langle \widehat{\gamma}(M_L), \widehat{\gamma}(M_R) \rangle \cdot L \Rightarrow_{\beta} \widehat{\gamma'}(M'_L)}{\widehat{\gamma}(\langle M_L, M_R \rangle \cdot L) \Rightarrow_{\beta} \widehat{\gamma'}(M'_L)} \quad 3
\end{array}$$

1 IH 2 Parallel-ProjL 3 fold $\widehat{\gamma}$

$$\boxed{\text{CASE}} \quad \frac{\gamma \Rightarrow_{\beta} \gamma' \quad \frac{M_R \Rightarrow_{\beta} M'_R}{\langle M_L, M_R \rangle \cdot R \Rightarrow_{\beta} M'_R} \text{ [Parallel-ProjR]}}{\widehat{\gamma}(\langle M_L, M_R \rangle \cdot R) \Rightarrow_{\beta} \widehat{\gamma'}(M'_R)}$$

$$\begin{array}{c}
\frac{\gamma \Rightarrow_{\beta} \gamma' \quad M_R \Rightarrow_{\beta} M'_R}{\widehat{\gamma}(M_R) \Rightarrow_{\beta} \widehat{\gamma'}(M'_R)} \quad 1 \\
\frac{\widehat{\gamma}(M_R) \Rightarrow_{\beta} \widehat{\gamma'}(M'_R)}{\langle \widehat{\gamma}(M_L), \widehat{\gamma}(M_R) \rangle \cdot R \Rightarrow_{\beta} \widehat{\gamma'}(M'_R)} \quad 2 \\
\frac{\langle \widehat{\gamma}(M_L), \widehat{\gamma}(M_R) \rangle \cdot R \Rightarrow_{\beta} \widehat{\gamma'}(M'_R)}{\widehat{\gamma}(\langle M_L, M_R \rangle \cdot R) \Rightarrow_{\beta} \widehat{\gamma'}(M'_R)} \quad 3
\end{array}$$

1 IH 2 Parallel-ProjR 3 fold $\widehat{\gamma}$

$$\boxed{\text{CASE}} \quad \frac{\gamma \Rightarrow_{\beta} \gamma' \quad \frac{M \Rightarrow_{\beta} M'}{\lambda x \Rightarrow M \Rightarrow_{\beta} \lambda x \Rightarrow M'} \text{ [Parallel-Lam]}}{\widehat{\gamma}(\lambda x \Rightarrow M) \Rightarrow_{\beta} \widehat{\gamma'}(\lambda x \Rightarrow M')}$$

$$\begin{array}{c}
\frac{\gamma \Rightarrow_{\beta} \gamma' \quad M \Rightarrow_{\beta} M'}{\widehat{\gamma}(M) \Rightarrow_{\beta} \widehat{\gamma'}(M')} \quad 1 \\
\frac{\lambda x \Rightarrow \widehat{\gamma}(M) \Rightarrow_{\beta} \lambda x \Rightarrow \widehat{\gamma'}(M')}{\widehat{\gamma}(\lambda x \Rightarrow M) \Rightarrow_{\beta} \widehat{\gamma'}(\lambda x \Rightarrow M')} \quad 2 \\
\quad 3
\end{array}$$

1 IH 2 Parallel-Lam 3 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$

$$\begin{array}{c}
\text{CASE} \quad \frac{\gamma \Rightarrow_{\beta} \gamma' \quad \frac{M \Rightarrow_{\beta} M' \quad N \Rightarrow_{\beta} N'}{M N \Rightarrow_{\beta} M' N'} \text{ [Parallel-App-Under]}}{\widehat{\gamma}(M N) \Rightarrow_{\beta} \widehat{\gamma'}(M' N')}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\gamma \Rightarrow_{\beta} \gamma' \quad M \Rightarrow_{\beta} M'}{\widehat{\gamma}(M) \Rightarrow_{\beta} \widehat{\gamma'}(M')} \quad 1 \quad \frac{\gamma \Rightarrow_{\beta} \gamma' \quad N \Rightarrow_{\beta} N'}{\widehat{\gamma}(N) \Rightarrow_{\beta} \widehat{\gamma'}(N')} \quad 2}{\widehat{\gamma}(M) \widehat{\gamma}(N) \Rightarrow_{\beta} \widehat{\gamma'}(M') \widehat{\gamma'}(N')} \quad 3 \\
\frac{\widehat{\gamma}(M) \widehat{\gamma}(N) \Rightarrow_{\beta} \widehat{\gamma'}(M') \widehat{\gamma'}(N')}{\widehat{\gamma}(M N) \Rightarrow_{\beta} \widehat{\gamma'}(M' N')} \quad 4
\end{array}$$

1 IH 2 IH 3 Parallel-App-Under 4 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$

$$\begin{array}{c}
\text{CASE} \quad \frac{\gamma \Rightarrow_{\beta} \gamma' \quad \frac{M \Rightarrow_{\beta} M' \quad N \Rightarrow_{\beta} N'}{\lambda x \Rightarrow M N \Rightarrow_{\beta} [N'/x](M')} \text{ [Parallel-App]}}{\widehat{\gamma}(\lambda x \Rightarrow M N) \Rightarrow_{\beta} \widehat{\gamma'}([N'/x](M'))}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\gamma \Rightarrow_{\beta} \gamma' \quad M \Rightarrow_{\beta} M'}{\widehat{\gamma}(M) \Rightarrow_{\beta} \widehat{\gamma'}(M')} \quad 1 \quad \frac{\gamma \Rightarrow_{\beta} \gamma' \quad N \Rightarrow_{\beta} N'}{\widehat{\gamma}(N) \Rightarrow_{\beta} \widehat{\gamma'}(N')} \quad 2}{\lambda x \Rightarrow \widehat{\gamma}(M) \widehat{\gamma}(N) \Rightarrow_{\beta} [\widehat{\gamma'}(N')/x](\widehat{\gamma'}(M'))} \quad 3 \\
\frac{\lambda x \Rightarrow \widehat{\gamma}(M) \widehat{\gamma}(N) \Rightarrow_{\beta} [\widehat{\gamma'}(N')/x](\widehat{\gamma'}(M'))}{\widehat{\gamma}(\lambda x \Rightarrow M N) \Rightarrow_{\beta} \widehat{\gamma'}([N'/x](M'))} \quad 4
\end{array}$$

1 IH 2 IH 3 Parallel-App 4 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$

$$\begin{array}{c}
\text{CASE} \quad \frac{\gamma \Rightarrow_{\beta} \gamma' \quad \frac{M \Rightarrow_{\beta} M'}{L \cdot M \Rightarrow_{\beta} L \cdot M'} \text{ [Parallel-InL]}}{\widehat{\gamma}(L \cdot M) \Rightarrow_{\beta} \widehat{\gamma'}(L \cdot M')}
\end{array}$$

$$\begin{array}{c}
\frac{\gamma \Rightarrow_\beta \gamma' \quad M \Rightarrow_\beta M'}{\widehat{\gamma}(M) \Rightarrow_\beta \widehat{\gamma'}(M')} \xrightarrow{1} \\
\frac{\widehat{\gamma}(M) \Rightarrow_\beta \widehat{\gamma'}(M')}{L \cdot \widehat{\gamma}(M) \Rightarrow_\beta L \cdot \widehat{\gamma'}(M')} \xrightarrow{2} \\
\frac{L \cdot \widehat{\gamma}(M) \Rightarrow_\beta L \cdot \widehat{\gamma'}(M')}{\widehat{\gamma}(L \cdot M) \Rightarrow_\beta \widehat{\gamma'}(L \cdot M')} \xrightarrow{3}
\end{array}$$

1 IH 2 Parallel-InL 3 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$

$$\boxed{\text{CASE}} \quad \frac{\gamma \Rightarrow_\beta \gamma' \quad \frac{M \Rightarrow_\beta M'}{R \cdot M \Rightarrow_\beta R \cdot M'} [\text{Parallel-InR}]}{\widehat{\gamma}(R \cdot M) \Rightarrow_\beta \widehat{\gamma'}(R \cdot M')}$$

$$\begin{array}{c}
\frac{\gamma \Rightarrow_\beta \gamma' \quad M \Rightarrow_\beta M'}{\widehat{\gamma}(M) \Rightarrow_\beta \widehat{\gamma'}(M')} \xrightarrow{1} \\
\frac{\widehat{\gamma}(M) \Rightarrow_\beta \widehat{\gamma'}(M')}{R \cdot \widehat{\gamma}(M) \Rightarrow_\beta R \cdot \widehat{\gamma'}(M')} \xrightarrow{2} \\
\frac{R \cdot \widehat{\gamma}(M) \Rightarrow_\beta R \cdot \widehat{\gamma'}(M')}{\widehat{\gamma}(R \cdot M) \Rightarrow_\beta \widehat{\gamma'}(R \cdot M')} \xrightarrow{3}
\end{array}$$

1 IH 2 Parallel-InR 3 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$

$$\boxed{\text{CASE}} \quad \frac{\gamma \Rightarrow_\beta \gamma' \quad \frac{M \Rightarrow_\beta M' \quad M_L \Rightarrow_\beta M'_L \quad M_R \Rightarrow_\beta M'_R}{\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_\beta \text{case } M' (x_L \Rightarrow M'_L \mid x_R \Rightarrow M'_R)} [\text{Parallel-Case-Under}]}{\widehat{\gamma}(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \Rightarrow_\beta \widehat{\gamma'}(\text{case } M' (x_L \Rightarrow M'_L \mid x_R \Rightarrow M'_R))}$$

$$\begin{array}{c}
\frac{\gamma \Rightarrow_\beta \gamma' \quad M \Rightarrow_\beta M'}{\widehat{\gamma}(M) \Rightarrow_\beta \widehat{\gamma'}(M')} \xrightarrow{1} \quad \frac{\gamma \Rightarrow_\beta \gamma' \quad M_L \Rightarrow_\beta M'_L}{\widehat{\gamma}(M_L) \Rightarrow_\beta \widehat{\gamma'}(M'_L)} \xrightarrow{2} \quad \frac{\gamma \Rightarrow_\beta \gamma' \quad M_R \Rightarrow_\beta M'_R}{\widehat{\gamma}(M_R) \Rightarrow_\beta \widehat{\gamma'}(M'_R)} \xrightarrow{3} \\
\frac{\text{case } \widehat{\gamma}(M) (x_L \Rightarrow \widehat{\gamma}(M_L) \mid x_R \Rightarrow \widehat{\gamma}(M_R)) \Rightarrow_\beta \text{case } \widehat{\gamma'}(M') (x_L \Rightarrow \widehat{\gamma'}(M'_L) \mid x_R \Rightarrow \widehat{\gamma'}(M'_R))}{\widehat{\gamma}(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \Rightarrow_\beta \widehat{\gamma'}(\text{case } M' (x_L \Rightarrow M'_L \mid x_R \Rightarrow M'_R))} \xrightarrow{4} \\
\widehat{\gamma}(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \Rightarrow_\beta \widehat{\gamma'}(\text{case } M' (x_L \Rightarrow M'_L \mid x_R \Rightarrow M'_R)) \xrightarrow{5}
\end{array}$$

1 IH 2 IH 3 IH 4 Parallel-Case-Under 5 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$

$$\boxed{\text{CASE}} \quad \frac{\gamma \Rightarrow_\beta \gamma' \quad \frac{M \Rightarrow_\beta M' \quad M_L \Rightarrow_\beta M'_L}{\text{case } L \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_\beta [M'/x_L](M'_L)} [\text{Parallel-Case-InL}]}{\widehat{\gamma}(\text{case } L \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \Rightarrow_\beta \widehat{\gamma'}([M'/x_L](M'_L))}$$

$$\begin{array}{c}
\frac{\gamma \Rightarrow_\beta \gamma' \quad M \Rightarrow_\beta M'}{\widehat{\gamma}(M) \Rightarrow_\beta \widehat{\gamma'}(M')} \quad 1 \quad \frac{\gamma \Rightarrow_\beta \gamma' \quad M_L \Rightarrow_\beta M'_L}{\widehat{\gamma}(M_L) \Rightarrow_\beta \widehat{\gamma'}(M'_L)} \quad 2 \\
\hline
\text{case } L \cdot \widehat{\gamma}(M) \ (x_L \Rightarrow \widehat{\gamma}(M_L) \mid x_R \Rightarrow \widehat{\gamma}(M_R)) \Rightarrow_\beta [\widehat{\gamma'}(M')/x_L](\widehat{\gamma'}(M'_L)) \quad 3 \\
\hline
\widehat{\gamma}(\text{case } L \cdot M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \Rightarrow_\beta \widehat{\gamma'}([M'/x_L](M'_L)) \quad 4
\end{array}$$

1 IH 2 IH 3 Parallel-Case-InL 4 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$

$$\begin{array}{c}
\text{CASE} \quad \frac{\gamma \Rightarrow_\beta \gamma' \quad \frac{M \Rightarrow_\beta M' \quad M_R \Rightarrow_\beta M'_R}{\text{case } R \cdot M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_\beta [M'/x_R](M'_R)} \text{ [Parallel-Case-InR]} \\
\hline
\widehat{\gamma}(\text{case } R \cdot M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \Rightarrow_\beta \widehat{\gamma'}([M'/x_R](M'_R)) \\
\hline
\frac{\gamma \Rightarrow_\beta \gamma' \quad M \Rightarrow_\beta M' \quad \frac{\gamma \Rightarrow_\beta \gamma' \quad M_R \Rightarrow_\beta M'_R}{\widehat{\gamma}(M_R) \Rightarrow_\beta \widehat{\gamma'}(M'_R)} \quad 1 \quad \frac{\gamma \Rightarrow_\beta \gamma' \quad M_R \Rightarrow_\beta M'_R}{\widehat{\gamma}(M_R) \Rightarrow_\beta \widehat{\gamma'}(M'_R)} \quad 2 \\
\hline
\text{case } R \cdot \widehat{\gamma}(M) \ (x_L \Rightarrow \widehat{\gamma}(M_L) \mid x_R \Rightarrow \widehat{\gamma}(M_R)) \Rightarrow_\beta [\widehat{\gamma'}(M')/x_R](\widehat{\gamma'}(M'_R)) \quad 3 \\
\hline
\widehat{\gamma}(\text{case } R \cdot M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \Rightarrow_\beta \widehat{\gamma'}([M'/x_R](M'_R)) \quad 4
\end{array}$$

1 IH 2 IH 3 Parallel-Case-InR 4 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$

$$\begin{array}{c}
\text{CASE} \quad \frac{\gamma \Rightarrow_\beta \gamma' \quad \overline{\text{zero} \Rightarrow_\beta \text{zero}} \text{ [Parallel-Zero]} \\
\hline
\widehat{\gamma}(\text{zero}) \Rightarrow_\beta \widehat{\gamma'}(\text{zero})
\end{array}$$

$$\frac{\overline{\text{zero} \Rightarrow_\beta \text{zero}} \quad 1}{\widehat{\gamma}(\text{zero}) \Rightarrow_\beta \widehat{\gamma'}(\text{zero})} \quad 2$$

1 Parallel-Zero 2 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$

$$\begin{array}{c}
\text{CASE} \quad \frac{\gamma \Rightarrow_\beta \gamma' \quad \frac{M \Rightarrow_\beta M'}{\text{succ } M \Rightarrow_\beta \text{succ } M'} \text{ [Parallel-Succ]} \\
\hline
\widehat{\gamma}(\text{succ } M) \Rightarrow_\beta \widehat{\gamma'}(\text{succ } M')
\end{array}$$

$$\begin{array}{c}
\frac{\gamma \Rightarrow_{\beta} \gamma' \quad M \Rightarrow_{\beta} M'}{\widehat{\gamma}(M) \Rightarrow_{\beta} \widehat{\gamma'}(M')} \text{ 1} \\
\frac{\widehat{\gamma}(M) \Rightarrow_{\beta} \widehat{\gamma'}(M')}{\text{succ } \widehat{\gamma}(M) \Rightarrow_{\beta} \text{succ } \widehat{\gamma'}(M')} \text{ 2} \\
\frac{\text{succ } \widehat{\gamma}(M) \Rightarrow_{\beta} \text{succ } \widehat{\gamma'}(M')}{\widehat{\gamma}(\text{succ } M) \Rightarrow_{\beta} \widehat{\gamma'}(\text{succ } M')} \text{ 3}
\end{array}$$

1 IH 2 Parallel-Succ 3 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$

$$\begin{array}{c}
\text{CASE } \frac{\gamma \Rightarrow_{\beta} \gamma' \quad \frac{M \Rightarrow_{\beta} M' \quad M_0 \Rightarrow_{\beta} M'_0 \quad M_S \Rightarrow_{\beta} M'_S}{\text{natrec } M (M_0 | x, y \Rightarrow M_S) \Rightarrow_{\beta} \text{natrec } M' (M'_0 | x, y \Rightarrow M'_S)} \text{ [Parallel-NatRec-Under]}}{\widehat{\gamma}(\text{natrec } M (M_0 | x, y \Rightarrow M_S)) \Rightarrow_{\beta} \widehat{\gamma'}(\text{natrec } M' (M'_0 | x, y \Rightarrow M'_S))} \\
\\
\frac{\frac{\gamma \Rightarrow_{\beta} \gamma' \quad M \Rightarrow_{\beta} M'}{\widehat{\gamma}(M) \Rightarrow_{\beta} \widehat{\gamma'}(M')} \text{ 1} \quad \frac{\gamma \Rightarrow_{\beta} \gamma' \quad M_0 \Rightarrow_{\beta} M'_0}{\widehat{\gamma}(M_0) \Rightarrow_{\beta} \widehat{\gamma'}(M'_0)} \text{ 2} \quad \frac{\gamma \Rightarrow_{\beta} \gamma' \quad M_S \Rightarrow_{\beta} M'_S}{\widehat{\gamma}(M_S) \Rightarrow_{\beta} \widehat{\gamma'}(M'_S)} \text{ 3}}{\text{natrec } \widehat{\gamma}(M) (\widehat{\gamma}(M_0) | x, y \Rightarrow \widehat{\gamma}(M_S)) \Rightarrow_{\beta} \text{natrec } \widehat{\gamma'}(M') (\widehat{\gamma'}(M'_0) | x, y \Rightarrow \widehat{\gamma'}(M'_S))} \text{ 4} \\
\frac{\text{natrec } \widehat{\gamma}(M) (\widehat{\gamma}(M_0) | x, y \Rightarrow \widehat{\gamma}(M_S)) \Rightarrow_{\beta} \text{natrec } \widehat{\gamma'}(M') (\widehat{\gamma'}(M'_0) | x, y \Rightarrow \widehat{\gamma'}(M'_S))}{\widehat{\gamma}(\text{natrec } M (M_0 | x, y \Rightarrow M_S)) \Rightarrow_{\beta} \widehat{\gamma'}(\text{natrec } M' (M'_0 | x, y \Rightarrow M'_S))} \text{ 5}
\end{array}$$

1 IH 2 IH 3 IH 4 Parallel-NatRec-Under 5 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$

$$\begin{array}{c}
\text{CASE } \frac{\gamma \Rightarrow_{\beta} \gamma' \quad \frac{M_0 \Rightarrow_{\beta} M'_0}{\text{natrec zero } (M_0 | x, y \Rightarrow M_S) \Rightarrow_{\beta} M'_0} \text{ [Parallel-NatRec-Zero]}}{\widehat{\gamma}(\text{natrec zero } (M_0 | x, y \Rightarrow M_S)) \Rightarrow_{\beta} \widehat{\gamma'}(M'_0)} \\
\\
\frac{\frac{\gamma \Rightarrow_{\beta} \gamma' \quad M_0 \Rightarrow_{\beta} M'_0}{\widehat{\gamma}(M_0) \Rightarrow_{\beta} \widehat{\gamma'}(M'_0)} \text{ 1}}{\text{natrec zero } (\widehat{\gamma}(M_0) | x, y \Rightarrow \widehat{\gamma}(M_S)) \Rightarrow_{\beta} \widehat{\gamma'}(M'_0)} \text{ 2} \\
\frac{\text{natrec zero } (\widehat{\gamma}(M_0) | x, y \Rightarrow \widehat{\gamma}(M_S)) \Rightarrow_{\beta} \widehat{\gamma'}(M'_0)}{\widehat{\gamma}(\text{natrec zero } (M_0 | x, y \Rightarrow M_S)) \Rightarrow_{\beta} \widehat{\gamma'}(M'_0)} \text{ 3}
\end{array}$$

1 IH 2 Parallel-NatRec-Zero 3 fold $\widehat{\gamma}$

$$\begin{array}{c}
\text{CASE } \frac{\gamma \Rightarrow_{\beta} \gamma' \quad \frac{M \Rightarrow_{\beta} M' \quad M_0 \Rightarrow_{\beta} M'_0 \quad M_S \Rightarrow_{\beta} M'_S}{\text{natrec succ } M (M_0 | x, y \Rightarrow M_S) \Rightarrow_{\beta} [M'/x, R'/y](M'_S)} \text{ [Parallel-NatRec-Succ]}}{\widehat{\gamma}(\text{natrec succ } M (M_0 | x, y \Rightarrow M_S)) \Rightarrow_{\beta} \widehat{\gamma'}([M'/x, R'/y](M'_S))}
\end{array}$$

Define $R' \triangleq \text{natrec } M' (M'_0 | x, y \Rightarrow M'_S)$ and $\widehat{R}' \triangleq \widehat{\gamma'}(R')$.

$$\begin{array}{c}
\frac{\gamma \Rightarrow_\beta \gamma' \quad M \Rightarrow_\beta M'}{\widehat{\gamma}(M) \Rightarrow_\beta \widehat{\gamma'}(M')} \xrightarrow{1} \quad \frac{\gamma \Rightarrow_\beta \gamma' \quad M_0 \Rightarrow_\beta M'_0}{\widehat{\gamma}(M_0) \Rightarrow_\beta \widehat{\gamma'}(M'_0)} \xrightarrow{2} \quad \frac{\gamma \Rightarrow_\beta \gamma' \quad M_S \Rightarrow_\beta M'_S}{\widehat{\gamma}(M_S) \Rightarrow_\beta \widehat{\gamma'}(M'_S)} \xrightarrow{3} \\
\frac{\widehat{\gamma}(M) \Rightarrow_\beta \widehat{\gamma'}(M') \quad \widehat{\gamma}(M_0) \Rightarrow_\beta \widehat{\gamma'}(M'_0) \quad \widehat{\gamma}(M_S) \Rightarrow_\beta \widehat{\gamma'}(M'_S)}{\text{natrec succ } \widehat{\gamma}(M) (\widehat{\gamma}(M_0) \mid x, y \Rightarrow \widehat{\gamma}(M_S)) \Rightarrow_\beta [\widehat{\gamma'}(M')/x, \widehat{\gamma'}(M'_0)/y](\widehat{\gamma'}(M'_S))} \xrightarrow{4} \\
\frac{\text{natrec succ } \widehat{\gamma}(M) (\widehat{\gamma}(M_0) \mid x, y \Rightarrow \widehat{\gamma}(M_S)) \Rightarrow_\beta [\widehat{\gamma'}(M')/x, \widehat{\gamma'}(M'_0)/y](\widehat{\gamma'}(M'_S))}{\widehat{\gamma}(\text{natrec succ } M (M_0 \mid x, y \Rightarrow M_S)) \Rightarrow_\beta \widehat{\gamma'}([M'/x, R'/y](M'_S))} \xrightarrow{5}
\end{array}$$

1 IH 2 IH 3 IH 4 Parallel-NatRec-Succ 5 fold $\widehat{\gamma}$ and $\widehat{\gamma'}$ □

Define the maximal parallel reduct $|M|^+$ by induction on the structure of the term M . In each elimination case, distinguish whether the term being eliminated is of the corresponding introduction form:

$ M ^+$	
$ x ^+$	$\triangleq x$
$ \langle \rangle ^+$	$\triangleq \langle \rangle$
$ \langle M_L, M_R \rangle ^+$	$\triangleq \langle M_L ^+, M_R ^+ \rangle$
$ \langle M_L, M_R \rangle \cdot L ^+$	$\triangleq M_L ^+$
$ M \cdot L ^+$	$\triangleq M ^+ \cdot L$
$ \langle M_L, M_R \rangle \cdot R ^+$	$\triangleq M_R ^+$
$ M \cdot R ^+$	$\triangleq M ^+ \cdot R$
$ \lambda x \Rightarrow M ^+$	$\triangleq \lambda x \Rightarrow M ^+$
$ \lambda x \Rightarrow M N ^+$	$\triangleq [N ^+/x](M ^+)$
$ M N ^+$	$\triangleq M ^+ N ^+$
$ L \cdot M ^+$	$\triangleq L \cdot M ^+$
$ R \cdot M ^+$	$\triangleq R \cdot M ^+$
$ \text{case } L \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) ^+$	$\triangleq [M ^+/x_L](M_L ^+)$
$ \text{case } R \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) ^+$	$\triangleq [M ^+/x_R](M_R ^+)$
$ \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) ^+$	$\triangleq \text{case } M ^+ (x_L \Rightarrow M_L ^+ \mid x_R \Rightarrow M_R ^+)$
$ \text{zero} ^+$	$\triangleq \text{zero}$
$ \text{succ } M ^+$	$\triangleq \text{succ } M ^+$
$ \text{natrec zero } (M_0 \mid x, y \Rightarrow M_S) ^+$	$\triangleq M_0 ^+$
$ \text{natrec succ } M (M_0 \mid x, y \Rightarrow M_S) ^+$	$\triangleq [M ^+/x, \text{natrec } M ^+ (M_0 ^+ \mid x, y \Rightarrow M_S ^+)/y](M_S ^+)$
$ \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) ^+$	$\triangleq \text{natrec } M ^+ (M_0 ^+ \mid x, y \Rightarrow M_S ^+)$

Parallel-Triangle

$$\frac{M \Rightarrow_\beta N}{N \Rightarrow_\beta |M|^+}$$

Proof by induction on $M \Rightarrow_\beta N$:

CASE $\frac{\overline{x \Rightarrow_\beta x} \text{ [Parallel-Var]}}{x \Rightarrow_\beta |x|^+}$

$$\frac{\overline{x \Rightarrow_{\beta} x} \quad 1}{x \Rightarrow_{\beta} |x|^{+}} \quad 2$$

1 Parallel-Var 2 fold $|x|^{+}$

$$\boxed{\text{CASE}} \quad \frac{\overline{\langle \rangle \Rightarrow_{\beta} \langle \rangle} \quad [\text{Parallel-Empty}]}{\langle \rangle \Rightarrow_{\beta} |\langle \rangle|^{+}}$$

$$\frac{\overline{\langle \rangle \Rightarrow_{\beta} \langle \rangle} \quad 1}{\langle \rangle \Rightarrow_{\beta} |\langle \rangle|^{+}} \quad 2$$

1 Parallel-Empty 2 fold $|\langle \rangle|^{+}$

$$\boxed{\text{CASE}} \quad \frac{\frac{M_L \Rightarrow_{\beta} M'_L \quad M_R \Rightarrow_{\beta} M'_R}{\langle M_L, M_R \rangle \Rightarrow_{\beta} \langle M'_L, M'_R \rangle} \quad [\text{Parallel-Pair}]}{\langle M'_L, M'_R \rangle \Rightarrow_{\beta} |\langle M_L, M_R \rangle|^{+}}$$

$$\frac{\frac{M_L \Rightarrow_{\beta} M'_L}{M'_L \Rightarrow_{\beta} |M_L|^{+}} \quad 1 \quad \frac{M_R \Rightarrow_{\beta} M'_R}{M'_R \Rightarrow_{\beta} |M_R|^{+}} \quad 2}{\langle M'_L, M'_R \rangle \Rightarrow_{\beta} |\langle M_L, M_R \rangle|^{+}} \quad 3$$

$$\frac{\langle M'_L, M'_R \rangle \Rightarrow_{\beta} |\langle M_L, M_R \rangle|^{+}}{\langle M'_L, M'_R \rangle \Rightarrow_{\beta} |\langle M_L, M_R \rangle|^{+}} \quad 4$$

1 IH 2 IH 3 Parallel-Pair 4 fold $|\langle M_L, M_R \rangle|^{+}$

$$\boxed{\text{CASE}} \quad \frac{\frac{M \Rightarrow_{\beta} M'}{M \cdot L \Rightarrow_{\beta} M' \cdot L} \quad [\text{Parallel-ProjL-Under}]}{M' \cdot L \Rightarrow_{\beta} |M \cdot L|^{+}}$$

$\boxed{\text{SUBCASE}} \quad M = \langle M_L, M_R \rangle$. By inversion, $M' = \langle M'_L, M'_R \rangle$.

$$\begin{array}{c}
\frac{\langle M_L, M_R \rangle \Rightarrow_{\beta} \langle M'_L, M'_R \rangle}{\langle M'_L, M'_R \rangle \Rightarrow_{\beta} |\langle M_L, M_R \rangle|^+} \xrightarrow{1} \\
\frac{}{M'_L \Rightarrow_{\beta} |M_L|^+} \xrightarrow{2} \\
\frac{}{\langle M'_L, M'_R \rangle \cdot L \Rightarrow_{\beta} |M_L|^+} \xrightarrow{3} \\
\frac{}{\langle M'_L, M'_R \rangle \cdot L \Rightarrow_{\beta} |\langle M_L, M_R \rangle \cdot L|^+} \xrightarrow{4}
\end{array}$$

1 IH 2 unfold $|\langle M_L, M_R \rangle|^+$ and inversion 3 Parallel-ProjL 4 fold $|\langle M_L, M_R \rangle \cdot L|^+$

SUBCASE M is not a pair.

$$\begin{array}{c}
\frac{}{M \Rightarrow_{\beta} M'} \xrightarrow{1} \\
\frac{}{M' \Rightarrow_{\beta} |M|^+} \xrightarrow{2} \\
\frac{}{M' \cdot L \Rightarrow_{\beta} |M|^+ \cdot L} \xrightarrow{3} \\
\frac{}{M' \cdot L \Rightarrow_{\beta} |M \cdot L|^+} \xrightarrow{3}
\end{array}$$

1 IH 2 Parallel-ProjL-Under 3 fold $|M \cdot L|^+$

CASE $\frac{M \Rightarrow_{\beta} M'}{M \cdot R \Rightarrow_{\beta} M' \cdot R} \text{ [Parallel-ProjR-Under]}$

$$M' \cdot R \Rightarrow_{\beta} |M \cdot R|^+$$

SUBCASE $M = \langle M_L, M_R \rangle$. By inversion, $M' = \langle M'_L, M'_R \rangle$.

$$\begin{array}{c}
\frac{\langle M_L, M_R \rangle \Rightarrow_{\beta} \langle M'_L, M'_R \rangle}{\langle M'_L, M'_R \rangle \Rightarrow_{\beta} |\langle M_L, M_R \rangle|^+} \xrightarrow{1} \\
\frac{}{M'_R \Rightarrow_{\beta} |M_R|^+} \xrightarrow{2} \\
\frac{}{\langle M'_L, M'_R \rangle \cdot R \Rightarrow_{\beta} |M_R|^+} \xrightarrow{3} \\
\frac{}{\langle M'_L, M'_R \rangle \cdot R \Rightarrow_{\beta} |\langle M_L, M_R \rangle \cdot R|^+} \xrightarrow{4}
\end{array}$$

1 IH 2 unfold $|\langle M_L, M_R \rangle|^+$ and inversion 3 Parallel-ProjR 4 fold $|\langle M_L, M_R \rangle \cdot R|^+$

SUBCASE M is not a pair.

$$\begin{array}{c}
M \Rightarrow_{\beta} M' \\
\hline
M' \Rightarrow_{\beta} |M|^{+} \quad 1 \\
\hline
M' \cdot R \Rightarrow_{\beta} |M|^{+} \cdot R \quad 2 \\
\hline
M' \cdot R \Rightarrow_{\beta} |M \cdot R|^{+} \quad 3
\end{array}$$

1 IH 2 Parallel-ProjR-Under 3 fold $|M \cdot R|^{+}$

$$\boxed{\text{CASE}} \quad \frac{\frac{M_L \Rightarrow_{\beta} M'_L}{\langle M_L, M_R \rangle \cdot L \Rightarrow_{\beta} M'_L} \text{ [Parallel-ProjL]}}{\dots} \quad \frac{\dots}{M'_L \Rightarrow_{\beta} |\langle M_L, M_R \rangle \cdot L|^{+}}$$

$$\begin{array}{c}
M_L \Rightarrow_{\beta} M'_L \\
\hline
M'_L \Rightarrow_{\beta} |M_L|^{+} \quad 1 \\
\hline
M'_L \Rightarrow_{\beta} |\langle M_L, M_R \rangle \cdot L|^{+} \quad 2
\end{array}$$

1 IH 2 fold $|\langle M_L, M_R \rangle \cdot L|^{+}$

$$\boxed{\text{CASE}} \quad \frac{\frac{M_R \Rightarrow_{\beta} M'_R}{\langle M_L, M_R \rangle \cdot R \Rightarrow_{\beta} M'_R} \text{ [Parallel-ProjR]}}{\dots} \quad \frac{\dots}{M'_R \Rightarrow_{\beta} |\langle M_L, M_R \rangle \cdot R|^{+}}$$

$$\begin{array}{c}
M_R \Rightarrow_{\beta} M'_R \\
\hline
M'_R \Rightarrow_{\beta} |M_R|^{+} \quad 1 \\
\hline
M'_R \Rightarrow_{\beta} |\langle M_L, M_R \rangle \cdot R|^{+} \quad 2
\end{array}$$

1 IH 2 fold $|\langle M_L, M_R \rangle \cdot R|^{+}$

$$\boxed{\text{CASE}} \quad \frac{\frac{M \Rightarrow_{\beta} M'}{\lambda x \Rightarrow M \Rightarrow_{\beta} \lambda x \Rightarrow M'} \text{ [Parallel-Lam]}}{\dots} \quad \frac{\dots}{\lambda x \Rightarrow M' \Rightarrow_{\beta} |\lambda x \Rightarrow M|^{+}}$$

$$\begin{array}{c}
M \Rightarrow_\beta M' \\
\hline
M' \Rightarrow_\beta |M|^+ \quad 1 \\
\hline
\lambda x \Rightarrow M' \Rightarrow_\beta \lambda x \Rightarrow |M|^+ \quad 2 \\
\hline
\lambda x \Rightarrow M' \Rightarrow_\beta |\lambda x \Rightarrow M|^+ \quad 3
\end{array}$$

1 IH 2 Parallel-Lam 3 fold $|\lambda x \Rightarrow M|^+$

CASE $\frac{M \Rightarrow_\beta M' \quad N \Rightarrow_\beta N'}{M N \Rightarrow_\beta M' N'} \text{ [Parallel-App-Under]}$

$$\begin{array}{c}
M' N' \Rightarrow_\beta |M N|^+
\end{array}$$

SUBCASE $M = \lambda x \Rightarrow V$. By inversion, $M' = \lambda x \Rightarrow V'$.

$$\begin{array}{c}
\lambda x \Rightarrow V \Rightarrow_\beta \lambda x \Rightarrow V' \\
\hline
\lambda x \Rightarrow V' \Rightarrow_\beta |\lambda x \Rightarrow V|^+ \quad 1 \\
\hline
V' \Rightarrow_\beta |V|^+ \quad 2 \\
\hline
\lambda x \Rightarrow V' N' \Rightarrow_\beta [|N|^+/x](|V|^+) \quad 3 \\
\hline
\lambda x \Rightarrow V' N' \Rightarrow_\beta |\lambda x \Rightarrow V N|^+ \quad 4 \\
\hline
\lambda x \Rightarrow V' N' \Rightarrow_\beta |\lambda x \Rightarrow V N|^+ \quad 5
\end{array}$$

1 IH 2 unfold $|\lambda x \Rightarrow V|^+$ and inversion 3 IH 4 Parallel-App 5 fold $|\lambda x \Rightarrow V N|^+$

SUBCASE M is not a lambda abstraction.

$$\begin{array}{c}
M \Rightarrow_\beta M' \quad N \Rightarrow_\beta N' \\
\hline
M' \Rightarrow_\beta |M|^+ \quad 1 \quad N' \Rightarrow_\beta |N|^+ \quad 2 \\
\hline
M' N' \Rightarrow_\beta |M|^+ |N|^+ \quad 3 \\
\hline
M' N' \Rightarrow_\beta |M N|^+ \quad 4
\end{array}$$

1 IH 2 IH 3 Parallel-App-Under 4 fold $|M N|^+$

CASE $\frac{M \Rightarrow_\beta M' \quad N \Rightarrow_\beta N'}{\lambda x \Rightarrow M N \Rightarrow_\beta [N'/x](M')} \text{ [Parallel-App]}$

$$\begin{array}{c}
[N'/x](M') \Rightarrow_\beta |\lambda x \Rightarrow M N|^+
\end{array}$$

1 IH 2 fold $[N'/x] \Rightarrow_{\beta} [|N|^{+}/x]$ 3 IH 4 Parallel-Hat-Subst 5 fold $|\lambda x \Rightarrow M N|^{+}$

$$\frac{\frac{M \Rightarrow_{\beta} M'}{M' \Rightarrow_{\beta} |M|^{+}} \quad 1}{L \cdot M' \Rightarrow_{\beta} L \cdot |M|^{+}} \quad 2$$

$$\frac{L \cdot M' \Rightarrow_{\beta} L \cdot |M|^{+}}{L \cdot M' \Rightarrow_{\beta} |L \cdot M|^{+}} \quad 3$$

$$\boxed{\text{CASE}} \quad \frac{\frac{M \Rightarrow_{\beta} M'}{R \cdot M \Rightarrow_{\beta} R \cdot M'} [\text{Parallel-InR}]}{R \cdot M' \Rightarrow_{\beta} |R \cdot M|^{+}}$$

$$\frac{\frac{M \Rightarrow_{\beta} M'}{M' \Rightarrow_{\beta} |M|^{+}} \quad 1}{R \cdot M' \Rightarrow_{\beta} R \cdot |M|^{+}} \quad 2$$

$$\frac{R \cdot M' \Rightarrow_{\beta} R \cdot |M|^{+}}{R \cdot M' \Rightarrow_{\beta} |R \cdot M|^{+}} \quad 3$$

$$\boxed{\text{CASE}} \quad \frac{\begin{array}{c} M \Rightarrow_{\beta} M' \quad M_L \Rightarrow_{\beta} M'_L \quad M_R \Rightarrow_{\beta} M'_R \\ \text{case } M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} \text{case } M' \ (x_L \Rightarrow M'_L \mid x_R \Rightarrow M'_R) \end{array}}{\text{case } M' \ (x_L \Rightarrow M'_L \mid x_R \Rightarrow M'_R) \Rightarrow_{\beta} |\text{case } M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)|^+} \quad [\text{Parallel-Case-Under}]$$

96

$$\begin{array}{c}
\frac{L \cdot V \Rightarrow_{\beta} L \cdot V'}{\frac{L \cdot V' \Rightarrow_{\beta} |L \cdot V|^{+}}{V' \Rightarrow_{\beta} |V|^{+}}} \xrightarrow{1} \frac{1}{2} \quad \frac{M_L \Rightarrow_{\beta} M'_L}{M'_L \Rightarrow_{\beta} |M_L|^{+}} \xrightarrow{3} \frac{3}{4} \\
\hline
\text{case } L \cdot V' (x_L \Rightarrow M'_L \mid x_R \Rightarrow M'_R) \Rightarrow_{\beta} [|V|^{+}/x_L](|M_L|^{+}) \quad 4 \\
\hline
\text{case } L \cdot V' (x_L \Rightarrow M'_L \mid x_R \Rightarrow M'_R) \Rightarrow_{\beta} |\text{case } L \cdot V (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)|^{+} \quad 5
\end{array}$$

1 IH 2 unfold $|L \cdot V|^{+}$ and inversion 3 IH 4 Parallel-Case-InL 5 fold
 $|\text{case } L \cdot V (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)|^{+}$

SUBCASE $M = R \cdot V$. By inversion, $M' = R \cdot V'$.

Entirely analogous to the previous subcase.

SUBCASE M is neither a left nor a right injection.

$$\begin{array}{c}
\frac{M \Rightarrow_{\beta} M'}{M' \Rightarrow_{\beta} |M|^{+}} \xrightarrow{1} \frac{1}{2} \quad \frac{M_L \Rightarrow_{\beta} M'_L}{M'_L \Rightarrow_{\beta} |M_L|^{+}} \xrightarrow{2} \frac{2}{3} \quad \frac{M_R \Rightarrow_{\beta} M'_R}{M'_R \Rightarrow_{\beta} |M_R|^{+}} \xrightarrow{3} \frac{3}{4} \\
\hline
\text{case } M' (x_L \Rightarrow M'_L \mid x_R \Rightarrow M'_R) \Rightarrow_{\beta} \text{case } |M|^{+} (x_L \Rightarrow |M_L|^{+} \mid x_R \Rightarrow |M_R|^{+}) \quad 4 \\
\hline
\text{case } M' (x_L \Rightarrow M'_L \mid x_R \Rightarrow M'_R) \Rightarrow_{\beta} |\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)|^{+} \quad 5
\end{array}$$

1 IH 2 IH 3 IH 4 Parallel-Case-Under 5 fold $|\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)|^{+}$

$$\begin{array}{c}
\frac{M \Rightarrow_{\beta} M' \quad M_L \Rightarrow_{\beta} M'_L}{\text{case } L \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} [M'/x_L](M'_L)} \quad [\text{Parallel-Case-InL}] \\
\hline
[M'/x_L](M'_L) \Rightarrow_{\beta} |\text{case } L \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)|^{+}
\end{array}$$

$$\begin{array}{c}
\frac{M \Rightarrow_{\beta} M'}{M' \Rightarrow_{\beta} |M|^{+}} \xrightarrow{1} \frac{1}{2} \quad \frac{M_L \Rightarrow_{\beta} M'_L}{M'_L \Rightarrow_{\beta} |M_L|^{+}} \xrightarrow{3} \frac{3}{4} \\
\frac{[M'/x_L] \Rightarrow_{\beta} [|M|^{+}/x_L]}{[M'/x_L](M'_L) \Rightarrow_{\beta} [|M|^{+}/x_L](|M_L|^{+})} \xrightarrow{2} \frac{2}{4} \\
\hline
[M'/x_L](M'_L) \Rightarrow_{\beta} |\text{case } L \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)|^{+} \quad 5
\end{array}$$

1 IH 2 fold $[M'/x_L] \Rightarrow_{\beta} [|M|^{+}/x_L]$ 3 IH 4 Parallel-Hat-Subst 5 fold
 $|\text{case } L \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)|^{+}$

$$\boxed{\text{CASE}} \quad \frac{\frac{M \Rightarrow_{\beta} M' \quad M_R \Rightarrow_{\beta} M'_R}{\text{case } R \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \Rightarrow_{\beta} [M'/x_R](M'_R)} \text{ [Parallel-Case-InR]}}{[M'/x_R](M'_R) \Rightarrow_{\beta} |\text{case } R \cdot M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)|^+}$$

Entirely analogous to the previous case.

$$\boxed{\text{CASE}} \quad \frac{\overline{\text{zero} \Rightarrow_{\beta} \text{zero}} \text{ [Parallel-Zero]}}{\text{zero} \Rightarrow_{\beta} |\text{zero}|^+}$$

$$\frac{\overline{\text{zero} \Rightarrow_{\beta} \text{zero}} \quad 1}{\text{zero} \Rightarrow_{\beta} |\text{zero}|^+ \quad 2}$$

1 Parallel-Zero 2 fold $|\text{zero}|^+$

$$\boxed{\text{CASE}} \quad \frac{\frac{M \Rightarrow_{\beta} M'}{\text{succ } M \Rightarrow_{\beta} \text{succ } M'} \text{ [Parallel-Succ]}}{\text{succ } M' \Rightarrow_{\beta} |\text{succ } M|^+}$$

$$\frac{\frac{\frac{M \Rightarrow_{\beta} M'}{\overline{\text{succ } M \Rightarrow_{\beta} \text{succ } M'}} \quad 1}{\text{succ } M' \Rightarrow_{\beta} |\text{succ } M|^+ \quad 2}}{\text{succ } M' \Rightarrow_{\beta} |\text{succ } M|^+ \quad 3}$$

1 IH 2 Parallel-Succ 3 fold $|\text{succ } M|^+$

$$\boxed{\text{CASE}} \quad \frac{\frac{M \Rightarrow_{\beta} M' \quad M_0 \Rightarrow_{\beta} M'_0 \quad M_S \Rightarrow_{\beta} M'_S}{\text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_{\beta} \text{natrec } M' (M'_0 \mid x, y \Rightarrow M'_S)} \text{ [Parallel-NatRec-Under]}}{\text{natrec } M' (M'_0 \mid x, y \Rightarrow M'_S) \Rightarrow_{\beta} |\text{natrec } M (M_0 \mid x, y \Rightarrow M_S)|^+}$$

SUBCASE $M = \text{zero}$. By inversion, $M' = \text{zero}$.

$$\frac{\frac{\frac{M_0 \Rightarrow_{\beta} M'_0}{\overline{\text{natrec zero } (M'_0 \mid x, y \Rightarrow M'_S)}} \quad 1}{\text{natrec zero } (M'_0 \mid x, y \Rightarrow M'_S) \Rightarrow_{\beta} |M_0|^+ \quad 2}}{\text{natrec zero } (M'_0 \mid x, y \Rightarrow M'_S) \Rightarrow_{\beta} |\text{natrec zero } (M_0 \mid x, y \Rightarrow M_S)|^+ \quad 3}$$

1 IH 2 Parallel-NatRec-Zero 3 fold $|\text{natrec zero } (M_0 \mid x, y \Rightarrow M_S)|^+$

SUBCASE $M = \text{succ } V$. By inversion, $M' = \text{succ } V'$.

Define $R^+ \triangleq \text{natrec } |V|^+ (|M_0|^+ \mid x, y \Rightarrow |M_S|^+)$.

$$\begin{array}{c}
 \frac{\text{succ } V \Rightarrow_\beta \text{succ } V'}{\text{succ } V' \Rightarrow_\beta |\text{succ } V|^+} \xrightarrow{1} \quad \frac{M_0 \Rightarrow_\beta M'_0}{M'_0 \Rightarrow_\beta |M_0|^+} \xrightarrow{2} \quad \frac{M_S \Rightarrow_\beta M'_S}{M'_S \Rightarrow_\beta |M_S|^+} \xrightarrow{3} \\
 \frac{V' \Rightarrow_\beta |V|^+}{\text{natrec succ } V' (M'_0 \mid x, y \Rightarrow M'_S) \Rightarrow_\beta [|V|^+ / x, R^+ / y] (|M_S|^+)} \xrightarrow{4} \quad \xrightarrow{5} \\
 \frac{\text{natrec succ } V' (M'_0 \mid x, y \Rightarrow M'_S) \Rightarrow_\beta [|V|^+ / x, R^+ / y] (|M_S|^+)}{\text{natrec succ } V' (M'_0 \mid x, y \Rightarrow M'_S) \Rightarrow_\beta |\text{natrec succ } V (M_0 \mid x, y \Rightarrow M_S)|^+} \xrightarrow{6}
 \end{array}$$

1 IH 2 unfold $|\text{succ } V|^+$ and inversion 3 IH 4 IH 5 Parallel-NatRec-Succ 6 fold $|\text{natrec succ } V (M_0 \mid x, y \Rightarrow M_S)|^+$

SUBCASE M is neither zero nor a successor.

$$\begin{array}{c}
 \frac{M \Rightarrow_\beta M'}{M' \Rightarrow_\beta |M|^+} \xrightarrow{1} \quad \frac{M_0 \Rightarrow_\beta M'_0}{M'_0 \Rightarrow_\beta |M_0|^+} \xrightarrow{2} \quad \frac{M_S \Rightarrow_\beta M'_S}{M'_S \Rightarrow_\beta |M_S|^+} \xrightarrow{3} \\
 \frac{\text{natrec } M' (M'_0 \mid x, y \Rightarrow M'_S) \Rightarrow_\beta \text{natrec } |M|^+ (|M_0|^+ \mid x, y \Rightarrow |M_S|^+)}{\text{natrec } M' (M'_0 \mid x, y \Rightarrow M'_S) \Rightarrow_\beta |\text{natrec } M (M_0 \mid x, y \Rightarrow M_S)|^+} \xrightarrow{4} \quad \xrightarrow{5}
 \end{array}$$

1 IH 2 IH 3 IH 4 Parallel-NatRec-Under 5 fold $|\text{natrec } M (M_0 \mid x, y \Rightarrow M_S)|^+$

$$\begin{array}{c}
 \frac{M_0 \Rightarrow_\beta M'_0}{\text{natrec zero } (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_\beta M'_0} \quad [\text{Parallel-NatRec-Zero}] \\
 \hline
 M'_0 \Rightarrow_\beta |\text{natrec zero } (M_0 \mid x, y \Rightarrow M_S)|^+
 \end{array}$$

$$\begin{array}{c}
 \frac{M_0 \Rightarrow_\beta M'_0}{M'_0 \Rightarrow_\beta |M_0|^+} \xrightarrow{1} \\
 \hline
 M'_0 \Rightarrow_\beta |\text{natrec zero } (M_0 \mid x, y \Rightarrow M_S)|^+ \quad 2
 \end{array}$$

1 IH 2 fold $|\text{natrec zero } (M_0 \mid x, y \Rightarrow M_S)|^+$

$$\begin{array}{c}
 \frac{M \Rightarrow_\beta M' \quad M_0 \Rightarrow_\beta M'_0 \quad M_S \Rightarrow_\beta M'_S}{\text{natrec succ } M (M_0 \mid x, y \Rightarrow M_S) \Rightarrow_\beta [M' / x, R' / y] (M'_S)} \quad [\text{Parallel-NatRec-Succ}] \\
 \hline
 [M' / x, R' / y] (M'_S) \Rightarrow_\beta |\text{natrec succ } M (M_0 \mid x, y \Rightarrow M_S)|^+
 \end{array}$$

Define $R' \triangleq \text{natrec } M' (M'_0 \mid x, y \Rightarrow M'_S)$ and $R^+ \triangleq \text{natrec } |M|^+ (|M_0|^+ \mid x, y \Rightarrow |M_S|^+)$.

$$\begin{array}{c}
\frac{M \Rightarrow_\beta M' \quad \frac{\frac{M_0 \Rightarrow_\beta M'_0 \quad M_S \Rightarrow_\beta M'_S}{M'_0 \Rightarrow_\beta |M_0|^+ \quad M'_S \Rightarrow_\beta |M_S|^+} \quad 5}{M' \Rightarrow_\beta |M|^+} \quad 2 \quad 4}{\frac{M' \Rightarrow_\beta |M|^+ \quad R' \Rightarrow_\beta R^+}{[M'/x, R'/y] \Rightarrow_\beta [|M|^+/x, R^+/y]} \quad 6} \quad 1 \\
\frac{[M'/x, R'/y] \Rightarrow_\beta [|M|^+/x, R^+/y] \quad \frac{M_S \Rightarrow_\beta M'_S}{M'_S \Rightarrow_\beta |M_S|^+} \quad 7}{[M'/x, R'/y](M'_S) \Rightarrow_\beta [|M|^+/x, R^+/y](|M_S|^+)} \quad 8 \\
\frac{[M'/x, R'/y](M'_S) \Rightarrow_\beta [|M|^+/x, R^+/y](|M_S|^+)}{[M'/x, R'/y](M'_S) \Rightarrow_\beta |\text{natrec succ } M (M_0 \mid x, y \Rightarrow M_S)|^+} \quad 9
\end{array}$$

1 IH 2 IH 3 IH 4 IH 5 Parallel-NatRec-Under 6 fold $[M'/x, R'/y] \Rightarrow_\beta [|M|^+/x, R^+/y]$ 7 IH
8 Parallel-Hat-Subst 9 fold $|\text{natrec succ } M (M_0 \mid x, y \Rightarrow M_S)|^+$ □

Parallel β -reduction has the diamond property as a direct corollary of [Parallel-Triangle]:

Parallel-Diamond

$$\frac{M \Rightarrow_\beta N \quad M \Rightarrow_\beta N'}{N \Rightarrow_\beta |M|^+ \wedge N' \Rightarrow_\beta |M|^+}$$

Proof:

$$\frac{\frac{M \Rightarrow_\beta N}{N \Rightarrow_\beta |M|^+} \quad 1 \quad \frac{M \Rightarrow_\beta N'}{N' \Rightarrow_\beta |M|^+} \quad 2}{N \Rightarrow_\beta |M|^+ \wedge N' \Rightarrow_\beta |M|^+} \quad 3$$

1 Parallel-Triangle 2 Parallel-Triangle 3 $\wedge$ -I □

To prove confluence for multi-step parallel β -reduction, we first prove the following strip lemma:

Parallel-Strip

$$\frac{M \Rightarrow_\beta N \quad M \Rightarrow_\beta^* N'}{\exists L. N \Rightarrow_\beta^* L \wedge N' \Rightarrow_\beta^* L}$$

Proof by induction on the derivation of $M \Rightarrow_\beta^* N'$:

$$\boxed{\text{CASE}} \quad \frac{M \Rightarrow_{\beta} N \quad \overline{M \Rightarrow_{\beta}^* M} \text{ [Parallel-Multi-Ref]}}{\exists L . N \Rightarrow_{\beta}^* L \wedge M \Rightarrow_{\beta}^* L}$$

$$\frac{\frac{\frac{M \Rightarrow_{\beta} N}{\overline{M \Rightarrow_{\beta}^* N}} \text{ 1} \quad \overline{M \Rightarrow_{\beta}^* N} \text{ 2}}{N \Rightarrow_{\beta}^* N \wedge M \Rightarrow_{\beta}^* N} \text{ 3}}{\exists L . N \Rightarrow_{\beta}^* L \wedge M \Rightarrow_{\beta}^* L} \text{ 4}$$

1 Parallel-Multi-Ref 2 Parallel-Multi-if-Single 3 $\wedge$ -I 4 $\exists$ -I(N)

$$\boxed{\text{CASE}} \quad \frac{M \Rightarrow_{\beta} N \quad \frac{M \Rightarrow_{\beta} P \quad P \Rightarrow_{\beta}^* N'}{M \Rightarrow_{\beta}^* N'} \text{ [Parallel-Multi-Cons]}}{\exists L . N \Rightarrow_{\beta}^* L \wedge N' \Rightarrow_{\beta}^* L}$$

$$\frac{\frac{\frac{M \Rightarrow_{\beta} N}{\overline{N \Rightarrow_{\beta}^* |M|^+}} \text{ 1} \quad \frac{\frac{M \Rightarrow_{\beta} P}{\overline{P \Rightarrow_{\beta}^* |M|^+}} \text{ 3} \quad P \Rightarrow_{\beta}^* N'}{\overline{\exists L . |M|^+ \Rightarrow_{\beta}^* L \wedge N' \Rightarrow_{\beta}^* L}} \text{ 4}}{\overline{|M|^+ \Rightarrow_{\beta}^* L \wedge N' \Rightarrow_{\beta}^* L}} \text{ 5}}{\overline{N \Rightarrow_{\beta}^* L \wedge N' \Rightarrow_{\beta}^* L}} \text{ 6}}{\overline{\exists K . N \Rightarrow_{\beta}^* K \wedge N' \Rightarrow_{\beta}^* K}} \text{ 7}$$

1 Parallel-Triangle 2 Parallel-Multi-if-Single 3 Parallel-Triangle 4 IH 5 pick L 6 Parallel-Multi-Trans
7 $\exists$ -I(L) □

$$\text{Parallel-Confluence}$$

$$\frac{M \Rightarrow_{\beta}^* N \quad M \Rightarrow_{\beta}^* N'}{\overline{\exists L . N \Rightarrow_{\beta}^* L \wedge N' \Rightarrow_{\beta}^* L}}$$

Proof by induction on the derivation of $M \Rightarrow_{\beta}^* N$:

$$\boxed{\text{CASE}} \quad \frac{\overline{M \Rightarrow_{\beta}^* M} \text{ [Parallel-Multi-Ref]} \quad M \Rightarrow_{\beta}^* N'}{\exists L . M \Rightarrow_{\beta}^* L \wedge N' \Rightarrow_{\beta}^* L}$$

$$\begin{array}{c}
\frac{M \Rightarrow_{\beta}^* N' \quad \overline{N' \Rightarrow_{\beta}^* N'}}{\frac{M \Rightarrow_{\beta}^* N' \wedge N' \Rightarrow_{\beta}^* N'}{\exists L . M \Rightarrow_{\beta}^* L \wedge N' \Rightarrow_{\beta}^* L}}
\end{array}
\begin{array}{l}
1 \\
2 \\
3
\end{array}$$

1 Parallel-Multi-Ref 2 $\wedge$ -I 3 $\exists$ -I(N')

$$\boxed{\text{CASE}} \quad \frac{M \Rightarrow_{\beta} P \quad P \Rightarrow_{\beta}^* N}{M \Rightarrow_{\beta}^* N} \text{ [Parallel-Multi-Cons]} \quad M \Rightarrow_{\beta}^* N'$$

$$\exists L . N \Rightarrow_{\beta}^* L \wedge N' \Rightarrow_{\beta}^* L$$

$$\begin{array}{c}
\frac{M \Rightarrow_{\beta} P \quad M \Rightarrow_{\beta}^* N'}{\frac{\frac{M \Rightarrow_{\beta} P \quad M \Rightarrow_{\beta}^* N'}{\exists Q . P \Rightarrow_{\beta}^* Q \wedge N' \Rightarrow_{\beta}^* Q}}{P \Rightarrow_{\beta}^* N \quad P \Rightarrow_{\beta}^* Q \wedge N' \Rightarrow_{\beta}^* Q}}
\end{array}
\begin{array}{l}
1 \\
2 \\
3
\end{array}$$

$$\frac{\exists L . (N \Rightarrow_{\beta}^* L \wedge Q \Rightarrow_{\beta}^* L) \wedge N' \Rightarrow_{\beta}^* Q}{N \Rightarrow_{\beta}^* L \wedge N' \Rightarrow_{\beta}^* L}$$

$$\frac{N \Rightarrow_{\beta}^* L \wedge N' \Rightarrow_{\beta}^* L}{\exists K . N \Rightarrow_{\beta}^* K \wedge N' \Rightarrow_{\beta}^* K}$$

4 5

1 Parallel-Strip 2 pick Q 3 IH 4 pick L and Parallel-Multi-Trans 5 $\exists$ -I(L)

□

Since the usual multi-step β -reduction is sound and complete with respect to the multi-step parallel β -reduction, the confluence of parallel β -reduction implies the confluence of β -reduction:

Confluence

$$\frac{M \rightarrow_{\beta}^* N \quad M \rightarrow_{\beta}^* N'}{\exists L . N \rightarrow_{\beta}^* L \wedge N' \rightarrow_{\beta}^* L}$$

Proof:

$$\begin{array}{c}
\frac{M \rightarrow_{\beta}^* N \quad M \rightarrow_{\beta}^* N'}{\frac{\frac{M \rightarrow_{\beta}^* N \quad M \rightarrow_{\beta}^* N'}{M \Rightarrow_{\beta}^* N} \quad M \Rightarrow_{\beta}^* N'}{\exists L . N \Rightarrow_{\beta}^* L \wedge N' \Rightarrow_{\beta}^* L}}
\end{array}
\begin{array}{l}
1 \\
2 \\
3 \\
4
\end{array}$$

1 Parallel-Multi-if-Usual-Multi 2 Parallel-Multi-if-Usual-Multi 3 Parallel-Confluence
4 Usual-Multi-if-Parallel-Multi twice

□

Chapter 6

Normalization

With [Confluence], we can prove the needed lemmas to finally arrive at the last pair of theorems regarding normalization:

$$\begin{array}{cc}
 \text{[norm-if-H]} & \text{[norm-if-N]} \\
 \frac{\Delta \xRightarrow{\mathcal{H}} M \in A}{\text{norm}_\beta(M)} & \frac{\Delta \xRightarrow{\mathcal{N}} M \in A}{\text{norm}_\beta(M)}
 \end{array}$$

The proof would be by simultaneous induction on $\Delta \xRightarrow{\mathcal{H}} M \in A$ and $\Delta \xRightarrow{\mathcal{N}} M \in A$. From the various induction cases, we collect the following lemmas that are needed for the proof:

$$\begin{array}{c}
 \text{[WN-Head-Expansion]} \\
 \frac{M \rightarrow_\beta^* M' \quad \text{norm}_\beta(M')}{\text{norm}_\beta(M)} \\
 \\
 \begin{array}{ccc}
 \text{[WN-VarM]} & \text{[WN-EmptyM]} & \text{[WN-ZeroM]} \\
 \frac{}{\text{norm}_\beta(x)} & \frac{}{\text{norm}_\beta(\langle \rangle)} & \frac{}{\text{norm}_\beta(\text{zero})}
 \end{array} \\
 \\
 \begin{array}{cc}
 \text{[WN-TimesT]} & \text{[WN-ArrowT]} \\
 \frac{\text{norm}_\beta(M \cdot L) \quad \text{norm}_\beta(M \cdot R)}{\text{norm}_\beta(M)} & \frac{\text{norm}_\beta(M \ x)}{\text{norm}_\beta(M)}
 \end{array}
 \end{array}$$

$$\begin{array}{c}
\begin{array}{ccc}
[\text{WN-SuccM}] & [\text{WN-InLM}] & [\text{WN-InRM}] \\
\frac{\text{norm}_\beta(M)}{\text{norm}_\beta(\text{succ } M)} & \frac{\text{norm}_\beta(M)}{\text{norm}_\beta(L \cdot M)} & \frac{\text{norm}_\beta(M)}{\text{norm}_\beta(R \cdot M)}
\end{array} \\
\begin{array}{cc}
[\text{WN-ProjLM}] & [\text{WN-ProjRM}] \\
\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R \quad \text{norm}_\beta(M)}{\text{norm}_\beta(M \cdot L)} & \frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R \quad \text{norm}_\beta(M)}{\text{norm}_\beta(M \cdot R)}
\end{array} \\
[\text{WN-Neutral-AppM}] \\
\frac{\Delta \xRightarrow{\mathcal{N}} M \in A \rightarrow B \quad \text{norm}_\beta(M) \quad \text{norm}_\beta(N)}{\text{norm}_\beta(M N)} \\
[\text{WN-Neutral-CaseM}] \\
\frac{\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L + A_R \quad \text{norm}_\beta(M)}{\text{norm}_\beta(M_L) \quad \text{norm}_\beta(M_R)}}{\text{norm}_\beta(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R))} \\
[\text{WN-Neutral-NatRecM}] \\
\frac{\frac{\Delta \xRightarrow{\mathcal{N}} M \in \text{nat} \quad \text{norm}_\beta(M)}{\text{norm}_\beta(M_0) \quad \text{norm}_\beta(M_S)}}{\text{norm}_\beta(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S))}
\end{array}$$

We can classify the above lemmas into several kinds. The first kind consists of only head expansion of normalizability:

$$[\text{WN-Head-Expansion}]$$

$$\frac{M \rightarrow_\beta^* M' \quad \text{norm}_\beta(M')}{\text{norm}_\beta(M)}$$

This lemma is intuitively true as if a term M' is normalizable, we may append any reduction $M \rightarrow_\beta^* M'$ to the front of the normalizing path to conclude that M is also normalizable.

The second kind of lemmas consists of those that demand constants be normalizable, which are true

because constants are already β -normal:

$$\begin{array}{ccc}
[\text{WN-VarM}] & [\text{WN-EmptyM}] & [\text{WN-ZeroM}] \\
\overrightarrow{\text{norm}_\beta(x)} & \overrightarrow{\text{norm}_\beta(\langle \rangle)} & \overrightarrow{\text{norm}_\beta(\text{zero})}
\end{array}$$

The third kind of lemmas consists of those that ask us to extract a normalizing path for a term M , given one or more normalizing paths for its elimination forms. As we have illustrated, the proof of $[\text{WN-TimesT}]$ demands that we show that normalizability is forward closed, which we were able to establish with $[\text{Confluence}]$:

$$\begin{array}{ccc}
[\text{WN-TimesT}] & & [\text{WN-ArrowT}] \\
\overrightarrow{\text{norm}_\beta(M \cdot L) \quad \text{norm}_\beta(M \cdot R)} & \overrightarrow{\text{norm}_\beta(M)} & \overrightarrow{\text{norm}_\beta(M \ x)} \\
& & \overrightarrow{\text{norm}_\beta(M)}
\end{array}$$

The fourth kind of lemmas consists of those that demand that we construct a normalizing path after applying introduction-form constructors to a normalizing term M :

$$\begin{array}{ccc}
[\text{WN-SuccM}] & [\text{WN-InLM}] & [\text{WN-InRM}] \\
\overrightarrow{\text{norm}_\beta(M)} & \overrightarrow{\text{norm}_\beta(M)} & \overrightarrow{\text{norm}_\beta(M)} \\
\overrightarrow{\text{norm}_\beta(\text{succ } M)} & \overrightarrow{\text{norm}_\beta(L \cdot M)} & \overrightarrow{\text{norm}_\beta(R \cdot M)}
\end{array}$$

The fifth, and final, kind of lemmas consists of showing spines are normalizable, given all of their immediate subterms are normalizable:

$$\begin{array}{ccc}
[\text{WN-ProjLM}] & & [\text{WN-ProjRM}] \\
\overrightarrow{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R \quad \text{norm}_\beta(M)} & \overrightarrow{\text{norm}_\beta(M \cdot L)} & \overrightarrow{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R \quad \text{norm}_\beta(M)} \\
& & \overrightarrow{\text{norm}_\beta(M \cdot R)} \\
[\text{WN-Neutral-AppM}] \\
\overrightarrow{\Delta \xRightarrow{\mathcal{N}} M \in A \rightarrow B \quad \text{norm}_\beta(M) \quad \text{norm}_\beta(N)} & \overrightarrow{\text{norm}_\beta(M \ N)} &
\end{array}$$

[WN-Neutral-CaseM]

$$\frac{\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L + A_R \quad \text{norm}_\beta(M)}{\text{norm}_\beta(M_L) \quad \text{norm}_\beta(M_R)}}{\text{norm}_\beta(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R))}$$

[WN-Neutral-NatRecM]

$$\frac{\frac{\Delta \xRightarrow{\mathcal{N}} M \in \text{nat} \quad \text{norm}_\beta(M)}{\text{norm}_\beta(M_0) \quad \text{norm}_\beta(M_S)}}{\text{norm}_\beta(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S))}$$

With these lemmas, we are able to prove both [norm-if-H] and [norm-if-N], with which we can then show that all well-typed terms are normalizable, since all well-typed terms are computable, by [FTLR]:

norm-if-WT

$$\frac{\Gamma \vdash M : A}{\text{norm}_\beta(M)}$$

Proof:

$$\frac{\frac{\Gamma \vdash M : A}{\Gamma \xRightarrow{\mathcal{H}} M \in A} \quad 1}{\text{norm}_\beta(M)} \quad 2$$

1 FTLR 2 norm-if-H □

Finally, we show that if a term M has some β -normal form T , then that T must be unique, as a corollary of [Confluence]:

Uniqueness-of-beta-Normal-Forms

$$\frac{M \rightarrow_\beta^* T \not\rightarrow_\beta \quad M \rightarrow_\beta^* T' \not\rightarrow_\beta}{T = T'}$$

Proof:

By [Confluence] on $M \rightarrow_\beta^* T$ and $M \rightarrow_\beta^* T'$, we know there exists a common reduct T'' such that $T \rightarrow_\beta^* T''$ and $T' \rightarrow_\beta^* T''$.

But at the same time we already have $T \not\rightarrow_\beta$ and $T' \not\rightarrow_\beta$, which forces $T = T''$ and $T' = T''$, and hence $T = T'$. $\square$

Therefore, combining [norm-if-WT] and [Uniqueness-of-beta-Normal-Forms], we can conclude that any well-typed term has a unique β -normal form.

The next section [Section 6.1 \(Proofs\)](#) contains the detailed proofs of the theorems and lemmas we discussed above.

6.1 Proofs

Recall the relevant definitions concerning normalizability:

$$\begin{aligned} M \not\rightarrow_\beta &\triangleq \forall M' . \neg (M \rightarrow_\beta M') \\ \text{norm}_\beta(M) &\triangleq \exists M' . M \rightarrow_\beta^* M' \wedge M' \not\rightarrow_\beta \end{aligned}$$

norm-if-H	norm-if-N
$\frac{\Delta \xRightarrow{\mathcal{H}} M \in A}{\text{norm}_\beta(M)}$	$\frac{\Delta \xRightarrow{\mathcal{N}} M \in A}{\text{norm}_\beta(M)}$

Proof by simultaneous induction on $\Delta \xRightarrow{\mathcal{H}} M \in A$ and $\Delta \xRightarrow{\mathcal{N}} M \in A$:

$$\boxed{\text{CASE}} \quad \frac{M \rightarrow_\beta^* \text{zero}}{\Delta \xRightarrow{\mathcal{H}} M \in \text{nat}} \quad [\text{H-ZeroM}]$$

$\text{norm}_\beta(M)$

$$\frac{M \rightarrow_\beta^* \text{zero} \quad \xRightarrow{\text{norm}_\beta(\text{zero})} 1}{\text{norm}_\beta(M)} \quad 2$$

1 WN-ZeroM 2 WN-Head-Expansion

$$\boxed{\text{CASE}} \quad \frac{M \rightarrow_\beta^* \text{succ } M' \quad \Delta \xRightarrow{\mathcal{H}} M' \in \text{nat}}{\Delta \xRightarrow{\mathcal{H}} M \in \text{nat}} \quad [\text{H-SuccM}]$$

$\text{norm}_\beta(M)$

$$\begin{array}{c}
\frac{\Delta \xRightarrow{\mathcal{H}} M' \in \text{nat}}{\text{norm}_\beta(M')} \xrightarrow{1} \\
\frac{M \rightarrow_\beta^* \text{succ } M' \quad \text{norm}_\beta(M')}{\text{norm}_\beta(\text{succ } M')} \xrightarrow{2} \\
\frac{}{\text{norm}_\beta(M)} \xrightarrow{3}
\end{array}$$

1 IH 2 WN-SuccM 3 WN-Head-Expansion

$$\boxed{\text{CASE}} \quad \frac{M \rightarrow_\beta^* \langle \rangle \quad \Delta \xRightarrow{\mathcal{H}} M \in 1}{\text{norm}_\beta(M)} \text{ [H-EmptyM]}$$

$$\begin{array}{c}
M \rightarrow_\beta^* \langle \rangle \quad \text{norm}_\beta(\langle \rangle) \xrightarrow{1} \\
\frac{}{\text{norm}_\beta(M)} \xrightarrow{2}
\end{array}$$

1 WN-EmptyM 2 WN-Head-Expansion

$$\boxed{\text{CASE}} \quad \frac{M \rightarrow_\beta^* M' \quad \Delta \xRightarrow{\mathcal{N}} M' \in 1}{\Delta \xRightarrow{\mathcal{H}} M \in 1} \text{ [H-Neutral-1T]}$$

$$\begin{array}{c}
\Delta \xRightarrow{\mathcal{N}} M' \in 1 \xrightarrow{1} \\
\frac{M \rightarrow_\beta^* M' \quad \text{norm}_\beta(M')}{\text{norm}_\beta(M)} \xrightarrow{2}
\end{array}$$

1 IH 2 WN-Head-Expansion

$$\boxed{\text{CASE}} \quad \frac{\Delta \xRightarrow{\mathcal{H}} M \cdot L \in A_L \quad \Delta \xRightarrow{\mathcal{H}} M \cdot R \in A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L \times A_R} \text{ [H-TimesT]}$$

$$\begin{array}{c}
\frac{\Delta \xRightarrow{\mathcal{H}} M \cdot L \in A_L}{\text{norm}_\beta(M \cdot L)} \xrightarrow{1} \quad \frac{\Delta \xRightarrow{\mathcal{H}} M \cdot R \in A_R}{\text{norm}_\beta(M \cdot R)} \xrightarrow{2} \\
\frac{}{\text{norm}_\beta(M)} \xrightarrow{3}
\end{array}$$

1 IH 2 IH 3 WN-TimesT

$$\begin{array}{c}
 \boxed{\text{CASE}} \quad \frac{x \notin \Delta \quad \Delta, x : A \xRightarrow{\mathcal{H}} M \quad x \in B}{\Delta \xRightarrow{\mathcal{H}} M \in A \rightarrow B} \text{ [H-ArrowT]} \\
 \text{-----} \\
 \text{norm}_\beta(M)
 \end{array}$$

$$\begin{array}{c}
 \Delta, x : A \xRightarrow{\mathcal{H}} M \quad x \in B \\
 \xrightarrow{\text{norm}_\beta(M \ x)} 1 \\
 \xrightarrow{\text{norm}_\beta(M)} 2
 \end{array}$$

1 IH 2 WN-ArrowT

$$\begin{array}{c}
 \boxed{\text{CASE}} \quad \frac{M \rightarrow_\beta^* L \cdot M_L \quad \Delta \xRightarrow{\mathcal{H}} M_L \in A_L}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R} \text{ [H-InLM]} \\
 \text{-----} \\
 \text{norm}_\beta(M)
 \end{array}$$

$$\begin{array}{c}
 \Delta \xRightarrow{\mathcal{H}} M_L \in A_L \\
 \xrightarrow{\text{norm}_\beta(M_L)} 1 \\
 M \rightarrow_\beta^* L \cdot M_L \quad \xrightarrow{\text{norm}_\beta(L \cdot M_L)} 2 \\
 \xrightarrow{\text{norm}_\beta(M)} 3
 \end{array}$$

1 IH 2 WN-InLM 3 WN-Head-Expansion

$$\begin{array}{c}
 \boxed{\text{CASE}} \quad \frac{M \rightarrow_\beta^* R \cdot M_R \quad \Delta \xRightarrow{\mathcal{H}} M_R \in A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R} \text{ [H-InRM]} \\
 \text{-----} \\
 \text{norm}_\beta(M)
 \end{array}$$

Entirely analogous to the previous case.

$$\begin{array}{c}
 \boxed{\text{CASE}} \quad \frac{M \rightarrow_\beta^* M' \quad \Delta \xRightarrow{\mathcal{N}} M' \in A_L + A_R}{\Delta \xRightarrow{\mathcal{H}} M \in A_L + A_R} \text{ [H-Neutral-SumT]} \\
 \text{-----} \\
 \text{norm}_\beta(M)
 \end{array}$$

$$\begin{array}{c}
\Delta \xRightarrow{\mathcal{N}} M' \in A_L + A_R \quad 1 \\
\hline
M \rightarrow_{\beta}^* M' \quad \text{norm}_{\beta}(M') \\
\hline
\text{norm}_{\beta}(M) \quad 2
\end{array}$$

1 IH 2 WN-Head-Expansion

$$\boxed{\text{CASE}} \quad \frac{M \rightarrow_{\beta}^* M' \quad \Delta \xRightarrow{\mathcal{N}} M' \in \text{nat}}{\Delta \xRightarrow{\mathcal{H}} M \in \text{nat} \quad \text{norm}_{\beta}(M)} \quad [\text{H-Neutral-NatT}]$$

$$\begin{array}{c}
\Delta \xRightarrow{\mathcal{N}} M' \in \text{nat} \quad 1 \\
\hline
M \rightarrow_{\beta}^* M' \quad \text{norm}_{\beta}(M') \\
\hline
\text{norm}_{\beta}(M) \quad 2
\end{array}$$

1 IH 2 WN-Head-Expansion

$$\boxed{\text{CASE}} \quad \frac{\Delta, x : A \xRightarrow{\mathcal{N}} x \in A}{\text{norm}_{\beta}(x)} \quad [\text{N-VarM}]$$

$$\begin{array}{c}
\hline
\text{norm}_{\beta}(x) \quad 1
\end{array}$$

1 WN-VarM

$$\boxed{\text{CASE}} \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R}{\Delta \xRightarrow{\mathcal{N}} M \cdot L \in A_L \quad \text{norm}_{\beta}(M \cdot L)} \quad [\text{N-projLM}]$$

$$\begin{array}{c}
\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R \quad 1 \\
\hline
\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R \quad \text{norm}_{\beta}(M) \\
\hline
\text{norm}_{\beta}(M \cdot L) \quad 2
\end{array}$$

1 IH 2 WN-ProjLM

$$\boxed{\text{CASE}} \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R}{\Delta \xRightarrow{\mathcal{N}} M \cdot R \in A_R} \text{ [N-projRM]}$$

$$\frac{}{\text{norm}_\beta(M \cdot R)}$$

$$\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R}{\text{norm}_\beta(M)} \xrightarrow{1}}{\text{norm}_\beta(M \cdot R)} \xrightarrow{2}$$

1 IH 2 WN-ProjRM

$$\boxed{\text{CASE}} \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A \rightarrow B \quad \Delta \xRightarrow{\mathcal{H}} N \in A}{\Delta \xRightarrow{\mathcal{N}} M N \in B} \text{ [N-AppM]}$$

$$\frac{}{\text{norm}_\beta(M N)}$$

$$\frac{\Delta \xRightarrow{\mathcal{N}} M \in A \rightarrow B \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A \rightarrow B}{\text{norm}_\beta(M)} \xrightarrow{1} \quad \frac{\Delta \xRightarrow{\mathcal{H}} N \in A}{\text{norm}_\beta(N)} \xrightarrow{2}}{\text{norm}_\beta(M N)} \xrightarrow{3}$$

1 IH 2 IH 3 WN-Neutral-AppM

$$\boxed{\text{CASE}} \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L + A_R \quad \frac{}{\Delta, x_L : A_L \xRightarrow{\mathcal{H}} M_L \in B} \quad \frac{}{\Delta, x_R : A_R \xRightarrow{\mathcal{H}} M_R \in B}}{\Delta \xRightarrow{\mathcal{N}} \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B} \text{ [N-CaseM]}$$

$$\frac{}{\text{norm}_\beta(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R))}$$

$$\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L + A_R \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L + A_R}{\text{norm}_\beta(M)} \xrightarrow{1} \quad \frac{}{\Delta, x_L : A_L \xRightarrow{\mathcal{H}} M_L \in B} \quad \frac{}{\Delta, x_R : A_R \xRightarrow{\mathcal{H}} M_R \in B}}{\text{norm}_\beta(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R))} \xrightarrow{4}$$

$$\frac{\text{norm}_\beta(M_L) \quad \text{norm}_\beta(M_R)}{\text{norm}_\beta(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R))} \xrightarrow{4}$$

1 IH 2 IH 3 IH 4 WN-Neutral-CaseM

$$\begin{array}{c}
\Delta \stackrel{\mathcal{N}}{\Rightarrow} M \in \text{nat} \\
\hline
\Delta \stackrel{\mathcal{H}}{\Rightarrow} M_0 \in A \\
\hline
\text{CASE} \quad \frac{\Delta, x : \text{nat}, y : A \stackrel{\mathcal{H}}{\Rightarrow} M_S \in A}{\Delta \stackrel{\mathcal{N}}{\Rightarrow} \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \in A} \text{ [N-NatRecM]} \\
\hline
\text{norm}_\beta(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S))
\end{array}$$

$$\begin{array}{c}
\Delta \stackrel{\mathcal{N}}{\Rightarrow} M \in \text{nat} \quad \frac{\Delta \stackrel{\mathcal{N}}{\Rightarrow} M \in \text{nat}}{\text{norm}_\beta(M)} \xrightarrow{1} \\
\hline
\Delta \stackrel{\mathcal{H}}{\Rightarrow} M_0 \in A \quad \Delta, x : \text{nat}, y : A \stackrel{\mathcal{H}}{\Rightarrow} M_S \in A \\
\frac{\Delta \stackrel{\mathcal{H}}{\Rightarrow} M_0 \in A}{\text{norm}_\beta(M_0)} \xrightarrow{2} \quad \frac{\Delta, x : \text{nat}, y : A \stackrel{\mathcal{H}}{\Rightarrow} M_S \in A}{\text{norm}_\beta(M_S)} \xrightarrow{3} \\
\frac{\text{norm}_\beta(M_0) \quad \text{norm}_\beta(M_S)}{\text{norm}_\beta(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S))} \xrightarrow{4}
\end{array}$$

1 IH 2 IH 3 IH 4 WN-Neutral-NatRecM □

WN-Head-Expansion

$$\frac{M \rightarrow_\beta^* M' \quad \text{norm}_\beta(M')}{\text{norm}_\beta(M)} \xrightarrow{\quad}$$

Proof by showing the following equivalent statement:

$$\frac{M \rightarrow_\beta^* M' \quad M' \rightarrow_\beta^* P \quad P \not\rightarrow_\beta}{\text{norm}_\beta(M)} \quad \frac{M \rightarrow_\beta^* M' \quad M' \rightarrow_\beta^* P}{M \rightarrow_\beta^* P} \xrightarrow{1} \quad \frac{M \rightarrow_\beta^* P \quad P \not\rightarrow_\beta}{\text{norm}_\beta(M)} \xrightarrow{2}$$

1 Beta-Multi-Trans 2 fold $\text{norm}_\beta(M)$ □

WN-VarM

$$\frac{}{\text{norm}_\beta(x)} \xrightarrow{\quad}$$

Proof:

$$\begin{array}{c}
 \frac{x \rightarrow_{\beta}^* x}{x \rightarrow_{\beta}^* x} \quad 1 \quad \frac{\frac{\frac{x \rightarrow_{\beta} M}{\perp} \quad 2}{\forall M . \neg (x \rightarrow_{\beta} M)} \quad 3}{x \not\rightarrow_{\beta} M} \quad 4 \\
 \hline
 \text{norm}_{\beta}(x) \quad 5
 \end{array}$$

1 Beta-Multi-Refl 2 inversion 3 $\forall / \Rightarrow$ -I 4 fold $x \not\rightarrow_{\beta}$ 5 fold $\text{norm}_{\beta}(x)$ □

WN-EmptyM

$$\xRightarrow{\text{norm}_{\beta}(\langle \rangle)}$$

WN-ZeroM

$$\xRightarrow{\text{norm}_{\beta}(\text{zero})}$$

Proof entirely analogous to that of [WN-VarM]. □

WN-SuccM

$$\xRightarrow{\text{norm}_{\beta}(\text{succ } M)}$$

Proof by showing the following equivalent statement:

$$\frac{M \rightarrow_{\beta}^* P \quad P \not\rightarrow_{\beta}}{\text{norm}_{\beta}(\text{succ } M)}$$

$$\begin{array}{c}
 \frac{M \rightarrow_{\beta}^* P}{\text{succ } M \rightarrow_{\beta}^* \text{succ } P} \quad 1 \quad \frac{\frac{\frac{\text{succ } P \rightarrow_{\beta} Q}{P \not\rightarrow_{\beta} \quad P \rightarrow_{\beta} P'}{\perp} \quad 2}{\forall Q . \neg (\text{succ } P \rightarrow_{\beta} Q)} \quad 3}{\text{succ } P \not\rightarrow_{\beta}} \quad 4 \\
 \hline
 \text{norm}_{\beta}(\text{succ } M) \quad 5
 \end{array}$$

1 β^* -under 2 inversion 3 unfold $P \not\rightarrow_\beta$ and $\forall / \Rightarrow$ -E 4 $\forall / \Rightarrow$ -I 5 fold succ $P \not\rightarrow_\beta$ 6 fold $\text{norm}_\beta(\text{succ } M)$ □

We prove a helper lemma, which shows that the property of being normalizable is forward closed, as a corollary of [Confluence]:

$$\text{norm-Forward-Closure} \quad \frac{\text{norm}_\beta(M) \quad M \rightarrow_\beta^* M'}{\text{norm}_\beta(M')}$$

Proof by showing the following equivalent statement:

$$\frac{M \rightarrow_\beta^* P \quad P \not\rightarrow_\beta \quad M \rightarrow_\beta^* M'}{\text{norm}_\beta(M')}$$

$$\frac{\frac{\frac{M \rightarrow_\beta^* P \quad M \rightarrow_\beta^* M'}{\exists Q . P \rightarrow_\beta^* Q \wedge M' \rightarrow_\beta^* Q} \xrightarrow{1} \quad \frac{P \rightarrow_\beta^* Q \wedge M' \rightarrow_\beta^* Q}{P \rightarrow_\beta^* Q \wedge M' \rightarrow_\beta^* Q} \xrightarrow{2} \quad P \not\rightarrow_\beta}{\frac{M' \rightarrow_\beta^* P \wedge P \not\rightarrow_\beta}{\text{norm}_\beta(M')} \xrightarrow{4} \quad \text{norm}_\beta(M')} \xrightarrow{3}$$

1 Confluence 2 pick Q 3 inversion on $P \rightarrow_\beta^* Q$ 4 fold $\text{norm}_\beta(M')$ □

$$\text{WN-TimesT} \quad \frac{\text{norm}_\beta(M \cdot L) \quad \text{norm}_\beta(M \cdot R)}{\text{norm}_\beta(M)}$$

Proof by showing the following equivalent statement:

$$\frac{M \cdot L \rightarrow_\beta^* P \quad P \not\rightarrow_\beta \quad \text{norm}_\beta(M \cdot R)}{\text{norm}_\beta(M)}$$

Proceed by induction on $M \cdot L \rightarrow_\beta^* P$.

CASE $\frac{\overline{M \cdot L \rightarrow_\beta^* M \cdot L} \quad [\text{Beta-Multi-Refl}] \quad M \cdot L \not\rightarrow_\beta \quad \text{norm}_\beta(M \cdot R)}{\text{norm}_\beta(M)}$

$$\begin{array}{c}
\frac{M \rightarrow_{\beta} M'}{M \cdot L \rightarrow_{\beta} M' \cdot L} \quad \frac{M \cdot L \rightarrow_{\beta} M' \cdot L}{\perp} \quad \frac{\perp}{\forall M' . \neg (M \rightarrow_{\beta} M')} \quad \frac{\forall M' . \neg (M \rightarrow_{\beta} M')}{M \not\rightarrow_{\beta}} \quad \frac{M \rightarrow_{\beta}^* M}{\text{norm}_{\beta}(M)} \quad \frac{M \not\rightarrow_{\beta}}{\text{norm}_{\beta}(M)}
\end{array}$$

2
3
4
5
6

1 Beta-Multi-Refl 2 ProjL-Under 3 unfold $M \cdot L \not\rightarrow_{\beta}$ and $\forall / \Rightarrow$ -E 4 $\forall / \Rightarrow$ -I 5 fold $M \not\rightarrow_{\beta}$ 6 fold $\text{norm}_{\beta}(M)$

$$\boxed{\text{CASE}} \quad \frac{M \cdot L \rightarrow_{\beta} N \quad N \rightarrow_{\beta}^* P}{M \cdot L \rightarrow_{\beta}^* P} [\text{Beta-Multi-Cons}] \quad \frac{P \not\rightarrow_{\beta} \quad \text{norm}_{\beta}(M \cdot R)}{\text{norm}_{\beta}(M)}$$

Inversion on $M \cdot L \rightarrow_{\beta} N$.

$$\boxed{\text{SUBCASE}} \quad \frac{\frac{M \rightarrow_{\beta} M'}{M \cdot L \rightarrow_{\beta} M' \cdot L} [\text{ProjL-Under}] \quad M' \cdot L \rightarrow_{\beta}^* P}{M \cdot L \rightarrow_{\beta}^* P} \quad \frac{P \not\rightarrow_{\beta} \quad \text{norm}_{\beta}(M \cdot R)}{\text{norm}_{\beta}(M)}$$

$$\begin{array}{c}
\frac{M \rightarrow_{\beta} M'}{M \cdot R \rightarrow_{\beta} M' \cdot R} \quad \frac{M \cdot R \rightarrow_{\beta} M' \cdot R}{M \cdot R \rightarrow_{\beta}^* M' \cdot R} \quad \frac{M \cdot R \rightarrow_{\beta}^* M' \cdot R}{\text{norm}_{\beta}(M' \cdot R)} \quad \frac{\text{norm}_{\beta}(M' \cdot R)}{\text{norm}_{\beta}(M')} \quad \frac{\text{norm}_{\beta}(M')}{\text{norm}_{\beta}(M)}
\end{array}$$

2
3
4
5
6

1 Beta-Multi-if-Single 2 ProjR-Under 3 Beta-Multi-if-Single 4 norm-Forward-Closure 5 IH 6 WN-Head-Expansion

$$\boxed{\text{SUBCASE}} \quad \frac{\frac{\langle M_L, M_R \rangle \cdot L \rightarrow_{\beta} M_L}{\langle M_L, M_R \rangle \cdot L \rightarrow_{\beta}^* P} [\text{ProjL-beta}] \quad M_L \rightarrow_{\beta}^* P}{\langle M_L, M_R \rangle \cdot L \rightarrow_{\beta}^* P} \quad \frac{P \not\rightarrow_{\beta} \quad \text{norm}_{\beta}(\langle M_L, M_R \rangle \cdot R)}{\text{norm}_{\beta}(\langle M_L, M_R \rangle)}$$

First, the right component is normalizable:

$$\begin{array}{c}
\frac{\frac{\frac{\langle M_L, M_R \rangle \cdot R \rightarrow_\beta M_R}{\text{norm}_\beta(\langle M_L, M_R \rangle \cdot R)} \quad \frac{\langle M_L, M_R \rangle \cdot R \rightarrow_\beta^* M_R}{\text{norm}_\beta(\langle M_L, M_R \rangle \cdot R)}}{\text{norm}_\beta(M_R)} \quad \frac{\text{norm}_\beta(M_R)}{M_R \rightarrow_\beta^* Q \wedge Q \not\rightarrow_\beta} \quad \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}$$

1 ProjR-beta 2 Beta-Multi-if-Single 3 norm-Forward-Closure 4 unfold $\text{norm}_\beta(M_R)$ and pick Q

Moreover:

$$\begin{array}{c}
\frac{\frac{P \not\rightarrow_\beta \quad Q \not\rightarrow_\beta \quad \frac{\langle P, Q \rangle \rightarrow_\beta R}{P \rightarrow_\beta P' \vee Q \rightarrow_\beta Q'}}{\perp} \quad \frac{\perp}{\forall R. \neg (\langle P, Q \rangle \rightarrow_\beta R)} \quad \frac{\forall R. \neg (\langle P, Q \rangle \rightarrow_\beta R)}{\langle P, Q \rangle \not\rightarrow_\beta} \quad \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}$$

1 inversion 2 $\vee$ -E 3 $\forall / \Rightarrow$ -I 4 fold $\langle P, Q \rangle \not\rightarrow_\beta$

Finally:

$$\begin{array}{c}
\frac{\frac{M_L \rightarrow_\beta^* P}{\langle M_L, M_R \rangle \rightarrow_\beta^* \langle P, M_R \rangle} \quad \frac{M_R \rightarrow_\beta^* Q}{\langle P, M_R \rangle \rightarrow_\beta^* \langle P, Q \rangle}}{\langle M_L, M_R \rangle \rightarrow_\beta^* \langle P, Q \rangle} \quad \frac{\langle M_L, M_R \rangle \rightarrow_\beta^* \langle P, Q \rangle}{\text{norm}_\beta(\langle M_L, M_R \rangle)} \quad \frac{\text{norm}_\beta(\langle M_L, M_R \rangle)}{\langle P, Q \rangle \not\rightarrow_\beta} \quad \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}$$

1 β^* -under 2 β^* -under 3 Beta-Multi-Trans 4 fold $\text{norm}_\beta(\langle M_L, M_R \rangle)$

□

Irrd-lam-if-Irrd-under

$$\frac{P \not\rightarrow_\beta}{\lambda x \Rightarrow P \not\rightarrow_\beta}$$

Proof:

$$\begin{array}{c}
\frac{\lambda x \Rightarrow P \rightarrow_{\beta} Q}{P \not\rightarrow_{\beta} \quad P \rightarrow_{\beta} P'} \quad 1 \\
\frac{}{\perp} \quad 2 \\
\frac{}{\forall Q . \neg (\lambda x \Rightarrow P \rightarrow_{\beta} Q)} \quad 3 \\
\frac{}{\lambda x \Rightarrow P \not\rightarrow_{\beta}} \quad 4
\end{array}$$

1 inversion 2 unfold $P \not\rightarrow_{\beta}$ and $\forall / \Rightarrow$ -E 3 $\forall / \Rightarrow$ -I 4 fold $\lambda x \Rightarrow P \not\rightarrow_{\beta}$ □

WN-ArrowT

$$\frac{\text{norm}_{\beta}(M x)}{\text{norm}_{\beta}(M)} \Rightarrow$$

Proof by showing the following equivalent statement:

$$\frac{M x \rightarrow_{\beta}^* P \quad P \not\rightarrow_{\beta}}{\text{norm}_{\beta}(M)}$$

Proceed by induction on $M x \rightarrow_{\beta}^* P$.

CASE $\frac{\overline{M x \rightarrow_{\beta}^* M x} \quad [\text{Beta-Multi-Refl}] \quad M x \not\rightarrow_{\beta}}{\text{norm}_{\beta}(M)}$

$$\begin{array}{c}
\frac{\frac{\frac{M \rightarrow_{\beta} M'}{M x \rightarrow_{\beta} M' x} \quad 2}{\perp} \quad 3}{\forall M' . \neg (M \rightarrow_{\beta} M')} \quad 4 \\
\frac{\overline{M \rightarrow_{\beta}^* M} \quad 1 \quad M \not\rightarrow_{\beta}}{\text{norm}_{\beta}(M)} \quad 5 \quad 6
\end{array}$$

1 Beta-Multi-Refl 2 App-Under-Fun 3 unfold $M x \not\rightarrow_{\beta}$ and $\forall / \Rightarrow$ -E 4 $\forall / \Rightarrow$ -I 5 fold $M \not\rightarrow_{\beta}$ 6 fold $\text{norm}_{\beta}(M)$

CASE $\frac{\frac{M x \rightarrow_{\beta} N \quad N \rightarrow_{\beta}^* P}{M x \rightarrow_{\beta}^* P} \quad [\text{Beta-Multi-Cons}] \quad P \not\rightarrow_{\beta}}{\text{norm}_{\beta}(M)}$

Inversion on $M x \rightarrow_{\beta} N$.

$$\begin{array}{c}
\text{SUBCASE} \quad \frac{\frac{M \rightarrow_{\beta} M'}{M x \rightarrow_{\beta} M' x} [\text{App-Under-Fun}] \quad M' x \rightarrow_{\beta}^* P}{M x \rightarrow_{\beta}^* P} \quad P \not\rightarrow_{\beta} \\
\hline
\text{norm}_{\beta}(M)
\end{array}$$

$$\begin{array}{c}
\frac{\frac{M \rightarrow_{\beta} M'}{M \rightarrow_{\beta}^* M'} \quad 1 \quad \frac{\frac{M' x \rightarrow_{\beta}^* P \quad P \not\rightarrow_{\beta}}{\text{norm}_{\beta}(M')} \rightarrow 2}{\text{norm}_{\beta}(M)} \rightarrow 3 \\
\text{norm}_{\beta}(M)
\end{array}$$

1 Beta-Multi-if-Single 2 IH 3 WN-Head-Expansion

$$\begin{array}{c}
\text{SUBCASE} \quad \frac{\frac{x \rightarrow_{\beta} x'}{M x \rightarrow_{\beta} M x'} [\text{App-Under-Arg}] \quad M x' \rightarrow_{\beta}^* P}{M x \rightarrow_{\beta}^* P} \quad P \not\rightarrow_{\beta} \\
\hline
\text{norm}_{\beta}(M)
\end{array}$$

$$\begin{array}{c}
\frac{x \rightarrow_{\beta} x'}{\perp} \quad 1 \\
\hline
\text{norm}_{\beta}(M) \quad 2
\end{array}$$

1 inversion 2 $\perp$ -E

$$\begin{array}{c}
\text{SUBCASE} \quad \frac{\frac{\lambda x \Rightarrow M' x \rightarrow_{\beta} M'}{\lambda x \Rightarrow M' x \rightarrow_{\beta}^* P} [\text{App-beta}] \quad M' \rightarrow_{\beta}^* P}{\lambda x \Rightarrow M' x \rightarrow_{\beta}^* P} \quad P \not\rightarrow_{\beta} \\
\hline
\text{norm}_{\beta}(\lambda x \Rightarrow M')
\end{array}$$

$$\begin{array}{c}
\frac{\frac{M' \rightarrow_{\beta}^* P}{\lambda x \Rightarrow M' \rightarrow_{\beta}^* \lambda x \Rightarrow P} \rightarrow 1 \quad \frac{P \not\rightarrow_{\beta}}{\lambda x \Rightarrow P \not\rightarrow_{\beta}} \rightarrow 2}{\text{norm}_{\beta}(\lambda x \Rightarrow M')}
\end{array}$$

1 β^* -under 2 Irrd-lam-if-Irrd-under

□

WN-InLM

$$\frac{\text{norm}_\beta(M)}{\text{norm}_\beta(L \cdot M)}$$

Proof by showing the following equivalent statement:

$$\frac{M \rightarrow_\beta^* P \quad P \not\rightarrow_\beta}{\text{norm}_\beta(L \cdot M)}$$

$$\frac{\frac{M \rightarrow_\beta^* P}{L \cdot M \rightarrow_\beta^* L \cdot P} \quad 1 \quad \frac{\frac{\frac{L \cdot P \rightarrow_\beta Q}{P \rightarrow_\beta P'} \quad 2}{P \not\rightarrow_\beta} \quad 3 \quad \frac{\perp}{\forall Q. \neg (L \cdot P \rightarrow_\beta Q)} \quad 4}{L \cdot P \not\rightarrow_\beta} \quad 5}{\text{norm}_\beta(L \cdot M)} \quad 6$$

1 β^* -under 2 inversion 3 unfold $P \not\rightarrow_\beta$ and $\forall / \Rightarrow$ -E 4 $\forall / \Rightarrow$ -I 5 fold $L \cdot P \not\rightarrow_\beta$ 6 fold $\text{norm}_\beta(L \cdot M)$ □

WN-InRM

$$\frac{\text{norm}_\beta(M)}{\text{norm}_\beta(R \cdot M)}$$

Proof entirely analogous to that of the previous lemma. □

Intro(M)

EmptyM-is-Intro

PairM-is-Intro

LamM-is-Intro

$\overline{\text{Intro}(\langle \rangle)}$

$\overline{\text{Intro}(\langle M_L, M_R \rangle)}$

$\overline{\text{Intro}(\lambda x \Rightarrow M)}$

InLM-is-Intro

InRM-is-Intro

ZeroM-is-Intro

SuccM-is-Intro

$\overline{\text{Intro}(L \cdot M)}$

$\overline{\text{Intro}(R \cdot M)}$

$\overline{\text{Intro}(\text{zero})}$

$\overline{\text{Intro}(\text{succ } M)}$

Spine(M)

$$\begin{array}{c}
\text{Spine-Var} \quad \text{Spine-ProjL} \quad \text{Spine-ProjR} \quad \text{Spine-App} \\
\frac{}{\text{Spine}(x)} \quad \frac{\text{Spine}(M)}{\text{Spine}(M \cdot L)} \quad \frac{\text{Spine}(M)}{\text{Spine}(M \cdot R)} \quad \frac{\text{Spine}(M)}{\text{Spine}(M \ N)} \\
\\
\text{Spine-Case} \quad \text{Spine-NatRec} \\
\frac{\text{Spine}(M)}{\text{Spine}(\text{case } M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R))} \quad \frac{\text{Spine}(M)}{\text{Spine}(\text{natrec } M \ (M_0 \mid x, y \Rightarrow M_S))}
\end{array}$$

Not-Intro-if-Spine

$$\frac{\text{Spine}(M)}{\neg\text{Intro}(M)}$$

Proof by induction on $\text{Spine}(M)$. In each case, introduce $\text{Intro}(M)$ and then immediately perform inversion on this judgement to derive $\perp$. $\square$

Spine-if-Neutral

$$\frac{\Delta \xRightarrow{\mathcal{N}} M \in A}{\text{Spine}(M)}$$

Proof by induction on $\Delta \xRightarrow{\mathcal{N}} M \in A$:

$$\boxed{\text{CASE}} \quad \frac{}{\Delta, x : A \xRightarrow{\mathcal{N}} x \in A} \text{[N-VarM]}$$

$$\frac{}{\text{Spine}(x)} \quad 1$$

1 Spine-Var

$$\boxed{\text{CASE}} \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R}{\Delta \xRightarrow{\mathcal{N}} M \cdot L \in A_L} \text{[N-projLM]}$$

$$\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R}{\frac{\text{Spine}(M)}{\text{Spine}(M \cdot L)}} \quad 1 \quad 2$$

1 IH 2 Spine-ProjL

$$\boxed{\text{CASE}} \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R}{\Delta \xRightarrow{\mathcal{N}} M \cdot R \in A_R} \text{ [N-projRM]}$$

$$\frac{}{\text{Spine}(M \cdot R)}$$

$$\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R}{\text{Spine}(M)} \xrightarrow{1}$$

$$\frac{}{\text{Spine}(M \cdot R)} \xrightarrow{2}$$

1 IH 2 Spine-ProjR

$$\boxed{\text{CASE}} \quad \frac{\Delta \xRightarrow{\mathcal{N}} M \in A \rightarrow B \quad \Delta \xRightarrow{\mathcal{H}} N \in A}{\Delta \xRightarrow{\mathcal{N}} M N \in B} \text{ [N-AppM]}$$

$$\frac{}{\text{Spine}(M N)}$$

$$\frac{\Delta \xRightarrow{\mathcal{N}} M \in A \rightarrow B}{\text{Spine}(M)} \xrightarrow{1}$$

$$\frac{}{\text{Spine}(M N)} \xrightarrow{2}$$

1 IH 2 Spine-App

$$\boxed{\text{CASE}} \quad \frac{\begin{array}{c} \Delta \xRightarrow{\mathcal{N}} M \in A_L + A_R \\ \sim \\ \Delta, x_L : A_L \xRightarrow{\mathcal{H}} M_L \in B \\ \sim \\ \Delta, x_R : A_R \xRightarrow{\mathcal{H}} M_R \in B \end{array}}{\Delta \xRightarrow{\mathcal{N}} \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \in B} \text{ [N-CaseM]}$$

$$\frac{}{\text{Spine}(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R))}$$

$$\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L + A_R}{\text{Spine}(M)} \xrightarrow{1}$$

$$\frac{}{\text{Spine}(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R))} \xrightarrow{2}$$

1 IH 2 Spine-Case

$$\begin{array}{c}
\Delta \stackrel{\mathcal{N}}{\Rightarrow} M \in \text{nat} \\
\hline
\Delta \stackrel{\mathcal{H}}{\Rightarrow} M_0 \in A \\
\hline
\text{CASE} \quad \frac{\Delta, x : \text{nat}, y : A \stackrel{\mathcal{H}}{\Rightarrow} M_S \in A}{\Delta \stackrel{\mathcal{N}}{\Rightarrow} \text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \in A} \text{ [N-NatRecM]} \\
\hline
\text{Spine}(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S))
\end{array}$$

$$\frac{\frac{\Delta \stackrel{\mathcal{N}}{\Rightarrow} M \in \text{nat}}{\text{Spine}(M)} 1}{\text{Spine}(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S))} 2$$

1 IH 2 Spine-NatRec

□

Spine-One-Step-Forward-Closure

$$\frac{\text{Spine}(M) \quad M \rightarrow_\beta M'}{\text{Spine}(M')}$$

Proof by induction on the derivation of $M \rightarrow_\beta M'$:

$$\begin{array}{c}
\text{CASE} \quad \frac{\text{Spine}(\langle M_L, M_R \rangle) \quad \frac{M_L \rightarrow_\beta M'_L}{\langle M_L, M_R \rangle \rightarrow_\beta \langle M'_L, M_R \rangle} \text{ [Pair-Under-L]} \\
\hline
\text{Spine}(\langle M'_L, M_R \rangle)
\end{array}$$

$$\frac{\frac{\text{Spine}(\langle M_L, M_R \rangle)}{\perp} 1}{\text{Spine}(\langle M'_L, M_R \rangle)} 2$$

1 inversion 2 $\perp$ -E

$$\begin{array}{c}
\text{CASE} \quad \frac{\text{Spine}(\langle M_L, M_R \rangle) \quad \frac{M_R \rightarrow_\beta M'_R}{\langle M_L, M_R \rangle \rightarrow_\beta \langle M_L, M'_R \rangle} \text{ [Pair-Under-R]} \\
\hline
\text{Spine}(\langle M_L, M'_R \rangle)
\end{array}$$

$$\frac{\frac{\text{Spine}(\langle M_L, M_R \rangle)}{\perp} 1}{\text{Spine}(\langle M_L, M'_R \rangle)} 2$$

1 inversion 2 $\perp$ -E

$$\boxed{\text{CASE}} \quad \frac{\text{Spine}(M \cdot L) \quad \frac{M \rightarrow_{\beta} M'}{M \cdot L \rightarrow_{\beta} M' \cdot L} [\text{ProjL-Under}]}{\text{Spine}(M' \cdot L)}$$

$$\frac{\frac{\text{Spine}(M \cdot L)}{\text{Spine}(M)} \quad 1 \quad M \rightarrow_{\beta} M'}{\frac{\text{Spine}(M')}{\text{Spine}(M' \cdot L)} \quad 3} \quad 2$$

1 inversion 2 IH 3 Spine-ProjL

$$\boxed{\text{CASE}} \quad \frac{\text{Spine}(M \cdot R) \quad \frac{M \rightarrow_{\beta} M'}{M \cdot R \rightarrow_{\beta} M' \cdot R} [\text{ProjR-Under}]}{\text{Spine}(M' \cdot R)}$$

$$\frac{\frac{\text{Spine}(M \cdot R)}{\text{Spine}(M)} \quad 1 \quad M \rightarrow_{\beta} M'}{\frac{\text{Spine}(M')}{\text{Spine}(M' \cdot R)} \quad 3} \quad 2$$

1 inversion 2 IH 3 Spine-ProjR

$$\boxed{\text{CASE}} \quad \frac{\text{Spine}(\langle M_L, M_R \rangle \cdot L) \quad \frac{}{\langle M_L, M_R \rangle \cdot L \rightarrow_{\beta} M_L} [\text{ProjL-beta}]}{\text{Spine}(M_L)}$$

$$\frac{\frac{\text{Spine}(\langle M_L, M_R \rangle \cdot L)}{\text{Spine}(\langle M_L, M_R \rangle)} \quad 1 \quad 2}{\frac{\perp}{\text{Spine}(M_L)} \quad 3}$$

1 inversion 2 inversion 3 $\perp$ -E

$$\boxed{\text{CASE}} \quad \frac{\text{Spine}(\langle M_L, M_R \rangle \cdot R) \quad \frac{}{\langle M_L, M_R \rangle \cdot R \rightarrow_{\beta} M_R} [\text{ProjR-beta}]}{\text{Spine}(M_R)}$$

$$\frac{\frac{\frac{\text{Spine}(\langle M_L, M_R \rangle \cdot R)}{\text{Spine}(\langle M_L, M_R \rangle)} \quad 1}{\perp} \quad 2}{\text{Spine}(M_R)} \quad 3$$

1 inversion 2 inversion 3 $\perp$ -E

$$\boxed{\text{CASE}} \quad \frac{\text{Spine}(\lambda x \Rightarrow M) \quad \frac{M \rightarrow_\beta M'}{\lambda x \Rightarrow M \rightarrow_\beta \lambda x \Rightarrow M'} \quad [\text{Lam-Under}]}{\text{Spine}(\lambda x \Rightarrow M')} \quad \text{-----}$$

$$\frac{\frac{\text{Spine}(\lambda x \Rightarrow M)}{\perp} \quad 1}{\text{Spine}(\lambda x \Rightarrow M')} \quad 2$$

1 inversion 2 $\perp$ -E

$$\boxed{\text{CASE}} \quad \frac{\text{Spine}(M N) \quad \frac{M \rightarrow_\beta M'}{M N \rightarrow_\beta M' N} \quad [\text{App-Under-Fun}]}{\text{Spine}(M' N)} \quad \text{-----}$$

$$\frac{\frac{\frac{\text{Spine}(M N)}{\text{Spine}(M)} \quad 1 \quad M \rightarrow_\beta M'}{\text{Spine}(M')} \quad 2}{\text{Spine}(M' N)} \quad 3$$

1 inversion 2 IH 3 Spine-App

$$\boxed{\text{CASE}} \quad \frac{\text{Spine}(M N) \quad \frac{N \rightarrow_\beta N'}{M N \rightarrow_\beta M N'} \quad [\text{App-Under-Arg}]}{\text{Spine}(M N')} \quad \text{-----}$$

$$\frac{\frac{\text{Spine}(M N)}{\text{Spine}(M)} \quad 1}{\text{Spine}(M N')} \quad 2$$

1 inversion 2 Spine-App

$$\boxed{\text{CASE}} \quad \frac{\text{Spine}(\lambda x \Rightarrow M N) \quad \overline{\lambda x \Rightarrow M N \rightarrow_{\beta} [N/x](M)}}{\text{Spine}([N/x](M))} \text{[App-beta]}$$

$$\frac{\frac{\frac{\text{Spine}(\lambda x \Rightarrow M N)}{\text{Spine}(\lambda x \Rightarrow M)} \quad 1}{\perp} \quad 2}{\text{Spine}([N/x](M))} \quad 3$$

1 inversion 2 inversion 3 $\perp$ -E

$$\boxed{\text{CASE}} \quad \frac{\text{Spine}(L \cdot M) \quad \frac{M \rightarrow_{\beta} M'}{L \cdot M \rightarrow_{\beta} L \cdot M'} \text{[InL-Under]}}{\text{Spine}(L \cdot M')} \text{[InL-Under]}$$

$$\frac{\frac{\text{Spine}(L \cdot M)}{\perp} \quad 1}{\text{Spine}(L \cdot M')} \quad 2$$

1 inversion 2 $\perp$ -E

$$\boxed{\text{CASE}} \quad \frac{\text{Spine}(R \cdot M) \quad \frac{M \rightarrow_{\beta} M'}{R \cdot M \rightarrow_{\beta} R \cdot M'} \text{[InR-Under]}}{\text{Spine}(R \cdot M')} \text{[InR-Under]}$$

$$\frac{\frac{\text{Spine}(R \cdot M)}{\perp} \quad 1}{\text{Spine}(R \cdot M')} \quad 2$$

1 inversion 2 $\perp$ -E

$$\boxed{\text{CASE}} \quad \frac{\frac{\text{Spine}(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R))}{M \rightarrow_{\beta} M'} \quad \text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_{\beta} \text{case } M' (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)}{\text{Spine}(\text{case } M' (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R))} \text{[Case-Under-Main]}$$

$$\begin{array}{c}
\text{Spine}(\text{case } M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \\
\hline
\text{Spine}(M) \quad 1 \quad M \rightarrow_\beta M' \\
\hline
\text{Spine}(M') \quad 2 \\
\hline
\text{Spine}(\text{case } M' \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \quad 3
\end{array}$$

1 inversion 2 IH 3 Spine-Case

$$\begin{array}{c}
\text{Spine}(\text{case } M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \\
\hline
M_L \rightarrow_\beta M'_L \\
\hline
\text{case } M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_\beta \text{case } M \ (x_L \Rightarrow M'_L \mid x_R \Rightarrow M_R) \quad [\text{Case-Under-L}] \\
\hline
\text{Spine}(\text{case } M \ (x_L \Rightarrow M'_L \mid x_R \Rightarrow M_R))
\end{array}$$

$$\begin{array}{c}
\text{Spine}(\text{case } M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \\
\hline
\text{Spine}(M) \quad 1 \\
\hline
\text{Spine}(\text{case } M \ (x_L \Rightarrow M'_L \mid x_R \Rightarrow M_R)) \quad 2
\end{array}$$

1 inversion 2 Spine-Case

$$\begin{array}{c}
\text{Spine}(\text{case } M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \\
\hline
M_R \rightarrow_\beta M'_R \\
\hline
\text{case } M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_\beta \text{case } M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M'_R) \quad [\text{Case-Under-R}] \\
\hline
\text{Spine}(\text{case } M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M'_R))
\end{array}$$

$$\begin{array}{c}
\text{Spine}(\text{case } M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \\
\hline
\text{Spine}(M) \quad 1 \\
\hline
\text{Spine}(\text{case } M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M'_R)) \quad 2
\end{array}$$

1 inversion 2 Spine-Case

$$\begin{array}{c}
\text{Spine}(\text{case } L \cdot M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \\
\hline
\text{case } L \cdot M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_\beta [M/x_L](M_L) \quad [\text{Case-InL-beta}] \\
\hline
\text{Spine}([M/x_L](M_L))
\end{array}$$

$$\begin{array}{c}
\text{Spine}(\text{case } L \cdot M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R)) \quad 1 \\
\hline
\text{Spine}(L \cdot M) \quad 2 \\
\hline
\perp \quad 3 \\
\hline
\text{Spine}([M/x_L](M_L))
\end{array}$$

1 inversion 2 inversion 3 $\perp$ -E

$$\boxed{\text{CASE}} \quad \frac{\frac{\text{Spine}(\text{case } R \cdot M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R))}{\text{case } R \cdot M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R) \rightarrow_\beta [M/x_R](M_R)} \text{ [Case-InR-beta]}}{\text{Spine}([M/x_R](M_R))}$$

$$\frac{\frac{\frac{\text{Spine}(\text{case } R \cdot M \ (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R))}{\text{Spine}(R \cdot M)} 1}{\perp} 2}{\text{Spine}([M/x_R](M_R))} 3$$

1 inversion 2 inversion 3 $\perp$ -E

$$\boxed{\text{CASE}} \quad \frac{\text{Spine}(\text{succ } M) \quad \frac{M \rightarrow_\beta M'}{\text{succ } M \rightarrow_\beta \text{succ } M'} \text{ [Succ-Under]}}{\text{Spine}(\text{succ } M')}$$

$$\frac{\frac{\text{Spine}(\text{succ } M)}{\perp} 1}{\text{Spine}(\text{succ } M')} 2$$

1 inversion 2 $\perp$ -E

$$\boxed{\text{CASE}} \quad \frac{\frac{\frac{\text{Spine}(\text{natrec } M \ (M_0 \mid x, y \Rightarrow M_S))}{M \rightarrow_\beta M'}}{\text{natrec } M \ (M_0 \mid x, y \Rightarrow M_S) \rightarrow_\beta \text{natrec } M' \ (M_0 \mid x, y \Rightarrow M_S)} \text{ [NatRec-Under-Main]}}{\text{Spine}(\text{natrec } M' \ (M_0 \mid x, y \Rightarrow M_S))}$$

$$\frac{\frac{\frac{\text{Spine}(\text{natrec } M \ (M_0 \mid x, y \Rightarrow M_S))}{\text{Spine}(M)} 1 \quad M \rightarrow_\beta M'}{\text{Spine}(M')} 2}{\text{Spine}(\text{natrec } M' \ (M_0 \mid x, y \Rightarrow M_S))} 3$$

1 inversion 2 IH 3 Spine-NatRec

$$\begin{array}{c}
\text{CASE} \quad \frac{\frac{\text{Spine}(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S))}{M_0 \rightarrow_\beta M'_0} \quad [\text{NatRec-Under-Zero}]}{\text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \rightarrow_\beta \text{natrec } M (M'_0 \mid x, y \Rightarrow M_S)} \\
\hline
\text{Spine}(\text{natrec } M (M'_0 \mid x, y \Rightarrow M_S))
\end{array}$$

$$\begin{array}{c}
\frac{\text{Spine}(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S))}{\text{Spine}(M)} \quad 1 \\
\hline
\text{Spine}(\text{natrec } M (M'_0 \mid x, y \Rightarrow M_S)) \quad 2
\end{array}$$

1 inversion 2 Spine-NatRec

$$\begin{array}{c}
\text{CASE} \quad \frac{\frac{\text{Spine}(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S))}{M_S \rightarrow_\beta M'_S} \quad [\text{NatRec-Under-Succ}]}{\text{natrec } M (M_0 \mid x, y \Rightarrow M_S) \rightarrow_\beta \text{natrec } M (M_0 \mid x, y \Rightarrow M'_S)} \\
\hline
\text{Spine}(\text{natrec } M (M_0 \mid x, y \Rightarrow M'_S))
\end{array}$$

$$\begin{array}{c}
\frac{\text{Spine}(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S))}{\text{Spine}(M)} \quad 1 \\
\hline
\text{Spine}(\text{natrec } M (M_0 \mid x, y \Rightarrow M'_S)) \quad 2
\end{array}$$

1 inversion 2 Spine-NatRec

$$\begin{array}{c}
\text{CASE} \quad \frac{\text{Spine}(\text{natrec zero } (M_0 \mid x, y \Rightarrow M_S)) \quad \text{natrec zero } (M_0 \mid x, y \Rightarrow M_S) \rightarrow_\beta M_0 \quad [\text{NatRec-Zero-beta}]}{\text{Spine}(M_0)} \\
\hline
\text{Spine}(M_0)
\end{array}$$

$$\begin{array}{c}
\frac{\text{Spine}(\text{natrec zero } (M_0 \mid x, y \Rightarrow M_S))}{\text{Spine}(\text{zero})} \quad 1 \\
\hline
\frac{\perp}{\text{Spine}(M_0)} \quad 2 \\
\hline
\text{Spine}(M_0) \quad 3
\end{array}$$

1 inversion 2 inversion 3 $\perp$ -E

$$\begin{array}{c}
\text{CASE} \quad \frac{\frac{\text{Spine}(\text{natrec succ } M (M_0 \mid x, y \Rightarrow M_S))}{\text{natrec succ } M (M_0 \mid x, y \Rightarrow M_S) \rightarrow_\beta [M/x, R/y](M_S)} \quad [\text{NatRec-Succ-beta}]}{\text{Spine}([M/x, R/y](M_S))} \\
\hline
\text{Spine}([M/x, R/y](M_S))
\end{array}$$

Define $R \triangleq \text{natrec } M (M_0 \mid x, y \Rightarrow M_S)$.

$$\begin{array}{c}
\text{Spine}(\text{natrec succ } M \ (M_0 \mid x, y \Rightarrow M_S)) \\
\hline
\text{Spine}(\text{succ } M) \quad 2 \\
\hline
\perp \quad 3 \\
\hline
\text{Spine}([M/x, R/y](M_S))
\end{array}$$

1 inversion 2 inversion 3 $\perp$ -E

□

Spine-Forward-Closure

$$\begin{array}{c}
\text{Spine}(M) \quad M \rightarrow_{\beta}^* M' \\
\hline
\text{Spine}(M')
\end{array}$$

Proof by induction on the derivation of $M \rightarrow_{\beta}^* M'$:

CASE $\text{Spine}(M) \quad \overline{M \rightarrow_{\beta}^* M} \quad [\text{Beta-Multi-Ref}]$

$$\begin{array}{c}
\text{Spine}(M) \\
\hline
\text{Spine}(M)
\end{array}$$

Spine(M)

CASE $\text{Spine}(M) \quad \frac{M \rightarrow_{\beta} P \quad P \rightarrow_{\beta}^* M'}{M \rightarrow_{\beta}^* M'} \quad [\text{Beta-Multi-Cons}]$

$$\begin{array}{c}
\text{Spine}(M) \\
\hline
\text{Spine}(M')
\end{array}$$

$$\begin{array}{c}
\text{Spine}(M) \quad M \rightarrow_{\beta} P \\
\hline
\text{Spine}(P) \quad 1 \\
\hline
\text{Spine}(M') \quad 2
\end{array}$$

1 Spine-One-Step-Forward-Closure 2 IH

□

Not-Intro-if-Neutral-Multi

$$\begin{array}{c}
\Delta \stackrel{\mathcal{N}}{\Rightarrow} M \in A \quad M \rightarrow_{\beta}^* P \\
\hline
\neg \text{Intro}(P)
\end{array}$$

Proof:

$$\begin{array}{c}
\Delta \stackrel{\mathcal{N}}{\Rightarrow} M \in A \\
\hline
\text{Spine}(M) \quad 1 \\
\hline
M \rightarrow_{\beta}^* P \quad 2 \\
\hline
\text{Spine}(P) \\
\hline
\neg\text{Intro}(P) \quad 3
\end{array}$$

1 Spine-if-Neutral 2 Spine-Forward-Closure 3 Not-Intro-if-Spine □

WN-ProjLM

$$\begin{array}{c}
\Delta \stackrel{\mathcal{N}}{\Rightarrow} M \in A_L \times A_R \quad \text{norm}_{\beta}(M) \\
\hline
\text{norm}_{\beta}(M \cdot L)
\end{array}$$

Proof by showing the following equivalent statement:

$$\frac{\Delta \stackrel{\mathcal{N}}{\Rightarrow} M \in A_L \times A_R \quad M \rightarrow_{\beta}^* P \quad P \not\rightarrow_{\beta}}{\text{norm}_{\beta}(M \cdot L)}$$

Claim $P \cdot L \not\rightarrow_{\beta}$. To show this, fix an arbitrary Q , assume $P \cdot L \rightarrow_{\beta} Q$, and we will derive $\perp$.

Inversion on $P \cdot L \rightarrow_{\beta} Q$:

$$\boxed{\text{SUBCASE}} \quad \frac{P \rightarrow_{\beta} P'}{P \cdot L \rightarrow_{\beta} P' \cdot L}$$

Hence $Q = P' \cdot L$.

$$\frac{P \not\rightarrow_{\beta} \quad P \rightarrow_{\beta} P'}{\perp} \quad 1$$

1 unfold $P \not\rightarrow_{\beta}$ and $\forall / \Rightarrow$ -E

$$\boxed{\text{SUBCASE}} \quad \overline{\langle P_L, P_R \rangle \cdot L \rightarrow_{\beta} P_L}$$

Hence $P = \langle P_L, P_R \rangle$ and $Q = P_L$.

$$\begin{array}{c}
\Delta \stackrel{\mathcal{N}}{\Rightarrow} M \in A_L \times A_R \quad M \rightarrow_{\beta}^* \langle P_L, P_R \rangle \\
\hline
\neg\text{Intro}(\langle P_L, P_R \rangle) \quad 1 \\
\hline
\text{Intro}(\langle P_L, P_R \rangle) \quad 2 \\
\hline
\perp \quad 3
\end{array}$$

1 Not-Intro-if-Neutral-Multi 2 PairM-is-Intro 3 $\Rightarrow$ -E

Hence shown $P \cdot L \not\rightarrow_\beta$.

Finally:

$$\frac{\frac{M \rightarrow_\beta^* P}{M \cdot L \rightarrow_\beta^* P \cdot L} \quad 1 \quad P \cdot L \not\rightarrow_\beta \quad 2}{\text{norm}_\beta(M \cdot L)} 2$$

1 β^* -under 2 fold $\text{norm}_\beta(M \cdot L)$ □

WN-ProjRM

$$\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L \times A_R \quad \text{norm}_\beta(M)}{\text{norm}_\beta(M \cdot R)}$$

Proof entirely analogous to that of the previous lemma. □

WN-Neutral-AppM

$$\frac{\Delta \xRightarrow{\mathcal{N}} M \in A \rightarrow B \quad \text{norm}_\beta(M) \quad \text{norm}_\beta(N)}{\text{norm}_\beta(M N)}$$

Proof by showing the following equivalent statement:

$$\frac{\Delta \xRightarrow{\mathcal{N}} M \in A \rightarrow B \quad M \rightarrow_\beta^* P \quad P \not\rightarrow_\beta \quad N \rightarrow_\beta^* Q \quad Q \not\rightarrow_\beta}{\text{norm}_\beta(M N)}$$

Claim $P Q \not\rightarrow_\beta$. To show this, fix an arbitrary R , assume $P Q \rightarrow_\beta R$, and we will derive $\perp$.

Inversion on $P Q \rightarrow_\beta R$:

SUBCASE $\frac{P \rightarrow_\beta P'}{P Q \rightarrow_\beta P' Q}$ [App-Under-Fun]

Hence $R = P' Q$.

$$\frac{P \not\rightarrow_\beta \quad P \rightarrow_\beta P'}{\perp} 1$$

1 unfold $P \not\rightarrow_\beta$ and $\forall / \Rightarrow$ -E

$$\boxed{\text{SUBCASE}} \quad \frac{Q \rightarrow_{\beta} Q'}{P Q \rightarrow_{\beta} P Q'} [\text{App-Under-Arg}]$$

Hence $R = P Q'$.

$$\frac{Q \not\rightarrow_{\beta} \quad Q \rightarrow_{\beta} Q'}{\perp} 1$$

1 unfold $Q \not\rightarrow_{\beta}$ and $\forall / \Rightarrow$ -E

$$\boxed{\text{SUBCASE}} \quad \frac{}{\lambda x \Rightarrow P' Q \rightarrow_{\beta} [Q/x](P')} [\text{App-beta}]$$

Hence $P = \lambda x \Rightarrow P'$ and $R = [Q/x](P')$.

$$\frac{\Delta \stackrel{\mathcal{N}}{\Rightarrow} M \in A \rightarrow B \quad M \rightarrow_{\beta}^* \lambda x \Rightarrow P'}{\neg\text{Intro}(\lambda x \Rightarrow P')} 1 \quad \frac{}{\text{Intro}(\lambda x \Rightarrow P')} 2$$

$$\frac{}{\perp} 3$$

1 Not-Intro-if-Neutral-Multi 2 LamM-is-Intro 3 $\Rightarrow$ -E

Hence shown $P Q \not\rightarrow_{\beta}$.

Finally:

$$\frac{\frac{\frac{M \rightarrow_{\beta}^* P}{MN \rightarrow_{\beta}^* P N} 1 \quad \frac{N \rightarrow_{\beta}^* Q}{P N \rightarrow_{\beta}^* P Q} 2}{MN \rightarrow_{\beta}^* P Q} 3 \quad P Q \not\rightarrow_{\beta}}{\text{norm}_{\beta}(M N)} 4$$

1 β^* -under 2 β^* -under 3 Beta-Multi-Trans 4 fold $\text{norm}_{\beta}(M N)$

□

WN-Neutral-CaseM

$$\frac{\Delta \stackrel{\mathcal{N}}{\Rightarrow} M \in A_L + A_R \quad \text{norm}_{\beta}(M)}{\frac{\text{norm}_{\beta}(M_L) \quad \text{norm}_{\beta}(M_R)}{\text{norm}_{\beta}(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R))}}$$

Proof by showing the following equivalent statement:

$$\begin{array}{c}
\Delta \stackrel{N}{\Rightarrow} M \in A_L + A_R \quad M \rightarrow_\beta^* P \quad P \not\rightarrow_\beta \\
\hline
M_L \rightarrow_\beta^* P_L \quad P_L \not\rightarrow_\beta \\
\hline
M_R \rightarrow_\beta^* P_R \quad P_R \not\rightarrow_\beta \\
\hline
\text{norm}_\beta(\text{case } M (x_L \Rightarrow M_L \mid x_R \Rightarrow M_R))
\end{array}$$

Claim $\text{case } P (x_L \Rightarrow P_L \mid x_R \Rightarrow P_R) \not\rightarrow_\beta$. To show this, fix an arbitrary Q , assume $\text{case } P (x_L \Rightarrow P_L \mid x_R \Rightarrow P_R) \rightarrow_\beta Q$, and we will derive $\perp$.

Inversion on $\text{case } P (x_L \Rightarrow P_L \mid x_R \Rightarrow P_R) \rightarrow_\beta Q$:

$$\boxed{\text{SUBCASE}} \quad \frac{P \rightarrow_\beta P'}{\text{case } P (x_L \Rightarrow P_L \mid x_R \Rightarrow P_R) \rightarrow_\beta \text{case } P' (x_L \Rightarrow P_L \mid x_R \Rightarrow P_R)} \text{ [Case-Under-Main]}$$

Hence $Q = \text{case } P' (x_L \Rightarrow P_L \mid x_R \Rightarrow P_R)$.

$$\frac{P \not\rightarrow_\beta \quad P \rightarrow_\beta P'}{\perp} 1$$

1 unfold $P \not\rightarrow_\beta$ and $\forall / \Rightarrow$ -E

$$\boxed{\text{SUBCASE}} \quad \frac{P_L \rightarrow_\beta P'_L}{\text{case } P (x_L \Rightarrow P_L \mid x_R \Rightarrow P_R) \rightarrow_\beta \text{case } P (x_L \Rightarrow P'_L \mid x_R \Rightarrow P_R)} \text{ [Case-Under-L]}$$

Hence $Q = \text{case } P (x_L \Rightarrow P'_L \mid x_R \Rightarrow P_R)$.

$$\frac{P_L \not\rightarrow_\beta \quad P_L \rightarrow_\beta P'_L}{\perp} 1$$

1 unfold $P_L \not\rightarrow_\beta$ and $\forall / \Rightarrow$ -E

$$\boxed{\text{SUBCASE}} \quad \frac{P_R \rightarrow_\beta P'_R}{\text{case } P (x_L \Rightarrow P_L \mid x_R \Rightarrow P_R) \rightarrow_\beta \text{case } P (x_L \Rightarrow P_L \mid x_R \Rightarrow P'_R)} \text{ [Case-Under-R]}$$

Hence $Q = \text{case } P (x_L \Rightarrow P_L \mid x_R \Rightarrow P'_R)$.

$$\frac{P_R \not\rightarrow_\beta \quad P_R \rightarrow_\beta P'_R}{\perp} 1$$

1 unfold $P_R \not\rightarrow_\beta$ and $\forall / \Rightarrow$ -E

SUBCASE $\frac{}{\text{case } L \cdot P'_L (x_L \Rightarrow P_L \mid x_R \Rightarrow P_R) \rightarrow_\beta [P'_L/x_L](P_L)} \text{ [Case-InL-beta]}$

Hence $P = L \cdot P'_L$ and $Q = [P'_L/x_L](P_L)$.

$$\frac{\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L + A_R \quad M \rightarrow_\beta^* L \cdot P'_L}{\neg \text{Intro}(L \cdot P'_L)} \quad 1 \quad \frac{}{\text{Intro}(L \cdot P'_L)} \quad 2}{\perp} \quad 3$$

1 Not-Intro-if-Neutral-Multi 2 InLM-is-Intro 3 $\Rightarrow$ -E

SUBCASE $\frac{}{\text{case } R \cdot P'_R (x_L \Rightarrow P_L \mid x_R \Rightarrow P_R) \rightarrow_\beta [P'_R/x_R](P_R)} \text{ [Case-InR-beta]}$

Hence $P = R \cdot P'_R$ and $Q = [P'_R/x_R](P_R)$.

$$\frac{\frac{\Delta \xRightarrow{\mathcal{N}} M \in A_L + A_R \quad M \rightarrow_\beta^* R \cdot P'_R}{\neg \text{Intro}(R \cdot P'_R)} \quad 1 \quad \frac{}{\text{Intro}(R \cdot P'_R)} \quad 2}{\perp} \quad 3$$

1 Not-Intro-if-Neutral-Multi 2 InRM-is-Intro 3 $\Rightarrow$ -E

Hence shown case $P (x_L \Rightarrow P_L \mid x_R \Rightarrow P_R) \not\rightarrow_\beta$.

Finally, write $C(U, U_L, U_R) \triangleq \text{case } U (x_L \Rightarrow U_L \mid x_R \Rightarrow U_R)$. Then:

$$\frac{\frac{M \rightarrow_\beta^* P \quad M_L \rightarrow_\beta^* P_L \quad M_R \rightarrow_\beta^* P_R}{C(M, M_L, M_R) \rightarrow_\beta^* C(P, P_L, P_R)} \quad 1 \quad C(P, P_L, P_R) \not\rightarrow_\beta \quad 2}{\text{norm}_\beta(C(M, M_L, M_R))}$$

1 β^* -under and Beta-Multi-Trans 2 fold $\text{norm}_\beta(C(M, M_L, M_R))$ □

WN-Neutral-NatRecM

$$\frac{\frac{\Delta \xRightarrow{\mathcal{N}} M \in \text{nat} \quad \text{norm}_\beta(M)}{\text{norm}_\beta(M_0) \quad \text{norm}_\beta(M_S)} \quad \text{norm}_\beta(M_0) \quad \text{norm}_\beta(M_S)}{\text{norm}_\beta(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S))}$$

Proof by showing the following equivalent statement:

$$\begin{array}{c}
\Delta \xRightarrow{\textcolor{red}{N}} M \in \text{nat} \quad M \rightarrow_{\beta}^* P \quad P \not\rightarrow_{\beta} \\
\hline
M_0 \rightarrow_{\beta}^* P_0 \quad P_0 \not\rightarrow_{\beta} \\
\hline
M_S \rightarrow_{\beta}^* P_S \quad P_S \not\rightarrow_{\beta} \\
\hline
\text{norm}_{\beta}(\text{natrec } M (M_0 \mid x, y \Rightarrow M_S))
\end{array}$$

Claim $\text{natrec } P (P_0 \mid x, y \Rightarrow P_S) \not\rightarrow_{\beta}$. To show this, fix an arbitrary Q , assume $\text{natrec } P (P_0 \mid x, y \Rightarrow P_S) \rightarrow_{\beta} Q$, and we will derive $\perp$.

Inversion on $\text{natrec } P (P_0 \mid x, y \Rightarrow P_S) \rightarrow_{\beta} Q$:

$$\boxed{\text{SUBCASE}} \quad \frac{P \rightarrow_{\beta} P'}{\text{natrec } P (P_0 \mid x, y \Rightarrow P_S) \rightarrow_{\beta} \text{natrec } P' (P_0 \mid x, y \Rightarrow P_S)} \text{ [NatRec-Under-Main]}$$

Hence $Q = \text{natrec } P' (P_0 \mid x, y \Rightarrow P_S)$.

$$\frac{P \not\rightarrow_{\beta} \quad P \rightarrow_{\beta} P'}{\perp} \textcolor{blue}{1}$$

$\textcolor{blue}{1}$ unfold $P \not\rightarrow_{\beta}$ and $\forall / \Rightarrow$ -E

$$\boxed{\text{SUBCASE}} \quad \frac{P_0 \rightarrow_{\beta} P'_0}{\text{natrec } P (P_0 \mid x, y \Rightarrow P_S) \rightarrow_{\beta} \text{natrec } P (P'_0 \mid x, y \Rightarrow P_S)} \text{ [NatRec-Under-Zero]}$$

Hence $Q = \text{natrec } P (P'_0 \mid x, y \Rightarrow P_S)$.

$$\frac{P_0 \not\rightarrow_{\beta} \quad P_0 \rightarrow_{\beta} P'_0}{\perp} \textcolor{blue}{1}$$

$\textcolor{blue}{1}$ unfold $P_0 \not\rightarrow_{\beta}$ and $\forall / \Rightarrow$ -E

$$\boxed{\text{SUBCASE}} \quad \frac{P_S \rightarrow_{\beta} P'_S}{\text{natrec } P (P_0 \mid x, y \Rightarrow P_S) \rightarrow_{\beta} \text{natrec } P (P_0 \mid x, y \Rightarrow P'_S)} \text{ [NatRec-Under-Succ]}$$

Hence $Q = \text{natrec } P (P_0 \mid x, y \Rightarrow P'_S)$.

$$\frac{P_S \not\rightarrow_{\beta} \quad P_S \rightarrow_{\beta} P'_S}{\perp} \textcolor{blue}{1}$$

$\textcolor{blue}{1}$ unfold $P_S \not\rightarrow_{\beta}$ and $\forall / \Rightarrow$ -E

$$\boxed{\text{SUBCASE}} \quad \frac{}{\text{natrec zero } (P_0 \mid x, y \Rightarrow P_S) \rightarrow_\beta P_0} [\text{NatRec-Zero-beta}]$$

Hence $P = \text{zero}$ and $Q = P_0$.

$$\frac{\frac{\Delta \xRightarrow{\textcolor{red}{N}} M \in \text{nat} \quad M \rightarrow_\beta^* \text{zero}}{\neg \text{Intro}(\text{zero})} \textcolor{blue}{1} \quad \frac{}{\text{Intro}(\text{zero})} \textcolor{blue}{2}}{\perp} \textcolor{blue}{3}$$

$\textcolor{blue}{1}$ Not-Intro-if-Neutral-Multi $\textcolor{blue}{2}$ ZeroM-is-Intro $\textcolor{blue}{3}$ $\Rightarrow$ -E

$$\boxed{\text{SUBCASE}} \quad \frac{}{\text{natrec succ } P' (P_0 \mid x, y \Rightarrow P_S) \rightarrow_\beta [P'/x, T/y](P_S)} [\text{NatRec-Succ-beta}]$$

Define $T \triangleq \text{natrec } P' (P_0 \mid x, y \Rightarrow P_S)$. Hence $P = \text{succ } P'$ and $Q = [P'/x, T/y](P_S)$.

$$\frac{\frac{\Delta \xRightarrow{\textcolor{red}{N}} M \in \text{nat} \quad M \rightarrow_\beta^* \text{succ } P'}{\neg \text{Intro}(\text{succ } P')} \textcolor{blue}{1} \quad \frac{}{\text{Intro}(\text{succ } P')} \textcolor{blue}{2}}{\perp} \textcolor{blue}{3}$$

$\textcolor{blue}{1}$ Not-Intro-if-Neutral-Multi $\textcolor{blue}{2}$ SuccM-is-Intro $\textcolor{blue}{3}$ $\Rightarrow$ -E

Hence shown $\text{natrec } P (P_0 \mid x, y \Rightarrow P_S) \not\rightarrow_\beta$.

Finally, write $R(U, U_0, U_S) \triangleq \text{natrec } U (U_0 \mid x, y \Rightarrow U_S)$. Then:

$$\frac{\frac{M \rightarrow_\beta^* P \quad M_0 \rightarrow_\beta^* P_0 \quad M_S \rightarrow_\beta^* P_S}{R(M, M_0, M_S) \rightarrow_\beta^* R(P, P_0, P_S)} \textcolor{blue}{1} \quad R(P, P_0, P_S) \not\rightarrow_\beta}{\text{norm}_\beta(R(M, M_0, M_S))} \textcolor{blue}{2}$$

$\textcolor{blue}{1}$ β^* -under and Beta-Multi-Trans $\textcolor{blue}{2}$ fold $\text{norm}_\beta(R(M, M_0, M_S))$

□

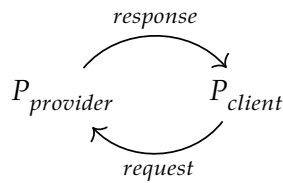
Part II

Termination of Process Calculus

Chapter 7

Process Calculus

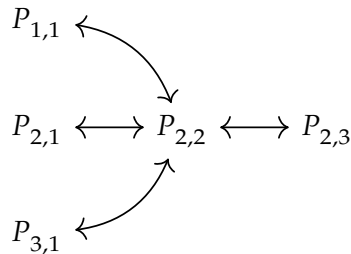
The process calculus we will be working with is based on intuitionistic linear logic session types [2, 8], in which a fundamental concept of message-passing communication is the distinction between the provider and the client, the two parties involved in a synchronous exchange:



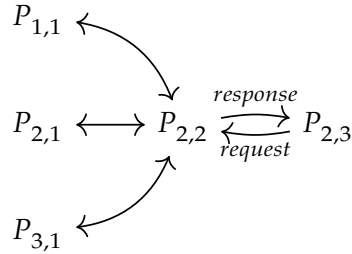
Without exception, a single communication is always between two parties, one a provider and the other a client, and consists of two actions, one request and one response. Hence, a crucial observation to be made here is that if any of these conditions is not met, if there is only one party, if there are two parties but the two actions do not match, or if one or both actions are missing, then a communication cannot take place.

Communication between a provider and a client is therefore strictly synchronous. Although it is possible to formulate the transition system so that communication becomes partly asynchronous, in the sense that sending actions are never blocked, we will focus primarily on synchronous exchange for the process calculus we are going to specify.

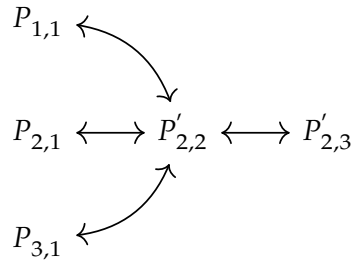
Next, an example shows that the connection structure might be much more complicated than one consisting of only two parties. There might be a case where there are more than two parties involved in a connection structure:



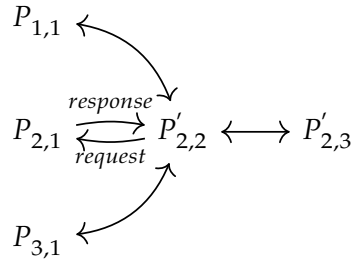
Here, a left-and-right / bidirectional arrow represents a connection between two parties which we will henceforth call processes. Let's observe the next step the process $P_{2,2}$ will take. First, it will respond to a request from $P_{2,3}$:



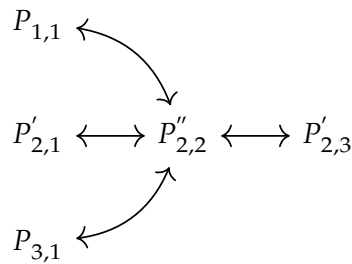
Afterwards, both of them transition into some new states $P'_{2,2}$ and $P'_{2,3}$:



Now, $P'_{2,2}$ may request (note the reversal) a service from $P_{2,1}$:



Similarly, both of them transition into some new states $P'_{2,1}$ and $P''_{2,2}$:



Note crucially that $P_{2,2}$ first acts as a provider, serving the request from $P_{2,3}$, but immediately afterwards, $P'_{2,2}$ acts as a client, requesting a service from $P_{2,1}$.

The role of a process depends on its next action: if it serves a request, then it is acting as a provider; if it requests a service, then it is acting as a client. Thus, a process shifts back and forth between being a provider and being a client during its lifetime.

7.1 Syntax

We work with the following syntax of a process calculus, where variables are more specifically called address variables, as substitutions are only mappings from such variables to addresses. Addresses are similar to channels, with the difference that addresses are utilized only by the client of a two-party communication, while the provider, as reflected in both the syntax and semantics to be defined later, remains agnostic of its address. Being agnostic means the semantic behaviors of a provider do not depend on the choice of a particular address.

AVar	$x, \dots$		
Choice	π	$::= R$	$ L$
STerm	S	$::=$	close $ \text{pick } \pi; S$ $ \text{case } (S_L S_R)$ $ \text{send } x; S$ $ \text{recv } x \Rightarrow S$ $ \text{spawn } S \stackrel{x}{\triangleright} S'$
			$ \text{wait}[x]; S$ $ \text{case}[x] (S_L S_R)$ $ \text{pick}[x] \pi; S$ $ \text{recv}[y] x \Rightarrow S$ $ \text{send}[y] x; S$ $ \text{fwd}[x]$
SType	A	$::=$	1 $ A_L \oplus A_R$ $ A_L \& A_R$ $ A_L \otimes A_R$ $ A_L \multimap A_R$

To streamline our presentation, we will focus mainly on binary types instead of labeled types. Here, then, to reflect the binary choice of a provider or client, we define π , which can be either the right R pick or the left L pick.

First, observe that there are three kinds of term constructors, the first of which consists of constructors without a destination address, such as close and those on the same vertical line except for $\text{spawn } S \overset{x}{\triangleright} S'$. These correspond to providing a service, as providers do not need to be aware of the addresses used by a client to refer to these providers. The second kind consists of constructors that contain an address variable, which will be replaced by an actual address at runtime. These correspond to clients, as they must specify the address to which they will send their request. This is similar to dialing a phone number: the caller needs to specify a number, whereas the callee simply receives the call and responds. The third kind consists of $\text{spawn } S \overset{x}{\triangleright} S'$ and $\text{fwd}[x]$, which are neither providers nor clients in the sense that they do not engage in communication. Instead, they work with their ambient environment, the connection structure, to change the topology of the connection, which we will illustrate in more detail later.

Next, observe that the process calculus supports three kinds of message-passing communication, depending on the nature of the message. The first kind involves sending and receiving closing signals and corresponds to the type 1 and the two term constructors close and $\text{wait}[x]; S$. The second kind involves sending and receiving binary choices, but there are two perspectives. The provider can send a choice to its client, which we call internal choice and which corresponds to the type $A_L \oplus A_R$ and the term constructors $\text{pick } \pi; S$ and $\text{case}[x] (S_L | S_R)$. Here, $\text{pick } \pi; S$ is the provider, which does not

specify an address, whereas $\text{case}[x] (S_L \mid S_R)$ is listening on an address and therefore waiting for the choice of the provider. From the opposite perspective, a provider can be listening for a client's binary choice, which we therefore call external (compare this with internal) choice and which corresponds to the type $A_L \& A_R$ and term constructors $\text{case} (S_L \mid S_R)$ and $\text{pick}[x] \pi; S$. Here, analogously, the provider is $\text{case} (S_L \mid S_R)$ and is listening for the choice of a client $\text{pick}[x] \pi; S$.

Now, to make the message-passing behaviors of processes more interesting, we also allow them to send and receive addresses, via which a connection structure consisting of processes can change. As with the sending and receiving of binary choices, for which there are two perspectives depending on whether it is the provider or the client sending the choice, we also have two perspectives on the sending and receiving of channels. We still first focus on the internal one, which means we first focus on the situation where a provider would like to send an address. This corresponds to types $A_L \otimes A_R$ and terms $\text{send } x; S$ and $\text{recv}[y] x \Rightarrow S$. Shifting perspective again, let's say now a provider is waiting to receive an address from its client, which then corresponds to types $A_L \multimap A_R$ and terms $\text{recv } x \Rightarrow S$ and $\text{send}[y] x; S$. Finally, we would like to emphasize again that $\text{spawn } S \overset{x}{\triangleright} S'$ and $\text{fwd}[x]$ do not engage in communication but rather are special terms that change the topology of a connection structure.

Next, we can specify how to type a session term using session types:

$\boxed{\Gamma \vdash S : A}$	
OneT-R	OneT-L
$\frac{}{\cdot \vdash \text{close} : 1}$	$\frac{\Gamma \vdash S : A}{\Gamma, x : 1 \vdash \text{wait}[x]; S : A}$
ICT-R	ICT-L
$\frac{\Gamma \vdash S : A_\pi}{\Gamma \vdash \text{pick } \pi; S : A_L \oplus A_R}$	$\frac{\Gamma, x : A_L \vdash S_L : B \quad \Gamma, x : A_R \vdash S_R : B}{\Gamma, x : A_L \oplus A_R \vdash \text{case}[x] (S_L \mid S_R) : B}$
ECT-R	ECT-L
$\frac{\Gamma \vdash S_L : A_L \quad \Gamma \vdash S_R : A_R}{\Gamma \vdash \text{case} (S_L \mid S_R) : A_L \& A_R}$	$\frac{\Gamma, x : A_\pi \vdash S : A}{\Gamma, x : A_L \& A_R \vdash \text{pick}[x] \pi; S : A}$
TPT-R	TPT-L
$\frac{\Gamma \vdash S : B}{\Gamma, y : A \vdash \text{send } y; S : A \otimes B}$	$\frac{\Gamma, y : A, x : B \vdash S : C}{\Gamma, x : A \otimes B \vdash \text{recv}[x] y \Rightarrow S : C}$
LolliT-R	LolliT-L
$\frac{\Gamma, z : A \vdash S : B}{\Gamma \vdash \text{recv } z \Rightarrow S : A \multimap B}$	$\frac{\Gamma, x : B \vdash S : C}{\Gamma, y : A, x : A \multimap B \vdash \text{send}[x] y; S : C}$

$$\begin{array}{c}
\text{Cut} \\
\frac{\Gamma \vdash S : A \quad \Gamma', x : A \vdash S' : B}{\Gamma, \Gamma' \vdash \text{spawn } S \overset{x}{\triangleright} S' : B} \\
\text{Forward} \\
\frac{}{x : A \vdash \text{fwd}[x] : A}
\end{array}$$

The type system is linear. In particular, unlike an affine type system, a linear type system based on session types requires that each type constructor be consumed exactly once, and hence a process cannot silently drop an address. Note here that linearity does NOT mean that a process can interact with an address only once. It means, rather, that every time a process interacts with an address, the outermost type constructor of the type inhabited by that address and hence by the corresponding process at that address is consumed, leaving a smaller type. Hence, another feature of our process calculus is that the addresses and the processes providing them have changing types, meaning that these types go down / become smaller after each interaction with these addresses.

7.2 Process Semantics

Previously, we mentioned that address variables are placeholders for addresses to be substituted during runtime. Therefore, to execute a session term, one must first turn it into something executable by replacing all the variables with addresses, which we assume are some infinite collection of names $a, b, c, \dots$. As with variables, we can freely decide if two addresses are equal or pick a fresh address that does not already occur in some collection of addresses.

$$\begin{array}{lcl}
\text{Address } a, \dots & & \\
\text{Process } P & ::= & \begin{array}{l}
\text{close} \quad | \quad \text{wait}[a]; P \\
| \quad \text{pick } \pi; P \quad | \quad \text{case}[a] (S_L \mid S_R) \\
| \quad \text{case } (S_L \mid S_R) \quad | \quad \text{pick}[a] \pi; P \\
| \quad \text{send } b; P \quad | \quad \text{recv}[a] x \Rightarrow P \\
| \quad \text{recv } x \Rightarrow P \quad | \quad \text{send}[a] b; P \\
| \quad \text{spawn } P \overset{x}{\triangleright} P' \quad | \quad \text{fwd}[a] \\
| \quad \mathbb{0}
\end{array}
\end{array}$$

Here, $\mathbb{0}$ stands for a process that has already terminated. Note that this is a semantic constructor, in the sense that we cannot write down a corresponding concrete $\mathbb{0}$ in source code. Further, note that the actual things being executed are processes, not session terms; processes are obtained from session terms by applying substitutions so that no more variables are present, after which the substituted terms are considered processes.

Next, we specify the operational semantics of these processes:

$P \xrightarrow{\alpha} P' \text{ with } P \xrightarrow{[b]\alpha} P'$

$$\begin{array}{c}
\text{LT-OneT-R} \\
\frac{}{\text{close} \xrightarrow{!()} \mathbb{0}} \\
\text{LT-OneT-L} \\
\frac{}{\text{wait}[a]; P \xrightarrow{[a]?()} P}
\end{array}$$

LT-ICT-R	LT-ICT-L
$\frac{}{\text{pick } \pi; P \xrightarrow{! \pi} P}$	$\frac{}{\text{case}[a] (S_L \mid S_R) \xrightarrow{[a]? \pi} S_\pi}$
LT-ECT-R	LT-ECT-L
$\frac{}{\text{case} (S_L \mid S_R) \xrightarrow{? \pi} S_\pi}$	$\frac{}{\text{pick}[a] \pi; P \xrightarrow{[a]! \pi} P}$
LT-TPT-R	LT-TPT-L
$\frac{}{\text{send } b; P \xrightarrow{! b} P}$	$\frac{}{\text{recv}[a] x \Rightarrow P \xrightarrow{[a]? b} [b/x]P}$
LT-LolliT-R	LT-LolliT-L
$\frac{}{\text{recv } x \Rightarrow P \xrightarrow{? b} [b/x]P}$	$\frac{}{\text{send}[a] b; P \xrightarrow{[a]! b} P}$

There are two kinds of actions corresponding to the notions of provider and client in a communication. The first kind consists of **right** actions with only two parts: a direction, where ! stands for send and ? for receive, and a corresponding payload, which is either a closing signal $()$, a binary choice π , or an address a . These are the actions of providers, since a provider simply responds upon receiving a request and therefore does not need to know the address. By the earlier analogy, a provider is like the callee in a phone call: it simply responds upon seeing a call. On the other hand, to dial a number and hence request a service, a client process must specify the number and thus the address to which it will send its request. These are the **left** actions, composed of three parts: an address enclosed in blue brackets, such as $[a]$, which is the destination to which the process will forward its request; a direction that is the same as the one specified in a **right** action; and a payload that is also the same as the one specified in a **right** action. Therefore, the syntax of these actions is specified as follows:

Direction	δ	$::=$	$! \mid ?$
Payload	ϕ	$::=$	$() \mid \pi \mid a$
RightAct	α	$\triangleq$	$\delta \phi$
LeftAct	$[b]\alpha$		

This completes the definition of the operational semantics of processes. Recall that earlier we said that for a single communication to take place, there must be a provider and a client present at the same time. This brings out another crucial aspect of message-passing communication, which is that a single process never transitions on its own. Processes always transition together as a collection, which will be the first topic of the next chapter. We call collections of processes configurations, and the actual execution, corresponding to the actual communication, always takes place between pairs of processes in a configuration. Note that this does not mean that, at any point in time, only a single pair may transition together; rather, it is possible to parallelize the transitions of many such pairs. The connection structure of configurations could also be formulated in several ways, but for this project, the formulation we picked uses finite maps. More precisely, configurations are formulated as pointed finite maps, where the point is a distinguished root node of the whole configuration, corresponding to the root provider of the whole collection.

Chapter 8

Logical Relations

As mentioned at the end of the last chapter, a single process never transitions on its own. Rather, processes always transition as provider-client pairs, so that one of them sends a request and the other responds to it. Moreover, there can be a connection structure consisting of many processes, which we call a configuration. In the first section that follows, we define the notion of configurations using finite maps. Here, any configuration has a distinguished root node process that does not have an address, whereas all the other processes in the finite map have addresses.

8.1 Configuration

Definition 8.1.1 (Configuration)

A configuration $\Omega : \mathbb{C}\text{onfig}$ *includes* a body which is a finite mapping $\mu : \mathbb{A}\text{ddress} \rightarrow \mathbb{P}\text{rocess}$. *In addition*, it has a distinguished root node P . Thus, a configuration is of the form:

$$\mathbb{C}\text{onfig} \quad \Omega \triangleq \mu \mid P$$

Here, the long vertical bar $\mid$ serves only as a notational separator between the finite map μ and the root node P .

With the notion of configurations, we can first specify what it means for a configuration to undergo a labeled transition. A configuration undergoes a labeled transition simply if any of its processes undergoes a labeled transition. However, for right actions, only the root node can perform them, as specified by the first rule [LT-Config-R-Act].

$$\boxed{\Omega \xrightarrow{\alpha} \Omega' \text{ with } \Omega \xrightarrow{[b]\alpha} \Omega'}$$

LT-Config-R-Act

$$\frac{P \xrightarrow{\alpha} P'}{\mu \mid P \xrightarrow{\alpha} \mu \mid P'}$$

LT-Config-L-Act-R

LT-Config-L-Act-L

$$\frac{P \xrightarrow{[b]\alpha} P'}{\mu \mid P \xrightarrow{[b]\alpha} \mu \mid P'} \quad \frac{P_L \xrightarrow{[b]\alpha} P'_L}{\mu, a \mapsto P_L \mid P_R \xrightarrow{[b]\alpha} \mu, a \mapsto P'_L \mid P_R}$$

Oftentimes, we want to give a configuration a name and turn it into a single finite map without a nameless root, which we do via the following naming operation:

Definition 8.1.2 (Naming Configurations)

The naming operation turns a configuration $\Omega : \text{Config}$ into a single finite map $\mu : \text{Address} \rightarrow \text{Process}$ by giving the root node P of Ω a name a , assigning a to P as its address.

$$\begin{aligned} (\mu \mid \mathbb{O})[a] &\triangleq \mu \\ (\mu \mid P)[a] &\triangleq \mu, a \mapsto P \quad \text{else if } P \neq \mathbb{O} \end{aligned}$$

We will also write $\Omega[a]$ to stand for such a naming operation without unfolding Ω .

This naming operation is used, for example, in the concatenation of two configurations Ω_L and Ω_R , where we first name Ω_L to turn it into a finite map $\Omega_L[a]$, so that we can then compose it with Ω_R . That is, parallel composition is always between a fully named configuration represented as a finite map and a regular nameless configuration with a distinguished nameless root process, as defined below:

Definition 8.1.3 (Concatenation of Maps and Configurations)

$$\mu \mid (\mu' \mid P) \triangleq \mu, \mu' \mid P$$

Later, we will write $\Omega_L[a] \mid \Omega_R$ to denote the concatenation of a named configuration $\Omega_L[a]$, which is a finite map, and a nameless one Ω_R .

We can then specify the operational semantics for configurations. The first rule [LT-Config-Comm] specifies when an actual communication takes place: a configuration Ω_R makes a request, and another configuration Ω_L happens to be able to respond. Note that the provider Ω_L must perform the complementary action: if the request is a sending action, then the provider must perform a receiving action, and vice versa. In the conclusion, we also give Ω_L the address b to match the destination address specified in the request $[b]\alpha$. However, this top-down reading is not intuitive. Rather, the rule should be read bottom-up: if we can already decompose a configuration into a provider Ω_L whose root node has the address b and another part Ω_R whose request contains b as its destination, then the two parts may interact.

$$\boxed{\Omega \rightarrow \Omega'}$$

LT-Config-Comm

$$\frac{\Omega_L \xrightarrow{\bar{\alpha}} \Omega'_L \quad \Omega_R \xrightarrow{[b]\alpha} \Omega'_R}{\Omega_L[b] \mid \Omega_R \rightarrow \Omega'_L[b] \mid \Omega'_R}$$

$$\begin{array}{c}
\text{LT-Config-Frame-L} \qquad \qquad \text{LT-Config-Frame-R} \\
\frac{\Omega_L \rightarrow \Omega'_L}{\Omega_L[a] \mid \Omega_R \rightarrow \Omega'_L[a] \mid \Omega_R} \qquad \frac{\Omega_R \rightarrow \Omega'_R}{\Omega_L[a] \mid \Omega_R \rightarrow \Omega_L[a] \mid \Omega'_R} \\
\\
\text{LT-Config-Spawn} \qquad \qquad \text{LT-Config-Fwd} \\
\frac{a \notin \text{dom}(\mu)}{\mu \mid \text{spawn } P \xrightarrow{x} P' \rightarrow (\mu, a \mapsto P) \mid [a/x]P'} \qquad \frac{}{a \mapsto P \mid \text{fwd}[a] \rightarrow \{P\}}
\end{array}$$

Here, we define complementary actions as follows:

$$\begin{array}{ccc}
\bar{!} & \triangleq & ? \\
\bar{?} & \triangleq & ! \\
\overline{\delta \phi} & \triangleq & \overline{\delta} \phi
\end{array}$$

Recall that any right action α consists of two parts: a direction δ , which is either $!$ or $?$, and a payload ϕ . Hence, to compute the complement of a right action, we simply flip its direction from send to receive or vice versa. The payload remains the same because the provider and the client must agree on the content of the message.

Next, both [LT-Config-Frame-L] and [LT-Config-Frame-R] specify that we may focus on some smaller part of a larger configuration and hence zoom in on it. If the smaller part is able to transition, then the whole configuration is able to transition.

The last two rules [LT-Config-Spawn] and [LT-Config-Fwd] are not communications, but rather special rules for the spawning and forwarding processes that change the topology of the whole connection structure they are part of. The spawning process transitions into a continuation configuration by adding a child to its environment finite map μ . The forwarding process simply drags, and hence forwards, its only child to become the root, and then disappears after being replaced by its child.

Next, we can define the multi-step transitions of configurations:

$$\boxed{\Omega \rightarrow^* \Omega'}$$

$$\begin{array}{c}
\text{LT-Config-Refl} \qquad \qquad \text{LT-Config-Cons} \\
\frac{}{\Omega \rightarrow^* \Omega} \qquad \frac{\Omega \rightarrow \Omega' \quad \Omega' \rightarrow^* \Omega''}{\Omega \rightarrow^* \Omega''} \\
\\
\text{LT-Config-Single-is-Multi} \qquad \text{LT-Config-Trans} \\
\frac{\Omega \rightarrow \Omega'}{\Omega \rightarrow^* \Omega'} \qquad \frac{\Omega \rightarrow^* \Omega' \quad \Omega' \rightarrow^* \Omega''}{\Omega \rightarrow^* \Omega''}
\end{array}$$

8.2 Inductive Logical Relation

With the notion of configurations and operational semantics on configurations in place, we can then specify our inductive logical relation. Since configurations are the actual entities that undergo transitions, the inhabitants of the inductive logical relation are the configurations. Moreover, these configurations are not required to be closed; rather, they may be arbitrary configurations containing free addresses that are specified in Δ . An address a is free in the sense that there is no corresponding provider configuration providing along a . As can be seen in rule [V-Cut], a free address is closed and therefore no longer appears in the conclusion once we compose the configuration Ω_R , which contains the free address a , with a provider configuration Ω_L that provides along a .

$$\boxed{\Delta \Rightarrow \Omega \in A} \quad \boxed{\Delta \Rightarrow \Omega \in A}$$

E-if-V

$$\frac{\Omega \rightarrow^* \Omega' \quad \Delta \Rightarrow \Omega' \in A}{\Delta \Rightarrow \Omega \in A}$$

V-1T-R

$$\frac{\Omega \xrightarrow{!()} \{\emptyset\}}{\emptyset \Rightarrow \Omega \in [1]}$$

V-1T-L

$$\frac{\Omega \xrightarrow{[a]?()} \Omega' \quad \Delta \Rightarrow \Omega' \in A}{\Delta, [a : 1] \Rightarrow \Omega \in A}$$

V-ICT-R

$$\frac{\Omega \xrightarrow{! \pi} \Omega' \quad \Delta \Rightarrow \Omega' \in A_\pi}{\Delta \Rightarrow \Omega \in [A_L \oplus A_R]}$$

V-ICT-L

$$\frac{\begin{array}{c} \Omega \xrightarrow{[a]?L} \Omega_L \quad \Delta, a : A_L \Rightarrow \Omega_L \in B \\ \hline \Omega \xrightarrow{[a]?R} \Omega_R \quad \Delta, a : A_R \Rightarrow \Omega_R \in B \end{array}}{\Delta, [a : A_L \oplus A_R] \Rightarrow \Omega \in B}$$

V-ECT-R

$$\frac{\begin{array}{c} \Omega \xrightarrow{?L} \Omega_L \quad \Delta \Rightarrow \Omega_L \in A_L \\ \hline \Omega \xrightarrow{?R} \Omega_R \quad \Delta \Rightarrow \Omega_R \in A_R \end{array}}{\Delta \Rightarrow \Omega \in [A_L \& A_R]}$$

V-ECT-L

$$\frac{\Omega \xrightarrow{[a]! \pi} \Omega' \quad \Delta, a : A_\pi \Rightarrow \Omega' \in B}{\Delta, [a : A_L \& A_R] \Rightarrow \Omega \in B}$$

V-TPT-R

$$\frac{\Omega \xrightarrow{!b} \Omega' \quad \Delta \Rightarrow \Omega' \in A_R}{\Delta, b : A_L \Rightarrow \Omega \in [A_L \otimes A_R]}$$

V-TPT-L

$$\frac{\Omega \xrightarrow{[a]?b} \Omega' \quad \Delta, b : A_L, a : A_R \Rightarrow \Omega' \in B}{\Delta, [a : A_L \otimes A_R] \Rightarrow \Omega \in B}$$

$$\begin{array}{c}
\text{V-LolliT-R} \qquad \qquad \qquad \text{V-LolliT-L} \\
\frac{\Omega \xrightarrow{?b} \Omega' \quad \Delta, b : A_L \Rightarrow \Omega' \in A_R}{\Delta \Rightarrow \Omega \in [A_L \multimap A_R]} \quad \frac{\Omega \xrightarrow{[a]!b} \Omega' \quad \Delta, a : A_R \Rightarrow \Omega' \in B}{\Delta, b : A_L, [a : A_L \multimap A_R] \Rightarrow \Omega \in B} \\
\text{V-Cut} \qquad \qquad \qquad \text{V-Fwd-Id} \\
\frac{\Delta_L \Rightarrow \Omega_L \in A \quad \Delta_R, a : A \Rightarrow \Omega_R \in B}{\Delta_L, \Delta_R \Rightarrow \Omega_L[a] \mid \Omega_R \in B} \quad \frac{}{a : A \Rightarrow \{\text{fwd}[a]\} \in A}
\end{array}$$

All the rules are standard in the sense that each configuration in the conclusion, once it undergoes some labeled transition, consumes a type constructor of either a free address in the context or the type at which it is providing a service. [V-LolliT-L] is the special rule that makes it possible to even formulate this inductive logical relation, as there is no negative occurrence of a universally quantified computable configuration. [V-Fwd-Id] is also a special rule as it is akin to the simplest kind of spine one can build in lambda calculus: a single variable x . Here, the forwarding process is akin to that single variable and can be viewed as the simplest kind of spine in the process calculus. Any configuration with at least one leaf forwarding process will inevitably become stuck at that leaf, as the forwarding process is forever waiting for a child unless one actually composes a child in parallel with the forwarding leaf.

8.3 Recursive Counterpart

As with the inductive logical relation for lambda calculus developed in [Part I \(Normalization of Lambda Calculus\)](#), where a recursive logical relation was developed to establish equivalence with the inductive logical relation and help restore its compositionality, here we likewise define a recursive logical relation for the inductive logical relation for the process calculus defined in the previous section.

However, whereas the recursive counterpart for lambda calculus is Kripke-style and therefore applicable to all open lambda terms, here we define a recursive logical relation applicable only to closed configurations; hence, it is not indexed by contexts. We define such a logical relation because we are mainly interested in the execution of closed configurations. We will use the following recursive logical relation to define closed computable complementary configurations. These are then composed in parallel with a single configuration computable according to the inductive logical relation, yielding a closed combined configuration. The following logical relation builds upon the one defined and explored in [11]:

$$\begin{array}{ll}
\mathcal{E}_A(\Omega) & \iff \exists \Omega' . \Omega \rightarrow^* \Omega' \wedge \mathcal{U}_A(\Omega') \\
\mathcal{U}_1(\Omega) & \iff \Omega \xrightarrow{!()} \{\emptyset\} \\
\mathcal{U}_{A_L \oplus A_R}(\Omega) & \iff \exists \pi . \exists \Omega' . \Omega \xrightarrow{!\pi} \Omega' \wedge \mathcal{E}_{A_\pi}(\Omega') \\
\mathcal{U}_{A_L \& A_R}(\Omega) & \iff \forall \pi . \exists \Omega' . \Omega \xrightarrow{?\pi} \Omega' \wedge \mathcal{E}_{A_\pi}(\Omega') \\
\mathcal{U}_{A_L \otimes A_R}(\Omega) & \iff \exists \Omega_L, b, \Omega_R . \Omega = \Omega_L[b] \mid \Omega_R \\
& \wedge \mathcal{E}_{A_L}(\Omega_L) \wedge \Omega_R \xrightarrow{!b} \Omega'_R \wedge \mathcal{E}_{A_R}(\Omega'_R) \\
\mathcal{U}_{A_L \multimap A_R}(\Omega) & \iff \forall b . \exists \Omega_b . \Omega \xrightarrow{?b} \Omega_b \\
& \wedge \forall \Omega_L . \mathcal{E}_{A_L}(\Omega_L) \implies \mathcal{E}_{A_R}(\Omega_L[b] \mid \Omega_b)
\end{array}$$

With these definitions of the logical relations and the notions of configurations and their operational

semantics, we are prepared to prove various properties of the logical relations and, therefore, of any configuration computable according to them. The theorems and their proofs will be presented in the next chapter.

Chapter 9

Properties of Logical Relations

This chapter shows the various properties of the logical relations. We start by proving that the recursive logical relation admits backward closure:

$$\begin{array}{c} \text{[E-Fun-BC]} \\ \Omega \rightarrow^* \Omega' \quad \mathcal{E}_A(\Omega') \\ \hline \mathcal{E}_A(\Omega) \end{array}$$

We then show various frame rules for multi-step transitions:

$$\begin{array}{ccc} \text{[Concat-Frame-L]} & \text{[Concat-Frame-R]} & \text{[Concat-Frame-R-Act]} \\ \frac{\Omega_L \rightarrow^* \Omega'_L}{\Omega_L[a] \mid \Omega_R \rightarrow^* \Omega'_L[a] \mid \Omega_R} & \frac{\Omega_R \rightarrow^* \Omega'_R}{\mu \mid \Omega_R \rightarrow^* \mu \mid \Omega'_R} & \frac{\Omega \xrightarrow{\alpha} \Omega'}{\mu \mid \Omega \xrightarrow{\alpha} \mu \mid \Omega'} \end{array}$$

Define complementary configurations:

$$\begin{array}{ll} \mathcal{E}_\Delta(\overline{\Omega}) & \triangleq \forall a \in \Delta. \mathcal{E}_{\Delta(a)}(\overline{\Omega}(a)) \\ \uplus[\overline{\Omega}, a \mapsto \Omega] & \triangleq \uplus[\overline{\Omega}], \Omega[a] \end{array}$$

The following pair of theorems is semantic in the sense that the right premise in each theorem statement demands that a configuration be computable according to the inductive logical relation.

$$\begin{array}{cc} \text{[Sem-FTLR-E]} & \text{[Sem-FTLR-V]} \\ \frac{\mathcal{E}_\Delta(\overline{\Omega}) \quad \Delta \Rightarrow \Omega \overset{\mathcal{E}}{\in} A}{\mathcal{E}_A(\uplus[\overline{\Omega}], \Omega)} & \frac{\mathcal{E}_\Delta(\overline{\Omega}) \quad \Delta \Rightarrow \Omega \overset{\mathcal{U}}{\in} A}{\mathcal{E}_A(\uplus[\overline{\Omega}], \Omega)} \end{array}$$

Define inhabitation of single types with respect to the inductive logical relation; in addition, define closed complementary configurations computable according to the inductive logical relation:

$$\begin{aligned}\Omega \overset{\mathcal{U}}{\in} A &\triangleq \emptyset \Rightarrow \Omega \overset{\mathcal{U}}{\in} A \\ \Omega \overset{\mathcal{E}}{\in} A &\triangleq \emptyset \Rightarrow \Omega \overset{\mathcal{E}}{\in} A \\ \overline{\Omega} \overset{\mathcal{E}}{\in} \Delta &\triangleq \forall a \in \Delta . \overline{\Omega}(a) \overset{\mathcal{E}}{\in} \Delta(a)\end{aligned}$$

With the semantic fundamental theorem, we can derive corollaries showing that closed configurations computable with respect to the inductive logical relation are also computable with respect to the recursive logical relation at the same type A :

[E-fun-if-E]	[E-fun-if-V]
$\frac{\Omega \overset{\mathcal{E}}{\in} A}{\mathcal{E}_A(\Omega)}$	$\frac{\Omega \overset{\mathcal{U}}{\in} A}{\mathcal{E}_A(\Omega)}$

Interestingly, we may also derive closing cut elimination in the sense that applying closing complementary configurations to a computable configuration, both of which are computable with respect to the inductive logical relation, yields a combined configuration that is computable according to the recursive logical relation:

[E-Cut-E]	[E-Cut-V]
$\frac{\overline{\Omega} \overset{\mathcal{E}}{\in} \Delta \quad \Delta \Rightarrow \Omega \overset{\mathcal{E}}{\in} A}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \Omega)}$	$\frac{\overline{\Omega} \overset{\mathcal{E}}{\in} \Delta \quad \Delta \Rightarrow \Omega \overset{\mathcal{U}}{\in} A}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \Omega)}$

This ends the discussion of the semantic fundamental theorem. We will prove a syntactic fundamental theorem, the right premise of whose statement is a syntactic typing judgement and is hence syntactic. First, we prove that the inductive logical relation also admits backward closure / expansion:

[E-BC]
$\frac{\Omega \rightarrow^* \Omega' \quad \Delta \Rightarrow \Omega' \overset{\mathcal{E}}{\in} A}{\Delta \Rightarrow \Omega \overset{\mathcal{E}}{\in} A}$

As a helper lemma, we show that any configuration that already satisfies a value interpretation also satisfies a term interpretation:

[E-if-V*]

$$\frac{\Delta \Rightarrow \Omega \overset{\textcolor{blue}{v}}{\in} A}{\Delta \Rightarrow \Omega \overset{\textcolor{red}{\varepsilon}}{\in} A}$$

To push through the spawn case of the proof of the syntactic fundamental theorem, we prove that semantic cut elimination is admissible for the term interpretation of the inductive logical relation, given that cut already holds for the value interpretation by definition:

[E-Cut]

$$\frac{\Delta_L \Rightarrow \Omega_L \overset{\textcolor{red}{\varepsilon}}{\in} A \quad \Delta_R, a : A \Rightarrow \Omega_R \overset{\textcolor{red}{\varepsilon}}{\in} B}{\Delta_L, \Delta_R \Rightarrow \Omega_L[a] \mid \Omega_R \overset{\textcolor{red}{\varepsilon}}{\in} B}$$

Then, we are able to show the syntactic fundamental theorem, which states that any session term S is automatically a computable configuration consisting of itself after substitutions:

[Syn-FTLR]

$$\frac{\Delta \vdash \gamma : \Gamma \quad \Gamma \vdash S : A}{\Delta \Rightarrow \{\hat{\gamma}(S)\} \overset{\textcolor{red}{\varepsilon}}{\in} A}$$

As corollaries of both the semantic and syntactic fundamental theorems, we are able to show the following combined fundamental theorem and its specialized versions for closed session terms:

[Chained-FTLR]

[Closed-FTLR-Ind]

[Closed-FTLR-Rec]

$$\frac{\overline{\Omega} \overset{\textcolor{red}{\varepsilon}}{\in} \Delta \quad \Delta \vdash \gamma : \Gamma \quad \Gamma \vdash S : A}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \hat{\gamma}(S))}$$

$$\frac{S : A}{\{S\} \overset{\textcolor{red}{\varepsilon}}{\in} A}$$

$$\frac{S : A}{\mathcal{E}_A(\{S\})}$$

Finally, all these enable us to prove termination of closed session terms at type 1:

[Adequacy-Termination]

$$\frac{S : 1}{\exists P . \{S\} \rightarrow^* \{P\} \wedge P \xrightarrow{!()} \emptyset}$$

The following sections contain the detailed proofs of the lemmas and theorems outlined above.

9.1 Semantic Fundamental Theorem

Next, we are ready to define complementary configurations:

$$\begin{aligned} \mathcal{E}_\Delta(\overline{\Omega}) &\triangleq \forall a \in \Delta. \mathcal{E}_{\Delta(a)}(\overline{\Omega}(a)) \\ \uplus[\overline{\Omega}, a \rightarrow \Omega] &\triangleq \uplus[\overline{\Omega}], \Omega[a] \end{aligned}$$

E-Fun-BC

$$\frac{\Omega \rightarrow^* \Omega' \quad \mathcal{E}_A(\Omega')}{\mathcal{E}_A(\Omega)}$$

Proof:

$$\frac{\mathcal{E}_A(\Omega')}{\exists \Omega'' . \Omega' \rightarrow^* \Omega'' \wedge \mathcal{U}_A(\Omega'')} \quad 1$$

1 unfold

$$\frac{\frac{\Omega \rightarrow^* \Omega' \quad \Omega' \rightarrow^* \Omega''}{\Omega \rightarrow^* \Omega''} \quad 1 \quad \mathcal{U}_A(\Omega'')}{\mathcal{E}_A(\Omega)} \quad 2$$

1 LT-Config-Trans 2 fold

□

We need some helper lemmas for proving the semantic fundamental theorem. They are the multi-step versions of the frame rules for configuration concatenations:

Concat-Frame-L

$$\frac{\Omega_L \rightarrow^* \Omega'_L}{\Omega_L[a] \mid \Omega_R \rightarrow^* \Omega'_L[a] \mid \Omega_R}$$

Concat-Frame-R

$$\frac{\Omega_R \rightarrow^* \Omega'_R}{\mu \mid \Omega_R \rightarrow^* \mu \mid \Omega'_R}$$

Proof by induction on $\Omega_L \rightarrow^* \Omega'_L$ for [Concat-Frame-L] and on $\Omega_R \rightarrow^* \Omega'_R$ for [Concat-Frame-R]; appeal to [LT-Config-Frame-L] or [LT-Config-Frame-R] in the inductive cases. □

Concat-Frame-R-Act

$$\frac{\Omega \xrightarrow{\alpha} \Omega'}{\mu \mid \Omega \xrightarrow{\alpha} \mu \mid \Omega'}$$

Proof by inversion on the premise $\Omega \xrightarrow{\alpha} \Omega'$, matching [LT-Config-R-Act], which tells us that the right action must be coming from the root node of Ω ; apply the same rule [LT-Config-R-Act] to finish the proof. $\square$

Sem-FTLR-E

Sem-FTLR-V

$$\frac{\mathcal{E}_{\Delta}(\overline{\Omega}) \quad \Delta \Rightarrow \Omega \overset{\mathcal{E}}{\in} A}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \Omega)} \quad \frac{\mathcal{E}_{\Delta}(\overline{\Omega}) \quad \Delta \Rightarrow \Omega \overset{\mathcal{V}}{\in} A}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \Omega)}$$

Proof by simultaneous induction on $\Delta \Rightarrow \Omega \overset{\mathcal{E}}{\in} A$ and $\Delta \Rightarrow \Omega \overset{\mathcal{V}}{\in} A$:

CASE $\mathcal{E}_{\Delta}(\overline{\Omega}) \quad \Delta \Rightarrow \Omega \overset{\mathcal{E}}{\in} A$

$$\frac{\Omega \rightarrow^* \Omega' \quad \Delta \Rightarrow \Omega' \overset{\mathcal{V}}{\in} A}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \Omega)} \text{ [E-if-V]}$$

$$\frac{\frac{\Omega \rightarrow^* \Omega'}{\uplus[\overline{\Omega}] \mid \Omega \rightarrow^* \uplus[\overline{\Omega}] \mid \Omega'} \xrightarrow{1} \quad \frac{\mathcal{E}_{\Delta}(\overline{\Omega}) \quad \Delta \Rightarrow \Omega' \overset{\mathcal{V}}{\in} A}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \Omega')} \xrightarrow{2}}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \Omega)} \xrightarrow{3}$$

1 Concat-Frame-R 2 IH 3 E-Fun-BC

CASE $\mathcal{E}_{\Delta_L, \Delta_R}(\overline{\Omega}) \quad \Delta_L, \Delta_R \Rightarrow \Omega_L[a] \mid \Omega_R \overset{\mathcal{V}}{\in} B$

$$\frac{\Delta_L \Rightarrow \Omega_L \overset{\mathcal{V}}{\in} A \quad \Delta_R, a : A \Rightarrow \Omega_R \overset{\mathcal{V}}{\in} B}{\mathcal{E}_{\Delta_L, \Delta_R}(\overline{\Omega}) \quad \Delta_L, \Delta_R \Rightarrow \Omega_L[a] \mid \Omega_R \overset{\mathcal{V}}{\in} B} \text{ [V-Cut]}$$

$$\mathcal{E}_B(\uplus[\overline{\Omega}] \mid \Omega_L[a] \mid \Omega_R)$$

• $\overline{\Omega} = \overline{\Omega}_{\Delta_L}, \overline{\Omega}_{\Delta_R}$

$$\begin{array}{c}
\frac{\mathcal{E}_{\Delta_L}(\overline{\Omega}_{\Delta_L}) \quad \Delta_L \Rightarrow \Omega_L \overset{\mathcal{U}}{\in} A}{\hspace{10em}} 1 \\
\frac{\mathcal{E}_{\Delta_R}(\overline{\Omega}_{\Delta_R}) \quad \mathcal{E}_A(\uplus[\overline{\Omega}_{\Delta_L}] \mid \Omega_L)}{\hspace{10em}} 2 \\
\frac{\mathcal{E}_{\Delta_R, a:A}(\overline{\Omega}_{\Delta_R'} a \mapsto \uplus[\overline{\Omega}_{\Delta_L}] \mid \Omega_L) \quad \Delta_R, a : A \Rightarrow \Omega_R \overset{\mathcal{U}}{\in} B}{\hspace{10em}} 3 \\
\mathcal{E}_B(\uplus[\overline{\Omega}] \mid \Omega_L[a] \mid \Omega_R)
\end{array}$$

1 IH 2 fold 3 IH

$$\begin{array}{c}
\text{CASE} \quad \frac{\mathcal{E}_{a:A}(\overline{\Omega}) \quad \frac{}{a : A \Rightarrow \{\text{fwd}[a]\} \overset{\mathcal{U}}{\in} A} [\text{V-Fwd-Id}]}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \{\text{fwd}[a]\})}
\end{array}$$

- $\overline{\Omega} = a \mapsto \Omega_a \wedge \uplus[\overline{\Omega}] = \Omega_a[a]$
- $\Omega_a \rightarrow^* \Omega'_a \wedge \mathcal{U}_A(\Omega'_a)$

$$\begin{array}{c}
\frac{\Omega_a \rightarrow^* \Omega'_a}{\hspace{10em}} 1 \quad \frac{}{\hspace{10em}} 2 \\
\frac{\Omega_a[a] \mid \{\text{fwd}[a]\} \rightarrow^* \Omega'_a[a] \mid \{\text{fwd}[a]\} \quad \Omega'_a[a] \mid \{\text{fwd}[a]\} \rightarrow^* \Omega'_a}{\hspace{10em}} 3 \\
\frac{\Omega_a[a] \mid \{\text{fwd}[a]\} \rightarrow^* \Omega'_a \quad \mathcal{U}_A(\Omega'_a)}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \{\text{fwd}[a]\})} 4
\end{array}$$

1 Concat-Frame-L 2 LT-Config-Fwd 3 LT-Config-Trans 4 fold

$$\begin{array}{c}
\text{CASE} \quad \frac{\mathcal{E}_{\emptyset}(\overline{\Omega}) \quad \frac{\Omega \xrightarrow{!()}\{\emptyset\}}{\emptyset \Rightarrow \Omega \overset{\mathcal{U}}{\in} [1]} [\text{V-1T-R}]}{\mathcal{E}_1(\uplus[\overline{\Omega}] \mid \Omega)}
\end{array}$$

$$\begin{array}{c}
\frac{\Omega \xrightarrow{!()}\{\emptyset\}}{\mathcal{E}_1(\Omega)} 1 \\
\frac{}{\mathcal{E}_1(\uplus[\overline{\Omega}] \mid \Omega)} 2
\end{array}$$

1 fold 2 $\uplus[\overline{\Omega}] = \emptyset$

$$\begin{array}{c}
\text{CASE} \quad \mathcal{E}_{\Delta, a:1}(\overline{\Omega}) \quad \frac{\Omega \xrightarrow{[a]?()} \Omega' \quad \Delta \Rightarrow \Omega' \overset{\mathcal{E}}{\in} A}{\Delta, [a:1] \Rightarrow \Omega \overset{\mathcal{U}}{\in} A} \text{ [V-1T-L]} \\
\hline
\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \Omega)
\end{array}$$

- $\overline{\Omega} = \overline{\Omega}_{\Delta}, a \mapsto \Omega_a$
- $\Omega_a \rightarrow^* \Omega'_a \wedge \Omega'_a \xrightarrow{!()} \{\emptyset\}$

$$\begin{array}{c}
\frac{\Omega_a \rightarrow^* \Omega'_a}{\Omega_a[a] \mid \Omega \rightarrow^* \Omega'_a[a] \mid \Omega} \xrightarrow{1} \quad \frac{\Omega'_a \xrightarrow{!()} \{\emptyset\} \quad \Omega \xrightarrow{[a]?()} \Omega'}{\Omega'_a[a] \mid \Omega \rightarrow \Omega'} \xrightarrow{2} \quad \frac{\Omega'_a[a] \mid \Omega \rightarrow \Omega'}{\Omega'_a[a] \mid \Omega \rightarrow^* \Omega'} \xrightarrow{3} \\
\frac{\Omega_a[a] \mid \Omega \rightarrow^* \Omega'}{\uplus[\overline{\Omega}_{\Delta}] \mid \Omega_a[a] \mid \Omega \rightarrow^* \uplus[\overline{\Omega}_{\Delta}] \mid \Omega'} \xrightarrow{4} \quad \frac{\Omega_a[a] \mid \Omega \rightarrow^* \Omega'}{\uplus[\overline{\Omega}_{\Delta}] \mid \Omega_a[a] \mid \Omega \rightarrow^* \uplus[\overline{\Omega}_{\Delta}] \mid \Omega'} \xrightarrow{5} \quad \frac{\mathcal{E}_{\Delta}(\overline{\Omega}_{\Delta}) \quad \Delta \Rightarrow \Omega' \overset{\mathcal{E}}{\in} A}{\mathcal{E}_A(\uplus[\overline{\Omega}_{\Delta}] \mid \Omega')} \xrightarrow{6} \\
\frac{\uplus[\overline{\Omega}_{\Delta}] \mid \Omega_a[a] \mid \Omega \rightarrow^* \uplus[\overline{\Omega}_{\Delta}] \mid \Omega'}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \Omega)} \xrightarrow{7}
\end{array}$$

1 Concat-Frame-L 2 LT-Config-Comm 3 LT-Config-Single-is-Multi 4 LT-Config-Trans
5 Concat-Frame-R 6 IH 7 E-Fun-BC

$$\begin{array}{c}
\text{CASE} \quad \mathcal{E}_{\Delta}(\overline{\Omega}) \quad \frac{\Omega \xrightarrow{!\pi} \Omega' \quad \Delta \Rightarrow \Omega' \overset{\mathcal{E}}{\in} A_{\pi}}{\Delta \Rightarrow \Omega \overset{\mathcal{U}}{\in} [A_L \oplus A_R]} \text{ [V-ICT-R]} \\
\hline
\mathcal{E}_{A_L \oplus A_R}(\uplus[\overline{\Omega}] \mid \Omega)
\end{array}$$

$$\begin{array}{c}
\frac{\Omega \xrightarrow{!\pi} \Omega'}{\uplus[\overline{\Omega}] \mid \Omega \xrightarrow{!\pi} \uplus[\overline{\Omega}] \mid \Omega'} \xrightarrow{1} \quad \frac{\mathcal{E}_{\Delta}(\overline{\Omega}) \quad \Delta \Rightarrow \Omega' \overset{\mathcal{E}}{\in} A_{\pi}}{\mathcal{E}_{A_{\pi}}(\uplus[\overline{\Omega}] \mid \Omega')} \xrightarrow{2} \\
\frac{\uplus[\overline{\Omega}] \mid \Omega \xrightarrow{!\pi} \uplus[\overline{\Omega}] \mid \Omega'}{\mathcal{E}_{A_L \oplus A_R}(\uplus[\overline{\Omega}] \mid \Omega)} \xrightarrow{3}
\end{array}$$

1 Concat-Frame-R-Act 2 IH 3 fold

$$\begin{array}{c}
\text{CASE} \quad \mathcal{E}_{\Delta, a:A_L \oplus A_R}(\overline{\Omega}) \quad \frac{\Omega \xrightarrow{[a]?L} \Omega_L \quad \Delta, a:A_L \Rightarrow \Omega_L \overset{\mathcal{E}}{\in} B}{\Omega \xrightarrow{[a]?R} \Omega_R \quad \Delta, a:A_R \Rightarrow \Omega_R \overset{\mathcal{E}}{\in} B} \text{ [V-ICT-L]} \\
\hline
\mathcal{E}_B(\uplus[\overline{\Omega}] \mid \Omega)
\end{array}$$

- $\overline{\Omega} = \overline{\Omega}_{\Delta}, a \mapsto \Omega_a$
- $\Omega_a \rightarrow^* \Omega'_a \wedge \Omega'_a \xrightarrow{! \pi} \Omega''_a \wedge \mathcal{E}_{A_\pi}(\Omega''_a)$
- $\Omega \xrightarrow{[a]? \pi} \Omega_\pi \wedge \Delta, a : A_\pi \Rightarrow \Omega_\pi \stackrel{\mathcal{E}}{\in} B$

$$\begin{array}{c}
\frac{\Omega_a \rightarrow^* \Omega'_a}{\Omega_a[a] \mid \Omega \rightarrow^* \Omega'_a[a] \mid \Omega} \xrightarrow{1} \frac{\Omega'_a \xrightarrow{! \pi} \Omega''_a \quad \Omega \xrightarrow{[a]? \pi} \Omega_\pi}{\Omega'_a[a] \mid \Omega \rightarrow \Omega''_a[a] \mid \Omega_\pi} \xrightarrow{2} \frac{\Omega'_a[a] \mid \Omega \rightarrow \Omega''_a[a] \mid \Omega_\pi}{\Omega'_a[a] \mid \Omega \rightarrow^* \Omega''_a[a] \mid \Omega_\pi} \xrightarrow{3} \frac{\Omega'_a[a] \mid \Omega \rightarrow^* \Omega''_a[a] \mid \Omega_\pi}{\Omega_a[a] \mid \Omega \rightarrow^* \Omega''_a[a] \mid \Omega_\pi} \xrightarrow{4} \frac{\Omega_a[a] \mid \Omega \rightarrow^* \Omega''_a[a] \mid \Omega_\pi}{\uplus[\overline{\Omega}_\Delta] \mid \Omega_a[a] \mid \Omega \rightarrow^* \uplus[\overline{\Omega}_\Delta] \mid \Omega''_a[a] \mid \Omega_\pi} \xrightarrow{5}
\end{array}$$

1 Concat-Frame-L 2 LT-Config-Comm 3 LT-Config-Single-is-Multi 4 LT-Config-Trans
5 Concat-Frame-R

$$\frac{\uplus[\overline{\Omega}_\Delta] \mid \Omega_a[a] \mid \Omega \rightarrow^* \uplus[\overline{\Omega}_\Delta] \mid \Omega''_a[a] \mid \Omega_\pi \quad \frac{\mathcal{E}_{\Delta, a : A_\pi}(\overline{\Omega}_{\Delta}, a \mapsto \Omega''_a) \quad \Delta, a : A_\pi \Rightarrow \Omega_\pi \stackrel{\mathcal{E}}{\in} B}{\mathcal{E}_B(\uplus[\overline{\Omega}_{\Delta}, a \mapsto \Omega''_a] \mid \Omega_\pi)} \xrightarrow{1} \frac{\mathcal{E}_B(\uplus[\overline{\Omega}_\Delta] \mid \Omega)}{\mathcal{E}_B(\uplus[\overline{\Omega}] \mid \Omega)} \xrightarrow{2}$$

1 IH 2 E-Fun-BC

$$\begin{array}{c}
\frac{\Omega \xrightarrow{?L} \Omega_L \quad \Delta \Rightarrow \Omega_L \stackrel{\mathcal{E}}{\in} A_L}{\Omega \xrightarrow{?R} \Omega_R \quad \Delta \Rightarrow \Omega_R \stackrel{\mathcal{E}}{\in} A_R} \text{ [V-ECT-R]} \\
\frac{\mathcal{E}_\Delta(\overline{\Omega}) \quad \Delta \Rightarrow \Omega \stackrel{\mathcal{U}}{\in} [A_L \& A_R]}{\mathcal{E}_{A_L \& A_R}(\uplus[\overline{\Omega}] \mid \Omega)}
\end{array}$$

$$\begin{array}{c}
\frac{\Omega \xrightarrow{?L} \Omega_L}{\uplus[\overline{\Omega}] \mid \Omega \xrightarrow{?L} \uplus[\overline{\Omega}] \mid \Omega_L} \xrightarrow{1} \frac{\mathcal{E}_\Delta(\overline{\Omega}) \quad \Delta \Rightarrow \Omega_L \stackrel{\mathcal{E}}{\in} A_L}{\mathcal{E}_{A_L}(\uplus[\overline{\Omega}] \mid \Omega_L)} \xrightarrow{2} \\
\frac{\Omega \xrightarrow{?R} \Omega_R}{\uplus[\overline{\Omega}] \mid \Omega \xrightarrow{?R} \uplus[\overline{\Omega}] \mid \Omega_R} \xrightarrow{3} \frac{\mathcal{E}_\Delta(\overline{\Omega}) \quad \Delta \Rightarrow \Omega_R \stackrel{\mathcal{E}}{\in} A_R}{\mathcal{E}_{A_R}(\uplus[\overline{\Omega}] \mid \Omega_R)} \xrightarrow{4} \frac{\mathcal{E}_{A_L \& A_R}(\uplus[\overline{\Omega}] \mid \Omega)}{\mathcal{E}_{A_L \& A_R}(\uplus[\overline{\Omega}] \mid \Omega)} \xrightarrow{5}
\end{array}$$

1 Concat-Frame-R-Act 2 IH 3 Concat-Frame-R-Act 4 IH 5 fold

$$\begin{array}{c}
\text{CASE} \quad \frac{\mathcal{E}_{\Delta, a: A_L \& A_R}(\overline{\Omega}) \quad \frac{\Omega \xrightarrow{[a]! \pi} \Omega' \quad \Delta, a : A_\pi \Rightarrow \Omega' \overset{\mathcal{E}}{\in} B}{\Delta, [a : A_L \& A_R] \Rightarrow \Omega \overset{\mathcal{U}}{\in} B} \text{ [V-ECT-L]}}{\mathcal{E}_B(\uplus[\overline{\Omega}] \mid \Omega)}
\end{array}$$

- $\overline{\Omega} = \overline{\Omega}_\Delta, a \mapsto \Omega_a$
- $\Omega_a \rightarrow^* \Omega'_a \wedge \Omega'_a \xrightarrow{? \pi} \Omega''_a \wedge \mathcal{E}_{A_\pi}(\Omega''_a)$

$$\begin{array}{c}
\frac{\Omega_a \rightarrow^* \Omega'_a}{\Omega_a[a] \mid \Omega \rightarrow^* \Omega'_a[a] \mid \Omega} \xrightarrow{1} \frac{\frac{\Omega'_a \xrightarrow{? \pi} \Omega''_a \quad \Omega \xrightarrow{[a]! \pi} \Omega'}{\Omega'_a[a] \mid \Omega \rightarrow \Omega''_a[a] \mid \Omega'} \xrightarrow{2} \frac{\Omega'_a[a] \mid \Omega \rightarrow \Omega''_a[a] \mid \Omega'}{\Omega'_a[a] \mid \Omega \rightarrow^* \Omega''_a[a] \mid \Omega'} \xrightarrow{3} \frac{\Omega'_a[a] \mid \Omega \rightarrow^* \Omega''_a[a] \mid \Omega'}{\Omega_a[a] \mid \Omega \rightarrow^* \Omega''_a[a] \mid \Omega'} \xrightarrow{4} \frac{\Omega_a[a] \mid \Omega \rightarrow^* \Omega''_a[a] \mid \Omega'}{\uplus[\overline{\Omega}_\Delta] \mid \Omega_a[a] \mid \Omega \rightarrow^* \uplus[\overline{\Omega}_\Delta] \mid \Omega''_a[a] \mid \Omega'} \xrightarrow{5}
\end{array}$$

1 Concat-Frame-L 2 LT-Config-Comm 3 LT-Config-Single-is-Multi 4 LT-Config-Trans
5 Concat-Frame-R

$$\frac{\uplus[\overline{\Omega}_\Delta] \mid \Omega_a[a] \mid \Omega \rightarrow^* \uplus[\overline{\Omega}_\Delta] \mid \Omega''_a[a] \mid \Omega' \quad \frac{\mathcal{E}_{\Delta, a: A_\pi}(\overline{\Omega}_\Delta, a \mapsto \Omega''_a) \quad \Delta, a : A_\pi \Rightarrow \Omega' \overset{\mathcal{E}}{\in} B}{\mathcal{E}_B(\uplus[\overline{\Omega}_\Delta, a \mapsto \Omega''_a] \mid \Omega')} \xrightarrow{1}}{\mathcal{E}_B(\uplus[\overline{\Omega}] \mid \Omega)} \xrightarrow{2}$$

1 IH 2 E-Fun-BC

$$\begin{array}{c}
\text{CASE} \quad \frac{\mathcal{E}_{\Delta, b: A_L}(\overline{\Omega}) \quad \frac{\Omega \xrightarrow{!b} \Omega' \quad \Delta \Rightarrow \Omega' \overset{\mathcal{E}}{\in} A_R}{\Delta, b : A_L \Rightarrow \Omega \overset{\mathcal{U}}{\in} [A_L \otimes A_R]} \text{ [V-TPT-R]}}{\mathcal{E}_{A_L \otimes A_R}(\uplus[\overline{\Omega}] \mid \Omega)}
\end{array}$$

- $\overline{\Omega} = \overline{\Omega}_\Delta, b \mapsto \Omega_b$

$$\frac{\frac{\Omega \xrightarrow{!b} \Omega'}{\Omega \xrightarrow{!b} \Omega'} \xrightarrow{1} \frac{\mathcal{E}_\Delta(\overline{\Omega}_\Delta) \quad \Delta \Rightarrow \Omega' \overset{\mathcal{E}}{\in} A_R}{\mathcal{E}_{A_R}(\uplus[\overline{\Omega}_\Delta] \mid \Omega')} \xrightarrow{2} \frac{\mathcal{E}_{A_L}(\Omega_b) \quad \uplus[\overline{\Omega}_\Delta] \mid \Omega \xrightarrow{!b} \uplus[\overline{\Omega}_\Delta] \mid \Omega'}{\mathcal{E}_{A_L \otimes A_R}(\uplus[\overline{\Omega}] \mid \Omega)} \xrightarrow{3}$$

1 Concat-Frame-R-Act 2 IH 3 fold

$$\begin{array}{c}
\text{CASE} \quad \frac{\mathcal{E}_{\Delta, a: A_L \otimes A_R}(\overline{\Omega}) \quad \frac{\Omega \xrightarrow{[a]?b} \Omega' \quad \Delta, b: A_L, a: A_R \Rightarrow \Omega' \overset{\mathcal{E}}{\in} B}{\Delta, [a: A_L \otimes A_R] \Rightarrow \Omega \overset{\mathcal{U}}{\in} B} [\text{V-TPT-L}]}{\mathcal{E}_B(\uplus[\overline{\Omega}] \mid \Omega)}
\end{array}$$

- $\overline{\Omega} = \overline{\Omega}_{\Delta}, a \mapsto \Omega_a$
- $\Omega_a \rightarrow^* \Omega_L[b] \mid \Omega_R \wedge \mathcal{E}_{A_L}(\Omega_L)$
- $\Omega_R \xrightarrow{!b} \Omega'_R \wedge \mathcal{E}_{A_R}(\Omega'_R)$

$$\begin{array}{c}
\frac{\Omega_a \rightarrow^* \Omega_L[b] \mid \Omega_R}{\Omega_a[a] \mid \Omega \rightarrow^* \Omega_L[b] \mid \Omega_R[a] \mid \Omega} \xrightarrow{1} \quad \frac{\frac{\Omega_R \xrightarrow{!b} \Omega'_R}{\Omega_L[b] \mid \Omega_R \xrightarrow{!b} \Omega_L[b] \mid \Omega'_R} \xrightarrow{2} \quad \frac{\Omega \xrightarrow{[a]?b} \Omega'}{\Omega_L[b] \mid \Omega_R[a] \mid \Omega \rightarrow \Omega_L[b] \mid \Omega'_R[a] \mid \Omega'} \xrightarrow{3}}{\Omega_L[b] \mid \Omega_R[a] \mid \Omega \rightarrow \Omega_L[b] \mid \Omega'_R[a] \mid \Omega'} \xrightarrow{4} \\
\frac{\Omega_a[a] \mid \Omega \rightarrow^* \Omega_L[b] \mid \Omega_R[a] \mid \Omega \quad \Omega_L[b] \mid \Omega_R[a] \mid \Omega \rightarrow \Omega_L[b] \mid \Omega'_R[a] \mid \Omega'}{\Omega_a[a] \mid \Omega \rightarrow^* \Omega_L[b] \mid \Omega'_R[a] \mid \Omega'} \xrightarrow{5} \\
\frac{\Omega_a[a] \mid \Omega \rightarrow^* \Omega_L[b] \mid \Omega'_R[a] \mid \Omega'}{\uplus[\overline{\Omega}] \mid \Omega \rightarrow^* \uplus[\overline{\Omega}_{\Delta}, b \mapsto \Omega_L, a \mapsto \Omega'_R] \mid \Omega'} \xrightarrow{6}
\end{array}$$

1 Concat-Frame-L 2 Concat-Frame-R-Act 3 LT-Config-Comm 4 LT-Config-Single-is-Multi
5 LT-Config-Trans 6 Concat-Frame-R

$$\frac{\mathcal{E}_{\Delta, b: A_L, a: A_R}(\overline{\Omega}_{\Delta}, b \mapsto \Omega_L, a \mapsto \Omega'_R) \quad \Delta, b: A_L, a: A_R \Rightarrow \Omega' \overset{\mathcal{E}}{\in} B}{\mathcal{E}_B(\uplus[\overline{\Omega}_{\Delta}, b \mapsto \Omega_L, a \mapsto \Omega'_R] \mid \Omega')} \xrightarrow{1}$$

1 IH

$$\frac{\uplus[\overline{\Omega}] \mid \Omega \rightarrow^* \uplus[\overline{\Omega}_{\Delta}, b \mapsto \Omega_L, a \mapsto \Omega'_R] \mid \Omega' \quad \mathcal{E}_B(\uplus[\overline{\Omega}_{\Delta}, b \mapsto \Omega_L, a \mapsto \Omega'_R] \mid \Omega')}{\mathcal{E}_B(\uplus[\overline{\Omega}] \mid \Omega)} \xrightarrow{1}$$

1 E-Fun-BC

$$\begin{array}{c}
\text{CASE} \quad \frac{\mathcal{E}_{\Delta}(\overline{\Omega}) \quad \frac{\Omega \xrightarrow{?b} \Omega' \quad \Delta, b: A_L \Rightarrow \Omega' \overset{\mathcal{E}}{\in} A_R}{\Delta \Rightarrow \Omega \overset{\mathcal{U}}{\in} [A_L \multimap A_R]} [\text{V-LolliT-R}]}{\mathcal{E}_{A_L \multimap A_R}(\uplus[\overline{\Omega}] \mid \Omega)}
\end{array}$$

- Fix an arbitrary $\mathcal{E}_{A_L}(\Omega_L)$

$$\begin{array}{c}
\Omega \xrightarrow{?b} \Omega' \\
\hline
\mathfrak{M}[\overline{\Omega}] \mid \Omega \xrightarrow{?b} \mathfrak{M}[\overline{\Omega}] \mid \Omega' \xrightarrow{1} \mathcal{E}_{\Delta}(\overline{\Omega}) \quad \mathcal{E}_{A_L}(\Omega_L) \quad \Delta, b : A_L \Rightarrow \Omega' \overset{\mathcal{E}}{\in} A_R \xrightarrow{2} \\
\hline
\mathcal{E}_{A_R}(\Omega_L[b] \mid \mathfrak{M}[\overline{\Omega}] \mid \Omega') \xrightarrow{3} \\
\hline
\mathcal{E}_{A_L \multimap A_R}(\mathfrak{M}[\overline{\Omega}] \mid \Omega)
\end{array}$$

1 Concat-Frame-R-Act 2 IH 3 fold

$$\begin{array}{c}
\Omega \xrightarrow{[a]!b} \Omega' \quad \Delta, a : A_R \Rightarrow \Omega' \overset{\mathcal{E}}{\in} B \quad [\text{V-LolliT-L}] \\
\hline
\text{CASE } \mathcal{E}_{\Delta, b : A_L, a : A_L \multimap A_R}(\overline{\Omega}) \quad \Delta, b : A_L, [a : A_L \multimap A_R] \Rightarrow \Omega \overset{\mathcal{U}}{\in} B \\
\hline
\mathcal{E}_B(\mathfrak{M}[\overline{\Omega}] \mid \Omega)
\end{array}$$

- $\overline{\Omega} = \overline{\Omega}_{\Delta}, b \mapsto \Omega_b, a \mapsto \Omega_a$
- $\mathcal{E}_{\Delta}(\overline{\Omega}_{\Delta}) \wedge \mathcal{E}_{A_L}(\Omega_b) \wedge \mathcal{E}_{A_L \multimap A_R}(\Omega_a)$
- $\Omega_a \rightarrow^* \Omega'_a \wedge \Omega'_a \xrightarrow{?b} \Omega''_a$
- $\mathcal{E}_{A_R}(\Omega_b[b] \mid \Omega''_a)$

$$\begin{array}{c}
\Omega'_a \xrightarrow{?b} \Omega''_a \quad \Omega \xrightarrow{[a]!b} \Omega' \quad 2 \\
\hline
\Omega_a \rightarrow^* \Omega'_a \quad \Omega'_a[a] \mid \Omega \rightarrow \Omega''_a[a] \mid \Omega' \quad 3 \\
\hline
\Omega_a[a] \mid \Omega \rightarrow^* \Omega'_a[a] \mid \Omega \quad \Omega'_a[a] \mid \Omega \rightarrow^* \Omega''_a[a] \mid \Omega' \quad 4 \\
\hline
\Omega_a[a] \mid \Omega \rightarrow^* \Omega''_a[a] \mid \Omega' \quad 5 \\
\hline
\mathfrak{M}[\overline{\Omega}] \mid \Omega \rightarrow^* \mathfrak{M}[\overline{\Omega}_{\Delta}, a \mapsto \Omega_b[b] \mid \Omega''_a] \mid \Omega'
\end{array}$$

1 Concat-Frame-L 2 LT-Config-Comm 3 LT-Config-Single-is-Multi 4 LT-Config-Trans
5 Concat-Frame-R

$$\begin{array}{c}
\mathcal{E}_{\Delta, a : A_R}(\overline{\Omega}_{\Delta}, a \mapsto \Omega_b[b] \mid \Omega''_a) \quad \Delta, a : A_R \Rightarrow \Omega' \overset{\mathcal{E}}{\in} B \\
\hline
\mathcal{E}_B(\mathfrak{M}[\overline{\Omega}_{\Delta}, a \mapsto \Omega_b[b] \mid \Omega''_a] \mid \Omega') \xrightarrow{1}
\end{array}$$

1 IH

$$\begin{array}{c}
\mathfrak{M}[\overline{\Omega}] \mid \Omega \rightarrow^* \mathfrak{M}[\overline{\Omega}_{\Delta}, a \mapsto \Omega_b[b] \mid \Omega''_a] \mid \Omega' \quad \mathcal{E}_B(\mathfrak{M}[\overline{\Omega}_{\Delta}, a \mapsto \Omega_b[b] \mid \Omega''_a] \mid \Omega') \\
\hline
\mathcal{E}_B(\mathfrak{M}[\overline{\Omega}] \mid \Omega) \xrightarrow{1}
\end{array}$$

1 E-Fun-BC

□

Definition 9.1.1 (Inhabitation of Single Types)

$$\begin{aligned}\Omega \overset{\textcolor{blue}{\mathcal{U}}}{\in} A &\triangleq \emptyset \Rightarrow \Omega \overset{\textcolor{blue}{\mathcal{U}}}{\in} A \\ \Omega \overset{\textcolor{red}{\mathcal{E}}}{\in} A &\triangleq \emptyset \Rightarrow \Omega \overset{\textcolor{red}{\mathcal{E}}}{\in} A \\ \overline{\Omega} \overset{\textcolor{red}{\mathcal{E}}}{\in} \Delta &\triangleq \forall a \in \Delta . \overline{\Omega}(a) \overset{\textcolor{red}{\mathcal{E}}}{\in} \Delta(a)\end{aligned}$$

Next, an easy-to-prove but highly non-trivial pair of corollaries resulting from the semantic fundamental theorem:

$$\begin{array}{cc}\text{E-fun-if-E} & \text{E-fun-if-V} \\ \frac{\Omega \overset{\textcolor{red}{\mathcal{E}}}{\in} A}{\mathcal{E}_A(\Omega)} & \frac{\Omega \overset{\textcolor{blue}{\mathcal{U}}}{\in} A}{\mathcal{E}_A(\Omega)}\end{array}$$

Proof of [E-fun-if-E] :

$$\frac{\frac{\emptyset \overset{\textcolor{red}{\mathcal{E}}}{\in} \emptyset \quad \emptyset \Rightarrow \Omega \overset{\textcolor{red}{\mathcal{E}}}{\in} A}{\mathcal{E}_A(\Omega)}}{1}$$

1 Sem-FTLR-E □

Proof of [E-fun-if-V] :

$$\frac{\frac{\emptyset \overset{\textcolor{red}{\mathcal{E}}}{\in} \emptyset \quad \emptyset \Rightarrow \Omega \overset{\textcolor{blue}{\mathcal{U}}}{\in} A}{\mathcal{E}_A(\Omega)}}{1}$$

1 Sem-FTLR-V □

We can further prove a pair of corollaries resulting from [E-fun-if-E] (recall that it is a corollary of the semantic fundamental theorem) and the semantic fundamental theorem itself:

$$\begin{array}{cc}\text{E-Cut-E} & \text{E-Cut-V} \\ \frac{\overline{\Omega} \overset{\textcolor{red}{\mathcal{E}}}{\in} \Delta \quad \Delta \Rightarrow \Omega \overset{\textcolor{red}{\mathcal{E}}}{\in} A}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \Omega)} & \frac{\overline{\Omega} \overset{\textcolor{red}{\mathcal{E}}}{\in} \Delta \quad \Delta \Rightarrow \Omega \overset{\textcolor{blue}{\mathcal{U}}}{\in} A}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \Omega)}\end{array}$$

Proof of [E-Cut-E] :

$$\frac{\frac{\overline{\Omega} \stackrel{\mathcal{E}}{\in} \Delta}{\mathcal{E}_{\Delta}(\overline{\Omega})} \xrightarrow{1} \Delta \Rightarrow \Omega \stackrel{\mathcal{E}}{\in} A}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \Omega)} \xrightarrow{2}$$

1 E-fun-if-E 2 Sem-FTLR-E

□

Proof of [E-Cut-V] :

$$\frac{\frac{\overline{\Omega} \stackrel{\mathcal{E}}{\in} \Delta}{\mathcal{E}_{\Delta}(\overline{\Omega})} \xrightarrow{1} \Delta \Rightarrow \Omega \stackrel{\mathcal{U}}{\in} A}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \Omega)} \xrightarrow{2}$$

1 E-fun-if-E 2 Sem-FTLR-V

□

9.2 Syntactic Fundamental Theorem

E-BC

$$\frac{\Omega \rightarrow^* \Omega' \quad \Delta \Rightarrow \Omega' \stackrel{\mathcal{E}}{\in} A}{\Delta \Rightarrow \Omega \stackrel{\mathcal{E}}{\in} A}$$

Proof:

$$\frac{\Delta \Rightarrow \Omega' \stackrel{\mathcal{E}}{\in} A}{\exists \Omega'' . \Omega' \rightarrow^* \Omega'' \wedge \Delta \Rightarrow \Omega'' \stackrel{\mathcal{U}}{\in} A} 1$$

1 inversion

$$\frac{\frac{\Omega \rightarrow^* \Omega' \quad \Omega' \rightarrow^* \Omega''}{\Omega \rightarrow^* \Omega''} \xrightarrow{1} \Delta \Rightarrow \Omega'' \stackrel{\mathcal{U}}{\in} A}{\Delta \Rightarrow \Omega \stackrel{\mathcal{E}}{\in} A} 2$$

1 LT-Config-Trans 2 E-if-V

□

E-if-V*

$$\frac{\Delta \Rightarrow \Omega \overset{\textcolor{blue}{v}}{\in} A}{\Delta \Rightarrow \Omega \overset{\textcolor{red}{e}}{\in} A}$$

Proof:

$$\frac{\overline{\Omega \rightarrow^* \Omega} \quad \overset{1}{\Delta \Rightarrow \Omega \overset{\textcolor{blue}{v}}{\in} A}}{\Delta \Rightarrow \Omega \overset{\textcolor{red}{e}}{\in} A} \quad \textcolor{blue}{2}$$

1 LT-Config-Refl 2 E-if-V

□

E-Cut

$$\frac{\Delta_L \Rightarrow \Omega_L \overset{\textcolor{red}{e}}{\in} A \quad \Delta_R, a : A \Rightarrow \Omega_R \overset{\textcolor{red}{e}}{\in} B}{\Delta_L, \Delta_R \Rightarrow \Omega_L[\textcolor{blue}{a}] \mid \Omega_R \overset{\textcolor{red}{e}}{\in} B}$$

Proof:

$$\frac{\Delta_L \Rightarrow \Omega_L \overset{\textcolor{red}{e}}{\in} A}{\exists \Omega'_L . \Omega_L \rightarrow^* \Omega'_L \wedge \Delta_L \Rightarrow \Omega'_L \overset{\textcolor{blue}{v}}{\in} A} \quad \textcolor{blue}{1}$$

1 inversion

$$\frac{\Delta_R, a : A \Rightarrow \Omega_R \overset{\textcolor{red}{e}}{\in} B}{\exists \Omega'_R . \Omega_R \rightarrow^* \Omega'_R \wedge \Delta_R, a : A \Rightarrow \Omega'_R \overset{\textcolor{blue}{v}}{\in} B} \quad \textcolor{blue}{1}$$

1 inversion

$$\frac{\frac{\Omega_L \rightarrow^* \Omega'_L}{\Omega_L[\textcolor{blue}{a}] \mid \Omega_R \rightarrow^* \Omega'_L[\textcolor{blue}{a}] \mid \Omega_R} \quad \frac{\Omega_R \rightarrow^* \Omega'_R}{\Omega'_L[\textcolor{blue}{a}] \mid \Omega_R \rightarrow^* \Omega'_L[\textcolor{blue}{a}] \mid \Omega'_R}}{\Omega_L[\textcolor{blue}{a}] \mid \Omega_R \rightarrow^* \Omega'_L[\textcolor{blue}{a}] \mid \Omega'_R} \quad \textcolor{blue}{3}$$

1 Concat-Frame-L 2 Concat-Frame-R 3 LT-Config-Trans

$$\begin{array}{c}
\frac{\Delta_L \Rightarrow \Omega'_L \overset{\textcolor{blue}{u}}{\in} A \quad \Delta_R, a : A \Rightarrow \Omega'_R \overset{\textcolor{blue}{u}}{\in} B}{\Delta_L, \Delta_R \Rightarrow \Omega'_L[a] \mid \Omega'_R \overset{\textcolor{blue}{u}}{\in} B} 1 \\
\frac{\Omega_L[a] \mid \Omega_R \rightarrow^* \Omega'_L[a] \mid \Omega'_R \quad \Delta_L, \Delta_R \Rightarrow \Omega'_L[a] \mid \Omega'_R \overset{\textcolor{blue}{u}}{\in} B}{\Delta_L, \Delta_R \Rightarrow \Omega_L[a] \mid \Omega_R \overset{\textcolor{red}{e}}{\in} B} 2
\end{array}$$

1 V-Cut 2 E-if-V

□

Define singleton configurations consisting of only root processes:

$$\{P\} \triangleq \emptyset \mid P$$

Syn-FTLR

$$\frac{\Delta \vdash \gamma : \Gamma \quad \Gamma \vdash S : A}{\Delta \Rightarrow \{\hat{\gamma}(S)\} \overset{\textcolor{red}{e}}{\in} A}$$

Proof by induction on the derivation of $\Gamma \vdash S : A$:

CASE $\frac{\Delta \vdash \gamma : \emptyset \quad \frac{}{\emptyset \vdash \text{close} : 1} [\text{OneT-R}]}{\Delta \Rightarrow \{\hat{\gamma}(\text{close})\} \overset{\textcolor{red}{e}}{\in} 1}$

- $\Delta = \emptyset \wedge \gamma = \emptyset$
- $\hat{\emptyset}(\text{close}) = \text{close}$

$$\begin{array}{c}
\frac{}{\text{close} \xrightarrow{!()} \emptyset} 1 \\
\frac{\text{close} \xrightarrow{!()} \emptyset}{\{\text{close}\} \xrightarrow{!()} \{\emptyset\}} 2 \\
\frac{\{\text{close}\} \xrightarrow{!()} \{\emptyset\}}{\emptyset \Rightarrow \{\text{close}\} \overset{\textcolor{blue}{u}}{\in} [1]} 3 \\
\frac{\emptyset \Rightarrow \{\text{close}\} \overset{\textcolor{blue}{u}}{\in} [1]}{\emptyset \Rightarrow \{\text{close}\} \overset{\textcolor{red}{e}}{\in} 1} 4
\end{array}$$

1 LT-OneT-R 2 LT-Config-R-Act 3 V-1T-R 4 E-if-V*

CASE $\frac{\Delta, a : 1 \vdash \gamma, a \leftarrow x : \Gamma, x : 1 \quad \frac{\Gamma \vdash S : A}{\Gamma, x : 1 \vdash \text{wait}[x]; S : A} [\text{OneT-L}]}{\Delta, a : 1 \Rightarrow \{(\gamma, \widehat{a \leftarrow x})(\text{wait}[x]; S)\} \overset{\textcolor{red}{e}}{\in} A}$

- $\Delta \vdash \gamma : \Gamma$
- $(\gamma, \widehat{a \leftarrow x})(\text{wait}[x]; S) = \text{wait}[a]; \hat{\gamma}(S)$

$$\begin{array}{c}
\frac{}{\text{wait}[a]; \hat{\gamma}(S) \xrightarrow{[a]?()} \hat{\gamma}(S)} \quad 1 \\
\frac{}{\{\text{wait}[a]; \hat{\gamma}(S)\} \xrightarrow{[a]?()} \{\hat{\gamma}(S)\}} \quad 2 \\
\frac{\Delta \vdash \gamma : \Gamma \quad \Gamma \vdash S : A}{\Delta \Rightarrow \{\hat{\gamma}(S)\} \in A} \quad 3 \\
\frac{}{\Delta, [a : 1] \Rightarrow \{\text{wait}[a]; \hat{\gamma}(S)\} \in A} \quad 4 \\
\frac{}{\Delta, [a : 1] \Rightarrow \{\hat{\gamma}(S)\} \in A} \quad 5
\end{array}$$

1 LT-OneT-L 2 LT-Config-L-Act-R 3 IH 4 V-1T-L 5 E-if-V*

CASE $\Delta \vdash \gamma : \Gamma \quad \frac{\Gamma \vdash S : A_\pi}{\Gamma \vdash \text{pick } \pi; S : A_L \oplus A_R} \text{ [ICT-R]}$

$$\Delta \Rightarrow \{\hat{\gamma}(\text{pick } \pi; S)\} \in A_L \oplus A_R$$

- $\hat{\gamma}(\text{pick } \pi; S) = \text{pick } \pi; \hat{\gamma}(S)$

$$\begin{array}{c}
\frac{}{\text{pick } \pi; \hat{\gamma}(S) \xrightarrow{! \pi} \hat{\gamma}(S)} \quad 1 \\
\frac{}{\{\text{pick } \pi; \hat{\gamma}(S)\} \xrightarrow{! \pi} \{\hat{\gamma}(S)\}} \quad 2 \\
\frac{\Delta \vdash \gamma : \Gamma \quad \Gamma \vdash S : A_\pi}{\Delta \Rightarrow \{\hat{\gamma}(S)\} \in A_\pi} \quad 3 \\
\frac{}{\Delta \Rightarrow \{\text{pick } \pi; \hat{\gamma}(S)\} \in [A_L \oplus A_R]} \quad 4 \\
\frac{}{\Delta \Rightarrow \{\text{pick } \pi; \hat{\gamma}(S)\} \in A_L \oplus A_R} \quad 5
\end{array}$$

1 LT-ICT-R 2 LT-Config-R-Act 3 IH 4 V-ICT-R 5 E-if-V*

CASE $\frac{\Delta, a : A_L \oplus A_R \vdash \gamma, a \leftarrow x : \Gamma, x : A_L \oplus A_R}{\frac{\Gamma, x : A_L \vdash S_L : B}{\Gamma, x : A_R \vdash S_R : B} \text{ [ICT-L]}} \text{ [ICT-L]}$

$$\Delta, a : A_L \oplus A_R \Rightarrow \{(\gamma, \widehat{a} \leftarrow x)(\text{case}[x] (S_L \mid S_R))\} \in B$$

- $\Delta \vdash \gamma : \Gamma$
- $\Delta, a : A_\pi \vdash \gamma, a \leftarrow x : \Gamma, x : A_\pi$
- $(\gamma, \widehat{a} \leftarrow x)(\text{case}[x] (S_L \mid S_R)) = \text{case}[a] ((\gamma, \widehat{a} \leftarrow x)(S_L) \mid (\gamma, \widehat{a} \leftarrow x)(S_R))$

$$\frac{\Delta, a : A_\pi \vdash \gamma, a \leftarrow x : \Gamma, x : A_\pi \quad \Gamma, x : A_\pi \vdash S_\pi : B}{\Delta, a : A_\pi \Rightarrow \{(\gamma, \widehat{a} \leftarrow x)(S_\pi)\} \in B} \quad 1$$

1 IH

$$\begin{array}{c}
\frac{}{\text{case}[a] ((\gamma, \widehat{a} \leftarrow x)(S_L) \mid (\gamma, \widehat{a} \leftarrow x)(S_R)) \xrightarrow{[a]? \pi} (\gamma, \widehat{a} \leftarrow x)(S_\pi)} \quad 1 \\
\frac{}{\{\text{case}[a] ((\gamma, \widehat{a} \leftarrow x)(S_L) \mid (\gamma, \widehat{a} \leftarrow x)(S_R))\} \xrightarrow{[a]? \pi} \{(\gamma, \widehat{a} \leftarrow x)(S_\pi)\}} \quad 2 \\
\frac{}{\Delta, a : A_\pi \Rightarrow \{(\gamma, \widehat{a} \leftarrow x)(S_\pi)\} \overset{\mathcal{E}}{\in} B} \quad 3 \\
\frac{}{\Delta, [a : A_L \oplus A_R] \Rightarrow \{\text{case}[a] ((\gamma, \widehat{a} \leftarrow x)(S_L) \mid (\gamma, \widehat{a} \leftarrow x)(S_R))\} \overset{\mathcal{U}}{\in} B} \quad 4 \\
\frac{}{\Delta, [a : A_L \oplus A_R] \Rightarrow \{\text{case}[a] ((\gamma, \widehat{a} \leftarrow x)(S_L) \mid (\gamma, \widehat{a} \leftarrow x)(S_R))\} \overset{\mathcal{E}}{\in} B}
\end{array}$$

1 LT-ICT-L 2 LT-Config-L-Act-R 3 V-ICT-L 4 E-if-V*

$$\begin{array}{c}
\frac{\Gamma \vdash S_L : A_L \quad \Gamma \vdash S_R : A_R}{\Gamma \vdash \text{case}(S_L \mid S_R) : A_L \& A_R} \text{ [ECT-R]} \\
\text{CASE } \frac{\Delta \vdash \gamma : \Gamma \quad \Gamma \vdash \text{case}(S_L \mid S_R) : A_L \& A_R}{\Delta \Rightarrow \{\hat{\gamma}(\text{case}(S_L \mid S_R))\} \overset{\mathcal{E}}{\in} A_L \& A_R}
\end{array}$$

- $\hat{\gamma}(\text{case}(S_L \mid S_R)) = \text{case}(\hat{\gamma}(S_L) \mid \hat{\gamma}(S_R))$

$$\begin{array}{c}
\frac{}{\text{case}(\hat{\gamma}(S_L) \mid \hat{\gamma}(S_R)) \xrightarrow{? \pi} \hat{\gamma}(S_\pi)} \quad 1 \\
\frac{}{\{\text{case}(\hat{\gamma}(S_L) \mid \hat{\gamma}(S_R))\} \xrightarrow{? \pi} \{\hat{\gamma}(S_\pi)\}} \quad 2 \\
\frac{\Delta \vdash \gamma : \Gamma \quad \Gamma \vdash S_\pi : A_\pi}{\Delta \Rightarrow \{\hat{\gamma}(S_\pi)\} \overset{\mathcal{E}}{\in} A_\pi} \quad 3 \\
\frac{}{\Delta \Rightarrow \{\text{case}(\hat{\gamma}(S_L) \mid \hat{\gamma}(S_R))\} \overset{\mathcal{U}}{\in} [A_L \& A_R]} \quad 4 \\
\frac{}{\Delta \Rightarrow \{\text{case}(\hat{\gamma}(S_L) \mid \hat{\gamma}(S_R))\} \overset{\mathcal{E}}{\in} A_L \& A_R} \quad 5
\end{array}$$

1 LT-ECT-R 2 LT-Config-R-Act 3 IH 4 V-ECT-R 5 E-if-V*

$$\begin{array}{c}
\frac{\Delta, a : A_L \& A_R \vdash (\gamma, a \leftarrow x) : \Gamma, x : A_L \& A_R}{\Gamma, x : A_\pi \vdash S : A} \text{ [ECT-L]} \\
\text{CASE } \frac{}{\Gamma, x : A_L \& A_R \vdash \text{pick}[x] \pi; S : A} \\
\frac{}{\Delta, a : A_L \& A_R \Rightarrow \{(\gamma, \widehat{a} \leftarrow x)(\text{pick}[x] \pi; S)\} \overset{\mathcal{E}}{\in} A}
\end{array}$$

- $\Delta \vdash \gamma : \Gamma \wedge \Delta, a : A_\pi \vdash \gamma, a \leftarrow x : \Gamma, x : A_\pi$
- $(\gamma, \widehat{a} \leftarrow x)(\text{pick}[x] \pi; S) = \text{pick}[a] \pi; (\gamma, \widehat{a} \leftarrow x)(S)$

$$\frac{\Delta, a : A_\pi \vdash \gamma, a \leftarrow x : \Gamma, x : A_\pi \quad \Gamma, x : A_\pi \vdash S : A}{\Delta, a : A_\pi \Rightarrow \{(\gamma, \widehat{a} \leftarrow x)(S)\} \overset{\mathcal{E}}{\in} A} \quad 1$$

1 IH

$$\begin{array}{c}
\frac{}{\text{pick}[a] \pi; (\gamma, \widehat{a} \leftarrow x)(S) \xrightarrow{[a]!\pi} (\gamma, \widehat{a} \leftarrow x)(S)} 1 \\
\frac{}{\{\text{pick}[a] \pi; (\gamma, \widehat{a} \leftarrow x)(S)\} \xrightarrow{[a]!\pi} \{(\gamma, \widehat{a} \leftarrow x)(S)\}} 2 \\
\frac{}{\Delta, a : A_\pi \Rightarrow \{(\gamma, \widehat{a} \leftarrow x)(S)\} \overset{\mathcal{E}}{\in} A} 3 \\
\frac{}{\Delta, [a : A_L \& A_R] \Rightarrow \{\text{pick}[a] \pi; (\gamma, \widehat{a} \leftarrow x)(S)\} \overset{\mathcal{U}}{\in} A} 4 \\
\frac{}{\Delta, a : A_L \& A_R \Rightarrow \{\text{pick}[a] \pi; (\gamma, \widehat{a} \leftarrow x)(S)\} \overset{\mathcal{E}}{\in} A}
\end{array}$$

1 LT-ECT-L 2 LT-Config-L-Act-R 3 V-ECT-L 4 E-if-V*

$$\begin{array}{c}
\text{CASE} \quad \frac{\Delta, a : A_L \vdash (\gamma, a \leftarrow x) : \Gamma, x : A_L \quad \frac{\Gamma \vdash S : A_R}{\Gamma, x : A_L \vdash \text{send } x; S : A_L \otimes A_R} [\text{TPT-R}]}{\Delta, a : A_L \Rightarrow \{(\gamma, \widehat{a} \leftarrow x)(\text{send } x; S)\} \overset{\mathcal{E}}{\in} A_L \otimes A_R}
\end{array}$$

- $\Delta \vdash \gamma : \Gamma$
- $(\gamma, \widehat{a} \leftarrow x)(\text{send } x; S) = \text{send } a; \hat{\gamma}(S)$

$$\begin{array}{c}
\frac{}{\text{send } a; \hat{\gamma}(S) \xrightarrow{!a} \hat{\gamma}(S)} 1 \\
\frac{}{\{\text{send } a; \hat{\gamma}(S)\} \xrightarrow{!a} \{\hat{\gamma}(S)\}} 2 \\
\frac{\Delta \vdash \gamma : \Gamma \quad \Gamma \vdash S : A_R}{\Delta \Rightarrow \{\hat{\gamma}(S)\} \overset{\mathcal{E}}{\in} A_R} 3 \\
\frac{}{\Delta, a : A_L \Rightarrow \{\text{send } a; \hat{\gamma}(S)\} \overset{\mathcal{U}}{\in} [A_L \otimes A_R]} 4 \\
\frac{}{\Delta, a : A_L \Rightarrow \{\text{send } a; \hat{\gamma}(S)\} \overset{\mathcal{E}}{\in} A_L \otimes A_R} 5
\end{array}$$

1 LT-TPT-R 2 LT-Config-R-Act 3 IH 4 V-TPT-R 5 E-if-V*

$$\begin{array}{c}
\frac{\Delta, a : A_L \otimes A_R \vdash \gamma, a \leftarrow x : \Gamma, x : A_L \otimes A_R}{\frac{\Gamma, y : A_L, x : A_R \vdash S : C}{\Gamma, x : A_L \otimes A_R \vdash \text{recv}[x] y \Rightarrow S : C} [\text{TPT-L}]} \\
\text{CASE} \quad \frac{}{\Delta, a : A_L \otimes A_R \Rightarrow \{(\gamma, \widehat{a} \leftarrow x)(\text{recv}[x] y \Rightarrow S)\} \overset{\mathcal{E}}{\in} C}
\end{array}$$

- $\Delta, b : A_L, a : A_R \vdash \gamma, b \leftarrow y, a \leftarrow x : \Gamma, y : A_L, x : A_R$
- $(\gamma, \widehat{a} \leftarrow x)(\text{recv}[x] y \Rightarrow S) = \text{recv}[a] y \Rightarrow (\gamma, \widehat{a} \leftarrow x)(S)$
- $[b/y](\gamma, \widehat{a} \leftarrow x)(S) = (\gamma, b \leftarrow \widehat{y}, a \leftarrow x)(S)$

$$\frac{\Delta, b : A_L, a : A_R \vdash \gamma, b \leftarrow y, a \leftarrow x : \Gamma, y : A_L, x : A_R \quad \Gamma, y : A_L, x : A_R \vdash S : C}{\Delta, b : A_L, a : A_R \Rightarrow \{[b/y](\gamma, \widehat{a} \leftarrow x)(S)\} \overset{\mathcal{E}}{\in} C} 1$$

1 IH

$$\begin{array}{c}
\frac{}{\text{recv}[a] y \Rightarrow (\gamma, \widehat{a \leftarrow x})(S)} \xrightarrow{[a]?b} [b/y](\gamma, \widehat{a \leftarrow x})(S) \quad 1 \\
\frac{}{\{\text{recv}[a] y \Rightarrow (\gamma, \widehat{a \leftarrow x})(S)\} \xrightarrow{[a]?b} \{[b/y](\gamma, \widehat{a \leftarrow x})(S)\}} \quad 2 \\
\hline
\Delta, b : A_L, a : A_R \Rightarrow \{[b/y](\gamma, \widehat{a \leftarrow x})(S)\} \stackrel{\mathcal{E}}{\in} C \quad 3 \\
\frac{}{\Delta, [a : A_L \otimes A_R] \Rightarrow \{\text{recv}[a] y \Rightarrow (\gamma, \widehat{a \leftarrow x})(S)\} \stackrel{\mathcal{U}}{\in} C} \quad 4 \\
\hline
\Delta, a : A_L \otimes A_R \Rightarrow \{\text{recv}[a] y \Rightarrow (\gamma, \widehat{a \leftarrow x})(S)\} \stackrel{\mathcal{E}}{\in} C
\end{array}$$

1 LT-TPT-L 2 LT-Config-L-Act-R 3 V-TPT-L 4 E-if-V*

CASE $\Delta \vdash \gamma : \Gamma \quad \frac{\Gamma, z : A_L \vdash S : A_R}{\Gamma \vdash \text{recv } z \Rightarrow S : A_L \multimap A_R} \text{ [LolliT-R]}$

$$\Delta \Rightarrow \{\hat{\gamma}(\text{recv } z \Rightarrow S)\} \stackrel{\mathcal{E}}{\in} A_L \multimap A_R$$

- Pick a fresh $b \notin \Delta$
- $\Delta, b : A_L \vdash \gamma, b \leftarrow z : \Gamma, z : A_L$
- $\hat{\gamma}(\text{recv } z \Rightarrow S) = \text{recv } z \Rightarrow \hat{\gamma}(S)$
- $[b/z]\hat{\gamma}(S) = (\gamma, \widehat{b \leftarrow z})(S)$

$$\frac{\Delta, b : A_L \vdash \gamma, b \leftarrow z : \Gamma, z : A_L \quad \Gamma, z : A_L \vdash S : A_R}{\Delta, b : A_L \Rightarrow \{[b/z]\hat{\gamma}(S)\} \stackrel{\mathcal{E}}{\in} A_R} \quad 1$$

1 IH

$$\begin{array}{c}
\frac{}{\text{recv } z \Rightarrow \hat{\gamma}(S)} \xrightarrow{?b} [b/z]\hat{\gamma}(S) \quad 1 \\
\frac{}{\{\text{recv } z \Rightarrow \hat{\gamma}(S)\} \xrightarrow{?b} \{[b/z]\hat{\gamma}(S)\}} \quad 2 \\
\hline
\Delta, b : A_L \Rightarrow \{[b/z]\hat{\gamma}(S)\} \stackrel{\mathcal{E}}{\in} A_R \quad 3 \\
\frac{}{\Delta \Rightarrow \{\text{recv } z \Rightarrow \hat{\gamma}(S)\} \stackrel{\mathcal{U}}{\in} [A_L \multimap A_R]} \quad 4 \\
\hline
\Delta \Rightarrow \{\text{recv } z \Rightarrow \hat{\gamma}(S)\} \stackrel{\mathcal{E}}{\in} A_L \multimap A_R
\end{array}$$

1 LT-LolliT-R 2 LT-Config-R-Act 3 V-LolliT-R 4 E-if-V*

CASE $\Delta, b : A_L, a : A_L \multimap A_R \vdash (\gamma, b \leftarrow y, a \leftarrow x) : \Gamma, y : A_L, x : A_L \multimap A_R$

$$\frac{\Gamma, x : A_R \vdash S : C}{\Gamma, y : A_L, x : A_L \multimap A_R \vdash \text{send}[x] y; S : C} \text{ [LolliT-L]}$$

$$\Delta, b : A_L, a : A_L \multimap A_R \Rightarrow \{(\gamma, b \leftarrow y, a \leftarrow x)(\text{send}[x] y; S)\} \stackrel{\mathcal{E}}{\in} C$$

- $\Delta \vdash \gamma : \Gamma$
- $\Delta, a : A_R \vdash \gamma, a \leftarrow x : \Gamma, x : A_R$
- $(\gamma, b \leftarrow \widehat{y}, a \leftarrow x)(\text{send}[x] y; S) = \text{send}[a] b; (\gamma, \widehat{a} \leftarrow x)(S)$

$$\frac{\Delta, a : A_R \vdash \gamma, a \leftarrow x : \Gamma, x : A_R \quad \Gamma, x : A_R \vdash S : C}{\Delta, a : A_R \Rightarrow \{(\gamma, \widehat{a} \leftarrow x)(S)\} \in C} \xrightarrow{1} \textcolor{red}{\mathcal{E}}$$

1 IH

$$\frac{\frac{\frac{\text{send}[a] b; (\gamma, \widehat{a} \leftarrow x)(S) \xrightarrow{\textcolor{blue}{[a]!b}} (\gamma, \widehat{a} \leftarrow x)(S)}{1}}{\textcolor{blue}{\{ \text{send}[a] b; (\gamma, \widehat{a} \leftarrow x)(S) \}} \xrightarrow{\textcolor{blue}{[a]!b}} \{(\gamma, \widehat{a} \leftarrow x)(S)\}} \quad \Delta, a : A_R \Rightarrow \{(\gamma, \widehat{a} \leftarrow x)(S)\} \in C}{\textcolor{blue}{\{ \text{send}[a] b; (\gamma, \widehat{a} \leftarrow x)(S) \}} \xrightarrow{\textcolor{blue}{[a]!b}} \{(\gamma, \widehat{a} \leftarrow x)(S)\}} \quad \Delta, a : A_R \Rightarrow \{(\gamma, \widehat{a} \leftarrow x)(S)\} \in C} \xrightarrow{3} \textcolor{blue}{\mathcal{U}}$$

$$\frac{\Delta, b : A_L, [a : A_L \multimap A_R] \Rightarrow \{ \text{send}[a] b; (\gamma, \widehat{a} \leftarrow x)(S) \} \in C}{\Delta, b : A_L, a : A_L \multimap A_R \Rightarrow \{ \text{send}[a] b; (\gamma, \widehat{a} \leftarrow x)(S) \} \in C} \xrightarrow{4} \textcolor{red}{\mathcal{E}}$$

1 LT-LolliT-L 2 LT-Config-L-Act-R 3 V-LolliT-L 4 E-if-V*

$$\boxed{\text{CASE}} \quad \frac{\Delta_L, \Delta_R \vdash \gamma_L, \gamma_R : \Gamma, \Gamma' \quad \frac{\Gamma \vdash S : A \quad \Gamma', x : A \vdash S' : B}{\Gamma, \Gamma' \vdash \text{spawn } S \overset{x}{\triangleright} S' : B} [\text{Cut}]}{\Delta_L, \Delta_R \Rightarrow \{(\gamma_L, \widehat{\gamma_R})(\text{spawn } S \overset{x}{\triangleright} S')\} \in B} \textcolor{red}{\mathcal{E}}$$

- $\Delta_L \vdash \gamma_L : \Gamma \wedge \Delta_R \vdash \gamma_R : \Gamma'$
- Pick a fresh a
- $\Delta_R, a : A \vdash \gamma_R, a \leftarrow x : \Gamma', x : A$
- $(\gamma_L, \widehat{\gamma_R})(\text{spawn } S \overset{x}{\triangleright} S') = \text{spawn } \widehat{\gamma_L}(S) \overset{x}{\triangleright} \widehat{\gamma_R}(S')$
- $[a/x] \widehat{\gamma_R}(S') = (\gamma_R, \widehat{a} \leftarrow x)(S')$

$$\frac{\frac{\text{spawn } \widehat{\gamma_L}(S) \overset{x}{\triangleright} \widehat{\gamma_R}(S') \rightarrow \{ \widehat{\gamma_L}(S) \} [a] \mid \{(\gamma_R, \widehat{a} \leftarrow x)(S')\}}{1}}{\textcolor{blue}{\{ \text{spawn } \widehat{\gamma_L}(S) \overset{x}{\triangleright} \widehat{\gamma_R}(S') \}} \rightarrow^* \{ \widehat{\gamma_L}(S) \} [a] \mid \{(\gamma_R, \widehat{a} \leftarrow x)(S')\}} \xrightarrow{2} \textcolor{red}{\mathcal{E}}$$

1 LT-Config-Spawn 2 LT-Config-Single-is-Multi

$$\begin{array}{c}
\frac{\Delta_L \vdash \gamma_L : \Gamma \quad \Gamma \vdash S : A}{\Delta_L \Rightarrow \{\hat{\gamma}_L(S)\} \in A} \xrightarrow{1} \quad \frac{\Delta_R, a : A \vdash \gamma_R, a \leftarrow x : \Gamma', x : A \quad \Gamma', x : A \vdash S' : B}{\Delta_R, a : A \Rightarrow \{(\gamma_R, \widehat{a} \leftarrow x)(S')\} \in B} \xrightarrow{2} \\
\frac{\Delta_L \Rightarrow \{\hat{\gamma}_L(S)\} \in A \quad \Delta_R, a : A \Rightarrow \{(\gamma_R, \widehat{a} \leftarrow x)(S')\} \in B}{\Delta_L, \Delta_R \Rightarrow \{\hat{\gamma}_L(S)[a] \mid \{(\gamma_R, \widehat{a} \leftarrow x)(S')\} \in B} \xrightarrow{3}
\end{array}$$

1 IH 2 IH 3 E-Cut

$$\begin{array}{c}
\{\text{spawn } \hat{\gamma}_L(S) \stackrel{x}{\triangleright} \hat{\gamma}_R(S')\} \rightarrow^* \{\hat{\gamma}_L(S)[a] \mid \{(\gamma_R, \widehat{a} \leftarrow x)(S')\} \\
\hline
\Delta_L, \Delta_R \Rightarrow \{\hat{\gamma}_L(S)[a] \mid \{(\gamma_R, \widehat{a} \leftarrow x)(S')\} \in B \\
\hline
\Delta_L, \Delta_R \Rightarrow \{\text{spawn } \hat{\gamma}_L(S) \stackrel{x}{\triangleright} \hat{\gamma}_R(S')\} \in B \xrightarrow{1}
\end{array}$$

1 E-BC

$$\begin{array}{c}
\boxed{\text{CASE}} \quad \frac{\Delta \vdash \gamma : x : A \quad \frac{}{x : A \vdash \text{fwd}[x] : A} [\text{Forward}]}{\Delta \Rightarrow \{\hat{\gamma}(\text{fwd}[x])\} \in A} \xrightarrow{\mathcal{E}}
\end{array}$$

- $\Delta = a : A \wedge \hat{\gamma}(\text{fwd}[x]) = \text{fwd}[a]$

$$\begin{array}{c}
\frac{}{a : A \Rightarrow \{\text{fwd}[a]\} \in A} \xrightarrow{\mathcal{U}} \xrightarrow{1} \\
\frac{}{a : A \Rightarrow \{\text{fwd}[a]\} \in A} \xrightarrow{2}
\end{array}$$

1 V-Fwd-Id 2 E-if-V*

□

9.3 Corollaries

Now, we can chain the semantic fundamental theorem and the syntactical fundamental theorem together:

$$\begin{array}{c}
\text{Chained-FTLR} \\
\frac{\overline{\Omega} \in \Delta \quad \Delta \vdash \gamma : \Gamma \quad \Gamma \vdash S : A}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \hat{\gamma}(S))} \xrightarrow{}
\end{array}$$

Proof:

$$\frac{\frac{\Delta \vdash \gamma : \Gamma \quad \Gamma \vdash S : A}{\Delta \Rightarrow \{\hat{\gamma}(S)\} \in A} \xrightarrow{1} \quad \overline{\Omega} \stackrel{\mathcal{E}}{\in} \Delta}{\mathcal{E}_A(\uplus[\overline{\Omega}] \mid \hat{\gamma}(S))} \xrightarrow{2}$$

1 Syn-FTLR 2 Sem-FTLR-E □

As a special case of [Syn-FTLR], we can show that any well-typed closed session term, hence process, is automatically computable / semantically well-typed with respect to the inductive logical relation:

Closed-FTLR-Ind

$$\frac{S : A}{\{S\} \stackrel{\mathcal{E}}{\in} A} \xrightarrow{\quad}$$

Proof:

$$\frac{\frac{\emptyset \vdash \emptyset : \emptyset \quad \emptyset \vdash S : A}{\{S\} \stackrel{\mathcal{E}}{\in} A} \xrightarrow{\quad} \xrightarrow{\quad} \xrightarrow{1}$$

1 Syn-FTLR □

We can likewise show any closed session term is computable with respect to the recursive counterpart:

Closed-FTLR-Rec

$$\frac{S : A}{\mathcal{E}_A(\{S\})} \xrightarrow{\quad}$$

Proof:

$$\frac{\frac{S : A}{\{S\} \stackrel{\mathcal{E}}{\in} A} \xrightarrow{1} \quad \xrightarrow{2}}{\mathcal{E}_A(\{S\})}$$

1 Closed-FTLR-Ind 2 E-fun-if-E □

Finally, we can show that any closed session term of the unit type takes zero or more internal steps to a singleton process that sends the closing signal and terminates:

Adequacy-Termination

$$\frac{S : 1}{\exists P . \{S\} \rightarrow^* \{P\} \wedge P \xrightarrow{!()}\mathbb{0}}$$

Proof:

$$\frac{\frac{\frac{S : 1}{\xrightarrow{\mathcal{E}_1(\{S\})} 1}}{\exists \Omega . \{S\} \rightarrow^* \Omega \wedge \Omega \xrightarrow{!()}\{\mathbb{0}\}} 2}{\exists P . \{S\} \rightarrow^* \{P\} \wedge P \xrightarrow{!()}\mathbb{0}} 3$$

1 Closed-FTLR-Rec 2 unfold $\mathcal{E}_1(\{S\})$ 3 inversion on $\Omega \xrightarrow{!()}\{\mathbb{0}\}$

□

Chapter 10

Related Work

The works on recursive logical relations for intuitionistic linear logic session types and on session logical relations for noninterference [1, 4] first explored the following formulation of the lolli case of the logical relations: a configuration inhabiting a lolli type receives a channel and adds it to the context governing all free channels currently used by the configuration. This formulation removes the negative occurrence of a universally quantified computable argument configuration, and hence makes it possible to formulate the entire logical relation inductively.

This motivates the development in [Part I \(Normalization of Lambda Calculus\)](#) of this thesis, where I likewise define the verification of an open function using the simple rule that it suffices to apply the function to a fresh variable and then verify that the resulting application remains computable. In [Part II \(Termination of Process Calculus\)](#), the development follows that of [1, 4] so that the rule for the lolli case of the inductive logical relation also simply states that it suffices to verify that, upon receiving any fresh address, the resulting configuration remains computable in the context extended with the fresh address. Different from [1, 4], the inductive logical relation for the process calculus developed in this thesis no longer demands that the inhabiting configurations be well-typed. Indeed, this thesis does not even define a notion of well-typedness for configurations; hence, the notion of computability for configurations relies entirely on the inductive logical relation.

[Part I \(Normalization of Lambda Calculus\)](#) of this thesis builds upon Kripke-Style Logical Relations for Normalization [5, 6] by formulating the logical relation using an additional Kripke structure which are worlds ranging over contexts. Part I also follows the presentation of neutral terms in [5], but differs in that the inductive logical relation developed in this thesis builds the neutral judgement into the logical relation, whereas [5] defines the notion of neutrally computable terms not as part of the logical relation.

Another respect in which this thesis differs is that its neutral judgement is indexed by worlds ranging over contexts, while [5] defines a neutral judgement that is not indexed by worlds. [5] requires that the argument of a neutral application already be normalizable, whereas the formulation in this thesis requires that argument to be hereditarily computable. This thesis also extends the type theory presented in [5] with two additional positive types: generic sums and natural numbers. Finally, this thesis proves normalization via the inductive logical relation, whereas [5] proves normalization via a recursive logical relation.

[Part II \(Termination of Process Calculus\)](#) builds upon the process calculus presented in [11], in which the notion of nameless configurations, which are configurations with a distinguished nameless root process, is already present. This notion was first defined and explored in [10]. This thesis further formulates

the two kinds of process actions, with right actions being nameless in a similar sense: such actions are not annotated with a destination address because they are carried out by providers that simply respond without knowing the addresses. Finally, the inductive logical relation for the process calculus defined in this thesis is Kripke-style in the sense of carrying an extra context governing the free addresses used by the inhabiting configurations, so that the inductive logical relation is applicable to all open configurations, whereas the two works [10, 11] define logical relations applicable only to closed configurations.

Chapter 11

Conclusion

Traditionally, logical relations are defined by induction on the structure of types. In this thesis, we defined two inductive logical relations using inference rules and applied them to two calculi: the simply-typed lambda calculus and a process calculus based on intuitionistic linear logic session types. We explored how to prove the fundamental theorems of the two calculi with respect to their inductive logical relations:

$$\frac{\Gamma \vdash M : A}{\Gamma \xRightarrow{\mathcal{H}} M \in A} \qquad \frac{\Delta \vdash \gamma : \Gamma \quad \Gamma \vdash S : A}{\Delta \Rightarrow \{\hat{\gamma}(S)\} \xrightarrow{\mathcal{E}} A}$$

Next, starting from the computability judgements that are the conclusions of the two fundamental theorems, we established additional theorems to derive the desired properties of lambda terms and configurations, respectively:

$$\frac{\Gamma \xRightarrow{\mathcal{H}} M \in A}{\text{norm}_\beta(M)} \qquad \frac{\mathcal{E}_\Delta(\bar{\Omega}) \quad \Delta \Rightarrow \Omega \xrightarrow{\mathcal{E}} A}{\mathcal{E}_A(\uplus[\bar{\Omega}] \mid \Omega)}$$

The former proves that any hereditarily computable term M is also normalizable. The latter proves that composing any configuration computable according to the inductive logical relation in parallel with complementary configurations computable according to the recursive logical relation yields a configuration computable according to the recursive one.

As a special case of the latter theorem, we proved that a computable configuration is terminating when it is already closed and inhabits the type 1, in the sense that it transitions to some process that then sends the closing signal and terminates.

Overall, we established two pipelines for proving properties of programs in the lambda calculus and the process calculus. As case studies, we proved normalization and termination, respectively, using the inductive logical relations as proof engines:

$$\frac{\Gamma \vdash M : A}{\Gamma \xRightarrow{\mathcal{H}} M \in A} \qquad \frac{\frac{S : 1}{\{S\} \xrightarrow{\mathcal{E}} 1}}{\exists P . \{S\} \rightarrow^* \{P\} \wedge P \xrightarrow{!()} \mathbb{0}} \text{norm}_\beta(M)$$

In addition to functioning properly as proof engines, the inductive logical relations offer several technical benefits. The inductive logical relation developed for the lambda calculus in [Part I \(Normalization of Lambda Calculus\)](#) streamlines the accommodation of natural numbers. Whereas a typical logical relation for the natural number type would be defined in addition to the main logical relation, which would then inline the additional predicate for natural numbers, the inductive logical relation for the lambda calculus in this thesis simply treats the natural number type in the same way as the other two positive types, unit and sums. It does so by stating that a term computable at the natural number type steps either to a canonical form or to a neutral spine. Crucially, the thesis does not rely on an additional computability predicate for natural numbers.

Next, the inductive logical relation for the lambda calculus makes it possible to formulate and prove a simpler fundamental theorem, in the sense that its statement does not involve computable substitutions, which would otherwise typically be required. The proof of the fundamental theorem is therefore easy to carry out. The inductive logical relation for the process calculus developed in [Part II \(Termination of Process Calculus\)](#) likewise makes it possible to decompose the traditional fundamental theorem into semantic and syntactic fundamental theorems, which could then be combined into the typical fundamental theorem. This is a more modular approach to proving the fundamental theorem, in which the roles of address substitutions and parallel composition are clearer: address substitutions inject a session term into a singleton computable configuration, while parallel composition makes it possible to close an open computable configuration by composing it in parallel with computable complementary configurations.

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