A Complete Axiomatization of Differential Game Logic for Hybrid Games

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Abstract

Differential game logic (dG$\mathcal{L}$) is a logic for specifying and verifying properties of *hybrid games*, i.e. games that combine discrete, continuous, and adversarial dynamics. Unlike hybrid systems, hybrid games allow choices in the system dynamics to be resolved adversarially by different players with different objectives. The logic dG$\mathcal{L}$ can be used to study the existence of winning strategies for such hybrid games, i.e. ways of resolving the player's choices in *some* way so that he wins by achieving his objective for *all* choices of the opponent. Hybrid games are determined, i.e. one player has a winning strategy from each state, yet their winning regions may require transfinite closure ordinals. The logic dG$\mathcal{L}$, nevertheless, has a sound and complete axiomatization relative to any expressive logic. Separating axioms are identified that distinguish hybrid games from hybrid systems. Finally, dG$\mathcal{L}$ is proved to be strictly more expressive than the corresponding logic of hybrid systems.

1 Introduction

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Hybrid systems [NK92, ACH$^+$95, Hen96, BBM98, DN00] are dynamical systems combining discrete dynamics and continuous dynamics. They are widely important, e.g., for modeling how computers control physical systems such as cars [DGV96], aircraft [UL07] and other cyber-physical systems. Hybrid systems combine difference equations (or discrete assignments) and differential equations with conditional switching, nondeterministic choices, and repetition. Hybrid systems are not semidecidable [Hen96], but nevertheless studied by many successful verification approaches. They have a complete axiomatization relative to differential equations in *differential dynamic logic* ($\mathsf{d}\mathcal{L}$) [Pla08, Pla12a], which extends Pratt's dynamic logic of conventional discrete programs [Pra76] to hybrid systems by adding differential equations and a reachability relation semantics on the real Euclidean space.

Hybrid games [NRY96, TPS98, HHM99, TLS00, DR06, BBC10, VPVD11] are games of two players on a hybrid system. Hybrid games add an adversarial dynamics to hybrid systems, i.e. an adversarial way of resolving the choices in the system dynamics. Both players can make their respective choices arbitrarily. They are not assumed to cooperate towards a common goal but may compete. The prototypical example of a hybrid game is RoboCup, where two (teams of) robots move continuously on a soccer field subject to the discrete decisions of their control programs, and they resolve their choices adversarially in active competition for scoring goals. Worst-case verification of many other situations leads to hybrid games. Two robots may already end up in a hybrid game if they do not know anything about each other's objectives, because worst-case analysis assumes they might compete. The former situation is *true competition*, the latter *analytic competition*, because possible competition was assumed for the sake of a worst-case analysis. UAVs etc. provide further natural scenarios for both true and analytic competition. Hybrid games are also fundamental for security questions about hybrid systems, which intrinsically involve more than one player.

This article studies a compositional model of hybrid games obtained from a compositional model of hybrid systems by adding the dual operator ^{d} for passing control between the players. The dual game α^d is the same as the hybrid game α with the roles of the players swapped, much like what happens when turning a chessboard around by $180°$ so that players black and white swap sides. Hybrid games without ^{d} are single player, like hybrid systems are, because ^{d} is the only operator where control passes to the other player. Hybrid games with ^{d} give both players control over their respective choices (indicated by ^{d}). They can play in reaction to the outcome that the previous choices by the players have had on the state of the system. The fact that ^{d} is an operator on hybrid games makes them fully symmetric. That is, they allow any combination of all operators at any nesting depth to define the game, not just a single fixed pattern like, e.g., the separation into a single loop of a continuous plant player and a discrete controller player that has been predominant in related work.

Hybrid games are game-theoretically reasonably tame sequential, non-cooperative, zero-sum, two-player games of perfect information with payoffs ± 1, except that they are played on hybrid systems, which makes reachability computations and the canonical game solution technique of backwards induction for winning regions on hybrid games more difficult, because they turn out to need infinite iterations with highly transfinite closure ordinals to terminate.

One of the most fundamental questions about a hybrid game is whether the player of interest has a *winning strategy*, i.e. a way of resolving his choices that will lead to a state in which that player wins, no matter how the opponent player resolves his respective choices. If the player has such a winning strategy, he can achieve his objectives no matter what the opponent does, otherwise he needs his opponent to cooperate.[1]

This article introduces a logic and proof calculus for hybrid games and thereby decouples the questions of truth (existence of winning strategies) and proof (winning strategy certificates) and proof search (automatic construction of winning strategies). It studies provability (existence of proofs) and the proof theory of hybrid games and identifies what the right proof rules for hybrid games are (soundness & completeness).

This article presents *differential game logic* (dG$\mathcal{L}$) and its axiomatization for studying the existence of winning strategies for hybrid games. It generalizes hybrid systems to hybrid games by adding the dual operator ^{d} and a winning strategy semantics on the real Euclidean space. Hybrid games simultaneously generalize hybrid systems [NK92, ACH+95] and discrete games [vNM55, Nas51]. Similarly, dG$\mathcal{L}$ simultaneously generalizes logics of hybrid systems and logics of discrete games. The logic dG$\mathcal{L}$ generalizes differential dynamic logic (d$\mathcal{L}$) [Pla08, Pla12a] from hybrid systems to hybrid games with their adversarial dynamics and, simultaneously, generalizes Parikh's propositional game logic [Par83, Par85, PP03] from games on finite-state discrete systems to games on hybrid systems with their differential equations, uncountable state spaces, uncountably many possible moves, and interacting discrete and continuous dynamics.

Every particular play of a hybrid game has exactly one winner (Section 2), exactly one player has a winning strategy from each state no matter how the opponent reacts (determinacy, Section 3), the dG$\mathcal{L}$ proof calculus can be used to find out which of the two players it is that has a winning strategy from which state (Section 4), and dG$\mathcal{L}$ for hybrid games is proved to be more expressive than d$\mathcal{L}$ for hybrid systems (Section 5).

The primary contributions of this article are as follows. The logic dG$\mathcal{L}$ identifies the logical essence of hybrid games and their game combinators.[2] This article introduces differential game logic for hybrid games with a simple modal semantics and a simple proof calculus, which is proved to be a sound and complete axiomatization relative to any expressive logic. Completeness for game logics is a subtle problem. Completeness of propositional discrete game logic has been an open problem for 30 years [Par83]. This article does not address the propositional case, but focuses on more general hybrid games and proves a generalization of Parikh's calculus to be relatively complete for hybrid games. The completeness proof is constructive and identifies a fixpoint-style proof technique, which can be considered a modal analogue of characterizations in the Calculus of Constructions [CH88]. This technique is practical for hybrid games, and easier for hybrid systems

[1] A closely related question is about ways to exhibit that winning strategy, for which existence is a prerequisite and a constructive proof an answer. As soon as one knows from which states a winning strategy exists, local search in the action space is enough.

[2] Hybrid games only lead to a minor syntactic change compared to hybrid systems (the addition of ^{d}), yet one that entails pervasive semantical reconsiderations, because the semantic basis for assigning meaning to any operator changes in the presence of adversarial resolutions. This change leads to more expressiveness. It is a sign of logical robustness that this results in a surprisingly small change in the axiomatization. Overall, the changes induced by dualities are in some ways radical, yet, in other ways smooth.

than previous complete proof techniques. These results suggest hybrid game versions of influential views of understanding program invariants as fixpoints [CC77, Cla79]. Harel's convergence rule [HMP77], which poses practical challenges for hybrid systems verification, now turns out to be unnecessary for hybrid games, hybrid systems, and programs. Separating axioms are identified that capture the logical difference of hybrid systems versus hybrid games. Hybrid games are proved to be determined, i.e. in every state, exactly one player has a winning strategy, which is the basis for assigning classical truth to logical formulas that refer to winning strategies of hybrid games. Winning regions of hybrid games are shown to need highly transfinite closure ordinals. Hybrid games are proved to be fundamentally more than hybrid systems by proving that the logic $\mathsf{dG}\mathcal{L}$ for hybrid games is strictly more expressive than the corresponding logic $\mathsf{d}\mathcal{L}$ for hybrid systems, which is related to long-standing questions in the propositional case [Par85, BGL07], some of which are still open.

2 Differential Game Logic

A robot is a canonical example of a hybrid system. Suppose a robot, W, is running around on a planet collecting trash. His dynamics is that of a hybrid system, because his continuous dynamics comes from his continuous physical motion in space, while his discrete dynamics comes from his computer-based control decisions about when to move in which direction and when to stop to gather trash. As soon as W meets another robot, E, however, her presence changes everything for him. If W neither knows how E is programmed nor exactly what her goal is, then the only safe thing he can assume about her is that she might do anything. It takes the study of a hybrid game to find out whether W can use his choices in some way to reach his goal, say, collecting trash and avoiding collisions with E, regardless of how E chooses her actions.

The hybrid games considered here have no draws. For any particular play of the W and E game, for example, either W achieves his objective or he does not. There is no in between. Following Zermelo [Zer13], games with draws can be turned into games without draws by considering draw outcomes pessimistically as losses for the player of interest. The actual draws then result from those states from which both players would lose when considering draws pessimistically as their respective losses.

When a hybrid game expects a player to move, but the rules of the game do not permit any of his moves from the current state, then that player loses right away (he *deadlocks*). If the game completes without deadlock, the player who reaches one of his winning states wins. Thus, exactly one player wins each (completed) game play for complementary winning states. The games are *zero-sum* games, i.e. if one player wins, the other one loses, with player payoffs ± 1. Losses or victories of different payoff are not considered, only whether a player wins or loses. The two players are classically called *Angel* and *Demon*. By considering aggregate players, these results generalize in the usual way to the case where Angel and Demon represent coalitions of agents that work together to achieve a common goal.

Hybrid games are non-cooperative and sequential games. In non-cooperative games, players do not negotiate binding contracts, but can choose to act arbitrarily according to the rules represented in the game. Sequential (or dynamic) games are games that proceed in a series of steps, where, at each step, exactly one of the players can choose an action based on the outcome of the game so far. Concurrent games, where both players choose actions simultaneously, as well as equivalent games of imperfect information, are interesting but, even though they are related, beyond the scope of this paper [vNM55, AHK02, BP09]. Imperfect information games lead to Henkin quantifiers, not first-order quantifiers.

2.1 Syntax

Differential game logic (dGL) is a logic for studying properties of hybrid games. The idea is to describe the game form, i.e. rules, dynamics, and choices of the particular hybrid game of interest, using a program notation and to then study its properties by proving the validity of logical formulas that refer to the existence of winning strategies for objectives of those hybrid games. Even though hybrid game forms only describe the game *form* with its dynamics and rules and choices, not the actual objective, they are still simply called hybrid games. The objective for a hybrid game is defined in the modal logical formula that refers to that hybrid game form.

Definition 1 (Hybrid games)**.** The *hybrid games of differential game logic* dGL are defined by the following grammar (α, β are hybrid games, x a vector of variables, θ a vector of (polynomial) terms of the same dimension, ψ is a dGL formula):

$$\alpha, \beta ::= x := \theta \mid x' = \theta \,\&\, \psi \mid ?\psi \mid \alpha \cup \beta \mid \alpha ; \beta \mid \alpha^* \mid \alpha^d$$

Definition 2 (dGL formulas)**.** The *formulas of differential game logic* dGL are defined by the following grammar (ϕ, ψ are dGL formulas, p is a predicate symbol of arity k, θ_i are (polynomial) terms, x a variable, and α is a hybrid game):

$$\phi, \psi ::= p(\theta_1, \ldots, \theta_k) \mid \theta_1 \geq \theta_2 \mid \neg\phi \mid \phi \wedge \psi \mid \exists x\, \phi \mid \langle\alpha\rangle\phi \mid [\alpha]\phi$$

Other operators $>, =, \leq, <, \vee, \rightarrow, \leftrightarrow, \forall x$ can be defined as usual, e.g., $\forall x\, \phi \equiv \neg\exists x\, \neg\phi$. The modal formula $\langle\alpha\rangle\phi$ expresses that Angel has a winning strategy to achieve ϕ in hybrid game α, i.e. Angel has a strategy to reach any of the states satisfying dGL formula ϕ when playing hybrid game α, no matter what strategy Demon chooses. The modal formula $[\alpha]\phi$ expresses that Demon has a winning strategy to achieve ϕ in hybrid game α, i.e. a strategy to reach any of the states satisfying ϕ, no matter what strategy Angel chooses. Note that the same game is played in $[\alpha]\phi$ as in $\langle\alpha\rangle\phi$ with the same choices resolved by the same players. The difference between both dGL formulas is the player whose winning strategy they refer to. Both use the set of states where dGL formula ϕ is true as the winning states for that player. The winning condition is defined by the modal formula, α only defines the hybrid game form, not when the game is won, which is what ϕ does. Hybrid game α defines the rules of the game, including conditions on state variables that, if violated, cause the present player to lose for violation of the rules of the game. The dGL formulas $\langle\alpha\rangle\phi$ and $[\alpha]\neg\phi$ consider complementary winning conditions for Angel and Demon.

The *atomic games* of dGL are assignments, continuous evolutions, and tests. In the *deterministic assignment game* $x := \theta$, the value of variable x changes instantly and deterministically to that of θ by a discrete jump without any choices to resolve. In the *continuous evolution game* $x' = \theta \,\&\, \psi$, the system follows the differential equation $x' = \theta$ where the duration is Angel's choice, but Angel is not allowed to choose a duration that would, at any time, take the state outside the region where formula ψ holds. In particular, Angel is deadlocked and loses immediately if ψ does not hold in the current state, because she cannot even evolve for duration 0 then without going outside ψ.[3] The *test game* or *challenge* $?\psi$ has no effect on the state, except that Angel loses the game immediately if dG$\mathcal{L}$ formula ψ does not hold in the current state.

The *compound games* of dG$\mathcal{L}$ are sequential composition, choice, repetition, and duals. The *sequential game* $\alpha;\beta$ is the hybrid game that first plays hybrid game α and, when hybrid game α terminates without a player having won already (so no challenge in α failed), continues by playing game β. When playing the *choice game* $\alpha \cup \beta$, Angel chooses whether to play hybrid game α or play hybrid game β. Like all the other choices, this choice is dynamic, i.e. every time $\alpha \cup \beta$ is played, Angel gets to choose again whether she wants to play α or β this time. The *repeated game* α^* plays hybrid game α repeatedly and Angel chooses, after each play of α that terminates without a player having won already, whether to play the game again or not, albeit she cannot choose to play indefinitely but has to stop repeating ultimately. Angel is also allowed to stop α^* right away after zero iterations of α. Most importantly, the *dual game* α^d is the same as playing the hybrid game α with the roles of the players swapped. That is Demon decides all choices in α^d that Angel has in α, and Angel decides all choices in α^d that Demon has in α. Players who are supposed to move but deadlock lose. Thus, while the test game $?\psi$ causes Angel to lose if formula ψ does not hold, the *dual test game* (or *dual challenge*) $(?\psi)^d$ causes Demon to lose if ψ does not hold. For example, if α describes the game of chess, then α^d is chess where the players switch sides. The dual operator ^{d} is the only syntactic difference of dG$\mathcal{L}$ for hybrid games compared to d$\mathcal{L}$ for hybrid systems [Pla08, Pla12a], but a fundamental one, because it is the only operator where control passes from Angel to Demon or back. Without ^{d} all choices are resolved uniformly by Angel without interaction. The presence of ^{d} requires a thorough semantic generalization throughout the logic.

The logic dG$\mathcal{L}$ only provides logically essential operators. Many other game interactions for games of perfect information can be defined from the elementary operators that dG$\mathcal{L}$ provides. *Demonic choice* between hybrid game α and β is $\alpha \cap \beta$, defined by $(\alpha^d \cup \beta^d)^d$, in which either the hybrid game α or the hybrid game β is played, by Demon's choice. *Demonic repetition* of hybrid game α is $\alpha^\times$, defined by $((\alpha^d)^*)^d$, in which α is repeated as often as Demon chooses to. In $\alpha^\times$, Demon chooses after each play of α whether to repeat the game, but cannot play indefinitely so he has to stop repeating ultimately. The *dual differential equation* $(x' = \theta \,\&\, \psi)^d$ follows the same dynamics as $x' = \theta \,\&\, \psi$ except that Demon chooses the duration, so he cannot choose a duration during which ψ stops to hold at any time. Hence he loses when ψ does not hold in the current state. Dual assignment $(x := \theta)^d$ is equivalent to $x := \theta$, because it involves no choices. Unary operators (including $^*, ^d, \forall x, [\alpha], \langle\alpha\rangle$) bind stronger than binary operators and let ; bind stronger than $\cup$ and $\cap$, so $\alpha;\beta \cup \gamma \equiv (\alpha;\beta) \cup \gamma$.

[3] Note that the most common case for ψ is a formula of first-order real arithmetic, but any dG$\mathcal{L}$ formula will work. In Section 3.2, evolution domain constraints ψ turn out to be unnecessary, because they can be defined using hybrid games. In the ordinary differential equation $x' = \theta$, the term x' denotes the time-derivative of x and θ is a polynomial term that is allowed to mention x and other variables. More general forms of differential equations are possible [Pla10a], but will not be considered explicitly.

Note that, quite unlike in the case of α^*, it is irrelevant whether Angel decides the duration for $x' = \theta \,\&\, \psi$ before or after that continuous evolution, because initial-value problems for $x' = \theta$ have unique solutions by Picard-Lindelöff as term θ is smooth.

Observe that every (completed) play of a game is won or lost by exactly one player. Even a play of repeated game α^* has only one winner, because the game stops as soon as one player has won, e.g., because his opponent failed a test. This is different than the repetition of whole game plays (including winning/losing), where the purpose is for the players to repeat the same game over and over again to completion, win and lose multiple times, and study who wins how often in the long run with mixed strategies. In this scenario, the overall game is played once (even if some part of it constitutes in repeating action choices) and it stops as soon as either Angel or Demon have won. In applications, the system is already in trouble even if it loses the game only once, because that may entail that a safety-critical property has already been violated.

Example 1 (Wall-E and Eve). Consider a game of the robots W and E moving on a (one-dimensional) planet. A similar game can be studied for robot motion in higher dimensions using dG$\mathcal{L}$.

$$\begin{aligned}(w-e)^2 \le 1 \land v = f \rightarrow \Big\langle \Big(&(u := 1 \cap u := -1);\\ &(g := 1 \cup g := -1);\\ &t := 0; (w' = v, v' = u, e' = f, f' = g, t' = 1 \,\&\, t \le 1)^d \Big)^\times\\ \Big\rangle\, &(w-e)^2 \le 1\end{aligned} \tag{1}$$

Robot W is at position w with velocity v and acceleration u and plays the part of Demon. Robot E is at e with velocity f and acceleration g and plays the part of Angel. The antecedent of (1) before the implication assumes that W and E start close to one another (distance at most 1) and with identical velocities. The objective of E, who plays Angel's part in (1), is to be close to W (i.e. $(w-e)^2 \le 1$) as specified after the $\langle\cdot\rangle$ modality in the succedent. The hybrid game proceeds as follows. Demon W controls how often the hybrid game repeats by operator $^\times$. In each iteration, Demon W first chooses ($\cap$) to accelerate ($u := 1$) or brake ($u := -1$), then Angel E chooses ($\cup$) whether to accelerate ($g := 1$) or brake ($g := -1$). Every time that the $^\times$ loop repeats, the players get to make that choice again. They are not bound by what they chose in the previous iterations. Yet, depending on the previous choices, the state will have evolved differently, which influences indirectly what moves a player needs to choose to win. After this sequence of choices of u and g by Demon and Angel, respectively, a clock variable t is reset to $t := 0$. Then the game follows a differential equation system such that the time-derivative of W's position w is the velocity v and the time-derivative of v is acceleration u, the time-derivative of E's position e is the velocity f and the time-derivative of f is acceleration g. The time-derivative of clock variable t is 1, yet the differential equation is restricted to the evolution domain $t \le 1$. Angel controls the duration of a differential equation. Yet, this differential equation is within a dual game by operator ^{d}, so Demon controls the duration of the continuous evolution. Here, both W and E evolve continuously but Demon W decides how long. He cannot chose any duration > 1, because that would make him violate the evolution domain constraint $t \le 1$.

Deeper nesting levels of hybrid game operators can be used to describe more complicated hybrid games with more levels of interaction (e.g., any number of nested $\cup$, ^{d} , *).

2.2 Semantics

The logic dGL has a denotational semantics. The dGL semantics defines, for each formula ϕ, the set $[\![\phi]\!]^I$ of states in which ϕ is true. For each hybrid game α and each set of winning states X, the dGL semantics defines the set $\varsigma_\alpha(X)$ of states from which Angel has a winning strategy to achieve X in hybrid game α, as well as the set $\delta_\alpha(X)$ of states from which Demon has a winning strategy to achieve X in α.

A *state* s is a mapping from variables to $\mathbb{R}$. An *interpretation* I assigns a relation $I(p) \subseteq \mathbb{R}^k$ to each predicate symbol p of arity k. The interpretation further determines the set of states $\mathcal{S}$, which is isomorphic to a Euclidean space $\mathbb{R}^n$ when n is the number of relevant variables. For a subset $X \subseteq \mathcal{S}$ the complement $\mathcal{S} \setminus X$ is denoted $X^\complement$. Let s_x^d denote the state that agrees with state s except for the interpretation of variable x, which is changed to $d \in \mathbb{R}$. The value of term θ in state s is denoted by $[\![\theta]\!]_s$. The denotational semantics of dGL formulas will be defined in Def. 3 by simultaneous induction along with the denotational semantics, $\varsigma_\alpha(\cdot)$ and $\delta_\alpha(\cdot)$, of hybrid games, defined in Def. 4, because dGL formulas are defined by simultaneous induction with hybrid games.

Definition 3 (dGL semantics)**.** The *semantics of a* dGL *formula* ϕ for each interpretation I with a corresponding set of states $\mathcal{S}$ is the subset $[\![\phi]\!]^I \subseteq \mathcal{S}$ of states in which ϕ is true. It is defined inductively as follows

1. $[\![p(\theta_1, \ldots, \theta_k)]\!]^I = \{s \in \mathcal{S} : ([\![\theta_1]\!]_s, \ldots, [\![\theta_k]\!]_s) \in I(p)\}$
2. $[\![\theta_1 \geq \theta_2]\!]^I = \{s \in \mathcal{S} : [\![\theta_1]\!]_s \geq [\![\theta_2]\!]_s\}$
3. $[\![\neg\phi]\!]^I = ([\![\phi]\!]^I)^\complement$
4. $[\![\phi \wedge \psi]\!]^I = [\![\phi]\!]^I \cap [\![\psi]\!]^I$
5. $[\![\exists x\, \phi]\!]^I = \{s \in \mathcal{S} : s_x^r \in [\![\phi]\!]^I \text{ for some } r \in \mathbb{R}\}$
6. $[\![\langle\alpha\rangle\phi]\!]^I = \varsigma_\alpha([\![\phi]\!]^I)$
7. $[\![[\alpha]\phi]\!]^I = \delta_\alpha([\![\phi]\!]^I)$

A dGL formula ϕ is *valid in* I, written $I \models \phi$, iff $[\![\phi]\!]^I = \mathcal{S}$. Formula ϕ is *valid*, $\vDash \phi$, iff $I \models \phi$ for all interpretations I.

Definition 4 (Semantics of hybrid games)**.** The *semantics of a hybrid game* α is a function $\varsigma_\alpha(\cdot)$ that, for each interpretation I and each set of Angel's winning states $X \subseteq \mathcal{S}$, gives the *winning region*, i.e. the set of states $\varsigma_\alpha(X)$ from which Angel has a winning strategy to achieve X (whatever strategy Demon chooses). It is defined inductively as follows[4]

1. $\varsigma_{x:=\theta}(X) = \{s \in \mathcal{S} : s_x^{[\![\theta]\!]_s} \in X\}$

[4] The semantics of a hybrid game is not merely a reachability relation between states as for hybrid systems [Pla12a], because the adversarial dynamic interactions and nested choices of the players have to be taken into account.

2. $\varsigma_{x'=\theta\,\&\,\psi}(X) = \{\varphi(0) \in \mathcal{S} : \varphi(r) \in X$ for some $r \in \mathbb{R}_{\geq 0}$ and (differentiable) $\varphi : [0, r] \to \mathcal{S}$ such that $\varphi(\zeta) \in [\![\psi]\!]^I$ and $\frac{\mathsf{d}\,\varphi(t)(x)}{\mathsf{d}t}(\zeta) = [\![\theta]\!]_{\varphi(\zeta)}$ for all $0 \leq \zeta \leq r\}$
3. $\varsigma_{?\psi}(X) = [\![\psi]\!]^I \cap X$
4. $\varsigma_{\alpha\cup\beta}(X) = \varsigma_\alpha(X) \cup \varsigma_\beta(X)$
5. $\varsigma_{\alpha;\beta}(X) = \varsigma_\alpha(\varsigma_\beta(X))$
6. $\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\}$
7. $\varsigma_{\alpha^d}(X) = (\varsigma_\alpha(X^\complement))^\complement$

The *winning region* of Demon, i.e. the set of states $\delta_\alpha(X)$ from which Demon has a winning strategy to achieve X (whatever strategy Angel chooses) is defined inductively as follows

1. $\delta_{x:=\theta}(X) = \{s \in \mathcal{S} : s_x^{[\![\theta]\!]_s} \in X\}$
2. $\delta_{x'=\theta\,\&\,\psi}(X) = \{\varphi(0) \in \mathcal{S} : \varphi(r) \in X$ for all $r \in \mathbb{R}_{\geq 0}$ and (differentiable) $\varphi : [0, r] \to \mathcal{S}$ such that $\varphi(\zeta) \in [\![\psi]\!]^I$ and $\frac{\mathsf{d}\,\varphi(t)(x)}{\mathsf{d}t}(\zeta) = [\![\theta]\!]_{\varphi(\zeta)}$ for all $0 \leq \zeta \leq r\}$
3. $\delta_{?\psi}(X) = ([\![\psi]\!]^I)^\complement \cup X$
4. $\delta_{\alpha\cup\beta}(X) = \delta_\alpha(X) \cap \delta_\beta(X)$
5. $\delta_{\alpha;\beta}(X) = \delta_\alpha(\delta_\beta(X))$
6. $\delta_{\alpha^*}(X) = \bigcup\{Z \subseteq \mathcal{S} : Z \subseteq X \cap \delta_\alpha(Z)\}$
7. $\delta_{\alpha^d}(X) = (\delta_\alpha(X^\complement))^\complement$

This notation uses $\varsigma_\alpha(X)$ instead of $\varsigma_\alpha^I(X)$ and $\delta_\alpha(X)$ instead of $\delta_\alpha^I(X)$, because the interpretation I that gives a semantics to predicate symbols in tests and evolution domains is clear from the context. Strategies do not occur explicitly in the dG$\mathcal{L}$ semantics, because it is based on the existence of winning strategies, not on the strategies themselves.

The semantics is *compositional*, i.e. the semantics of a compound dG$\mathcal{L}$ formula is a simple function of the semantics of its pieces, and the semantics of a compound hybrid game is a function of the semantics of its pieces. This enables us to identify a compositional proof calculus. Furthermore, existence of a strategy in hybrid game α to achieve X is independent of any game and dG$\mathcal{L}$ formula surrounding α, but just depends on the remaining game α itself and the goal X. By a simple inductive argument, this shows that one can focus on memoryless strategies, because the existence of strategies does not depend on the context, hence, by working bottom up, the strategy itself cannot depend on past states and choices, only the current state, remaining game, and goal. This follows from a generalization of a classical result [Zer13], but is directly apparent in a logical setting. Furthermore, the semantics is monotone, i.e. larger sets of winning states induce larger winning regions.

Lemma 1 (Monotonicity)**.** *The semantics is* monotone*, i.e.* $\varsigma_\alpha(X) \subseteq \varsigma_\alpha(Y)$ *and* $\delta_\alpha(X) \subseteq \delta_\alpha(Y)$ *for all* $X \subseteq Y$.

Proof. A simple check based on the observation that X only occurs with an even number of negations in the semantics. For example, $\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\} \subseteq \bigcap\{Z \subseteq \mathcal{S} : Y \cup \varsigma_\alpha(Z) \subseteq Z\} = \varsigma_{\alpha^*}(Y)$ if $X \subseteq Y$. Likewise, $X \subseteq Y$ implies $X^\complement \supseteq Y^\complement$, hence $\varsigma_\alpha(X^\complement) \supseteq \varsigma_\alpha(Y^\complement)$, so $\varsigma_{\alpha^d}(X) = (\varsigma_\alpha(X^\complement))^\complement \subseteq (\varsigma_\alpha(Y^\complement))^\complement = \varsigma_{\alpha^d}(Y)$. □

Monotonicity implies that the least fixpoint in $\varsigma_{\alpha^*}(X)$ and the greatest fixpoint in $\delta_{\alpha^*}(X)$ are well-defined [HKT00, Lemma 1.7]. The semantics of $\varsigma_{\alpha^*}(X)$ is a least fixpoint, which results in a well-founded repetition of α, i.e. Angel can repeat any number of times but she ultimately needs to stop at a state in X in order to win. The semantics of $\delta_{\alpha^*}(X)$ is a greatest fixpoint, instead, for which Demon needs to achieve a state in X after every number of repetitions, because Angel could choose to stop at any time, but Demon still wins if he only postpones $X^\complement$ forever, because Angel ultimately has to stop repeating. Thus, for the formula $\langle\alpha^*\rangle\phi$, Demon already has a winning strategy if he only has a strategy that is not losing by preventing ϕ indefinitely, because Angel eventually has to stop repeating anyhow and will then end up in a state not satisfying ϕ, which makes her lose. The situation for $[\alpha^*]\phi$ is dual.

Hybrid games branch finitely when the players decide which game to play in $\alpha \cup \beta$ and $\alpha \cap \beta$, respectively. The games α^* and $\alpha^\times$ also branch finitely, because, after each repetition of α, the respective player (Angel for α^* and Demon for $\alpha^\times$) may decide whether to repeat again or stop. Repeated games may still lead to infinitely many branches, because a repeated game can be repeated any number of times. The game branches uncountably infinitely, however, when the players decide how long to evolve along differential equations in $x' = \theta \,\&\, \psi$ and $(x' = \theta \,\&\, \psi)^d$, because uncountably many nonnegative real number could be chosen as a duration (unless the system leaves ψ immediately). These choices can be made explicit by relating the simple denotational modal semantics of dG$\mathcal{L}$ to an equivalent operational game semantics that is technically much more involved but directly exposes the interactive intuition of game play. For reference, this approach has been made explicit in Appendix C.

Example 2. The following simple dG$\mathcal{L}$ formula

$$\langle(x := x + 1; (x' = x^2)^d \cup x := x - 1)^*\rangle\,(0 \leq x < 1) \tag{2}$$

is true in all states from which there is a winning strategy for Angel to reach [0,1). It is Angel's choice whether to repeat (*) and, ever time she does, it is her choice ($\cup$) whether to increase x by 1 and then (after ;) give Demon control over the duration of the differential equation $x' = x^2$ (left game) or whether to instead decrease x by 1 (right game). Formula (2) is valid, because Angel has the winning strategy of choosing the left action until $x \geq 0$, which will ultimately happen, followed by the right action until $0 \leq x < 1$. The following minor variation, however, is not valid:

$$\langle(x := x + 1; (x' = x^2)^d \cup (x := x - 1 \cap x := x - 2))^*\rangle(0 \leq x < 1)$$

because Demon can spoil Angel's efforts by choosing $x := x - 2$ in his choice ($\cap$) to make x negative whenever $1 \leq x < 2$, and then increasing x to 1.5 again via $(x' = x^2)^d$ when Angel takes the left choice. Angel will never reach $0 \leq x < 1$ that way unless this was true initially already. This phenomenon is examined in Section 3.1 in more detail.

Example 3 (Wall-E and Eve). The dG$\mathcal{L}$ formula (1) from Example 1 is valid, because Angel E indeed has a winning strategy to get close to W by mimicking Demon's choices. Recall that Demon W controls the repetition $^\times$, so the fact that the hybrid game starts E off close to W is not sufficient for E to win the game. Note that the hybrid game in (1) would be trivial if Angel were to control the repetition (because she would then win by simply choosing not to repeat) or to control the differential equation (because she would then win by only ever evolving for duration 0). Finally, the analysis of (1) requires more careful proofs if the first two lines in the hybrid game are swapped so that Angel E chooses g before Demon W chooses u.

3 Meta-Properties

This section analyzes meta-properties and semantical properties of dG$\mathcal{L}$ and hybrid games, including determinacy of hybrid games, hybrid game equivalences, and closure ordinals of hybrid games.

3.1 Determinacy

Every particular game play in a hybrid game is won by exactly one player, because hybrid games are zero-sum and there are no draws. That alone does not imply determinacy, i.e. that, from any initial situation, either one of the players always has a winning strategy to force a win, regardless of how the other player chooses to play.

In order to understand the importance of determinacy for classical logics, consider the semantics of repetition, defined as a *least* fixpoint, which is crucial because that gives a well-founded repetition. Otherwise, the *filibuster formula* would not have a well-defined truth-value:

$$\langle (x := 0 \cap x := 1)^* \rangle x = 0 \tag{3}$$

It is Angel's choice whether to repeat (*), but every time Angel repeats, it is Demon's choice ($\cap$) whether to play $x := 0$ or $x := 1$. The game in this formula never deadlocks, because every player always has a remaining move (here even two). But, without the least fixpoint, the game would have perpetual checks, because no strategy helps either player win the game; see Fig. 1.

Demon can move $x := 1$ and would win, but Angel observes this and decides to repeat, so Demon can again move $x := 1$. Thus (unless Angel is lucky starting from an initial state where she has won already) every strategy that one player has to reach $x = 0$ or $x = 1$ could be spoiled by the other player so the game would not be determined, i.e. no player has a winning strategy. Every player can let his opponent win, but would not have a strategy to win himself. Because of the least fixpoint $\varsigma_{\alpha^*}(X)$ in the semantics, however, repetitions are well-founded and, thus, have to stop eventually (after an arbitrary unbounded number of rounds). Hence, in the example in Fig. 1, Demon still wins and formula (3) is *false*, unless $x = 0$ holds initially. In other words, the formula in (3) is equivalent to $x = 0$. The same phenomenon happens in Example 2. Likewise, the dual filibuster game formula

$$x = 0 \rightarrow \langle (x := 0 \cup x := 1)^\times \rangle x = 0 \tag{4}$$

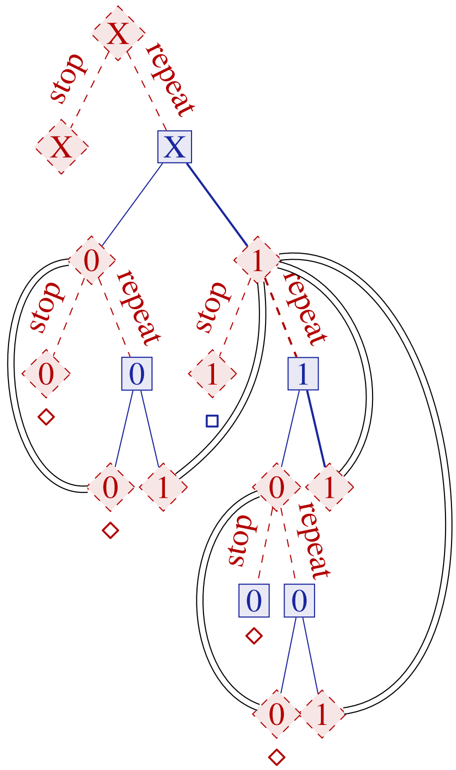

Figure 1: The filibuster game formula $\langle(x := 0 \cap x := 1)^*\rangle x = 0$ is false (unless $x = 0$ initially), but would be non-determined without least fixpoints (strategies follow thick actions). Angel's action choices are illustrated by dashed edges from dashed diamonds, Demon's action choices by solid edges from solid squares, and double lines indicate identical states with the same continuous state and a subgame of the same structure of subsequent choices. States where Angel wins are marked $\diamond$ and states where Demon wins by $\square$.

is (determined and) valid, because Demon has to stop repeating $^\times$ eventually so that Angel wins if she patiently plays $x := 0$ each time. Similarly, the game in the following hybrid filibuster formula would not be determined without the least fixpoint semantics

$$\langle(x := 0; x' = 1^d)^*\rangle x = 0$$

because Demon could always evolve continuously to some state where $x > 0$ and Angel would never want to stop. Since Angel will have to stop eventually, she loses and the formula is *false* unless $x = 0$ holds initially.

It is important as well that Angel can only choose real durations $r \in \mathbb{R}_{\geq 0}$ for a continuous evolution game $x' = \theta \,\&\, \psi$, not infinity ∞, so she ultimately stops. Otherwise

$$\langle(x' = 1^d; x := 0)^*\rangle x = 0 \tag{5}$$

would not be determined, because Angel wants to repeat (unless $x = 0$ initially) and $x := 0$ will make her win once she stops after any nonzero number of repetitions. Yet, if Demon could choose

∞ as the duration for the continuous evolution game $x' = 1^d$, Angel will never get to play the subsequent $x := 0$ to win. Since durations need to be real numbers, however, each continuous evolution ultimately has to stop, so the formula in (5) is valid.

In order to make sure that dG$\mathcal{L}$ is a classical two-valued modal logic, hybrid games have no draws during any game play. But, because modalities refer to the existence of winning strategies, they only receive classical truth values if, from each state, one of the players has a winning strategy for complementary winning conditions of a hybrid game α. The logical setup of dG$\mathcal{L}$ makes this determinacy proof very simple, without the need to use, e.g., the deep Borel determinacy theorem for winning conditions that are Borel in the product topology induced on game trees by the discrete topology of actions [Mar75].

Theorem 2 (Consistency & determinacy)**.** *Hybrid games are consistent and determined, i.e.*

$$\vDash \neg\langle\alpha\rangle\neg\phi \leftrightarrow [\alpha]\phi$$

Proof. The proof shows by induction on the structure of α that $\varsigma_\alpha(X^\complement)^\complement = \delta_\alpha(X)$ for all $X \subseteq \mathcal{S}$ and all I with some set of states $\mathcal{S}$, which implies the validity of $\neg\langle\alpha\rangle\neg\phi \leftrightarrow [\alpha]\phi$ using $X \overset{\text{def}}{=} [\![\phi]\!]^I$.

1. $\varsigma_{x:=\theta}(X^\complement)^\complement = \{s \in \mathcal{S} : s_x^{[\![\theta]\!]_s} \notin X\}^\complement = \varsigma_{x:=\theta}(X) = \delta_{x:=\theta}(X)$
2. $\varsigma_{x'=\theta\,\&\,\psi}(X^\complement)^\complement = \{\varphi(0) \in \mathcal{S} : \varphi(r) \notin X$ for some $0 \le r \in \mathbb{R}$ and some (differentiable) $\varphi : [0,r] \to \mathcal{S}$ such that $\frac{\mathrm{d}\,\varphi(t)(x)}{\mathrm{d}t}(\zeta) = [\![\theta]\!]_{\varphi(\zeta)}$ and $\varphi(\zeta) \in [\![\psi]\!]^I$ for all $0 \le \zeta \le r\}^\complement$ $= \delta_{x'=\theta\,\&\,\psi}(X)$, because the set of states from which there is no winning strategy for Angel to reach a state in $X^\complement$ prior to leaving $[\![\psi]\!]^I$ along $x' = \theta \,\&\, \psi$ is exactly the set of states from which $x' = \theta \,\&\, \psi$ always stays in X (until leaving $[\![\psi]\!]^I$ in case that ever happens).
3. $\varsigma_{?\psi}(X^\complement)^\complement = ([\![\psi]\!]^I \cap X^\complement)^\complement = ([\![\psi]\!]^I)^\complement \cup (X^\complement)^\complement = \delta_{?\psi}(X)$
4. $\varsigma_{\alpha\cup\beta}(X^\complement)^\complement = (\varsigma_\alpha(X^\complement) \cup \varsigma_\beta(X^\complement))^\complement = \varsigma_\alpha(X^\complement)^\complement \cap \varsigma_\beta(X^\complement)^\complement = \delta_\alpha(X) \cap \delta_\beta(X) = \delta_{\alpha\cup\beta}(X)$
5. $\varsigma_{\alpha;\beta}(X^\complement)^\complement = \varsigma_\alpha(\varsigma_\beta(X^\complement))^\complement = \varsigma_\alpha(\delta_\beta(X)^\complement)^\complement = \delta_\alpha(\delta_\beta(X)) = \delta_{\alpha;\beta}(X)$
6. $\varsigma_{\alpha^*}(X^\complement)^\complement = \left(\bigcap\{Z \subseteq \mathcal{S} : X^\complement \cup \varsigma_\alpha(Z) \subseteq Z\}\right)^\complement = \left(\bigcap\{Z \subseteq \mathcal{S} : (X \cap \varsigma_\alpha(Z)^\complement)^\complement \subseteq Z\}\right)^\complement$ $= \left(\bigcap\{Z \subseteq \mathcal{S} : (X \cap \delta_\alpha(Z^\complement))^\complement \subseteq Z\}\right)^\complement = \bigcup\{Z \subseteq \mathcal{S} : Z \subseteq X \cap \delta_\alpha(Z)\} = \delta_{\alpha^*}(X)$. [5]
7. $\varsigma_{\alpha^d}(X^\complement)^\complement = (\varsigma_\alpha((X^\complement)^\complement)^\complement)^\complement = \delta_\alpha(X^\complement)^\complement = \delta_{\alpha^d}(X)$ □

One direction of Theorem 2 implies $\vDash \neg\langle\alpha\rangle\neg\phi \to [\alpha]\phi$, i.e. $\vDash \langle\alpha\rangle\neg\phi \vee [\alpha]\phi$, whose validity means that, from any initial state, either Angel has a winning strategy to achieve $\neg\phi$ or Demon has a winning strategy to achieve ϕ. That is, hybrid games are determined, because there are no states from which none of the players has a winning strategy (for the same hybrid game α and complementary winning conditions $\neg\phi$ and ϕ, respectively). At least one player, thus, has a winning strategy for

[5]The penultimate equation follows from the μ-calculus equivalence $\nu Z.\Upsilon(Z) \equiv \neg\mu Z.\neg\Upsilon(\neg Z)$ and the fact that least pre-fixpoints are fixpoints and that greatest post-fixpoints are fixpoints for monotone functions.

complementary winning conditions. The other direction of Theorem 2 implies $\vDash [\alpha]\phi \rightarrow \neg\langle\alpha\rangle\neg\phi$, i.e. $\vDash \neg([\alpha]\phi \wedge \langle\alpha\rangle\neg\phi)$, whose validity means that there is no state from which Demon has a winning strategy to achieve ϕ and, simultaneously, Angel has a winning strategy to achieve $\neg\phi$. It cannot be that both players have a winning strategy for complementary conditions from the same state. That is, hybrid games are *consistent*, because at most one player has a winning strategy for complementary winning conditions. Along with modal congruence rules, which hold for dG$\mathcal{L}$, Theorem 2 makes dG$\mathcal{L}$ a classical (multi)modal logic [Che80], yet with modalities indexed by hybrid games.

Instead of giving a semantics to $[\cdot]$ in terms of the existence of a winning strategy for Demon, Theorem 2 could have been used as a definition of $[\cdot]$. That would have been easier, but would have obscured determinacy and the role of $[\cdot]$ as the winning strategy operator for Demon.

3.2 Hybrid Game Equivalences

As usual, the same hybrid game can have multiple different syntactical representations. Some equivalence transformations on hybrid games can be useful to transform hybrid games into a simpler form.

Definition 5 (Hybrid game equivalence)**.** Hybrid games α and β are *equivalent*, denoted $\alpha \equiv \beta$, if $\varsigma_\alpha(X) = \varsigma_\beta(X)$ for all X and all I.

By Theorem 2, α and β are equivalent iff $\delta_\alpha(X) = \delta_\beta(X)$ for all X and all I.

Remark 1. The equivalences

$$(\alpha \cup \beta)^d \equiv \alpha^d \cap \beta^d, \quad (\alpha;\beta)^d \equiv \alpha^d;\beta^d, \quad (\alpha^*)^d \equiv (\alpha^d)^\times, \quad \alpha^{dd} \equiv \alpha$$

on hybrid games can transform every hybrid game α into an equivalent hybrid game in which ^{d} only occurs right after atomic games or as part of the definition of the derived operators $\cap$ and $^\times$. Other equivalences include $(x' = \theta)^* \equiv x' = \theta$ and $(x' = \theta \,\&\, \psi)^* \equiv ?true \cup x' = \theta \,\&\, \psi$.

Quite unlike in hybrid systems and (poor test) differential dynamic logic [Pla08, Pla12a], every hybrid game containing a differential equation $x' = \theta \,\&\, \psi$ with evolution domain constraints ψ can be replaced equivalently by a hybrid game without evolution domain constrains (even using poor tests, i.e. each test $?\psi$ uses only first-order formulas ψ). Evolution domains are definable in hybrid games and can, thus, be removed equivalently.

Lemma 3 (Domain reduction)**.** *Evolution domains of differential equations are definable as hybrid games: For every hybrid game there is an equivalent hybrid game that has no evolution domain constraints, i.e. all continuous evolutions are of the form $x' = \theta$.*

Proof. For notational convenience, assume the (vectorial) differential equation $x' = \theta(x)$ to contain a clock $x_0' = 1$ and that t_0 and z are fresh variables. Then $x' = \theta(x) \,\&\, \psi(x)$ is equivalent to the hybrid game:

$$t_0 := x_0; x' = \theta(x); (z := x; z' = -\theta(z))^d; ?(z_0 \geq t_0 \rightarrow \psi(z)) \tag{6}$$

See Fig. 2 for an illustration. Suppose the current player is Angel. The idea behind game equiva-

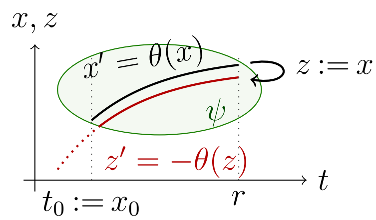

Figure 2: "There and back again game": Angel evolves x forwards in time along $x' = \theta(x)$, Demon checks evolution domain backwards in time along $z' = -\theta(z)$ on a copy z of the state vector x

lence (6) is that the fresh variable t_0 remembers the initial time x_0, and Angel then evolves forward along $x' = \theta(x)$ for any amount of time (Angel's choice). Afterwards, the opponent Demon copies the state x into a fresh variable (vector) z that he can evolve backwards along $(z' = -\theta(z))^d$ for any amount of time (Demon's choice). The original player Angel must then pass the challenge $?(z_0 \geq t_0 \rightarrow \psi(z))$, i.e. Angel loses immediately if Demon was able to evolve backwards and leave region $\psi(z)$ while satisfying $z_0 \geq t_0$, which checks that Demon did not evolve backward for longer than Angel evolved forward. Otherwise, when Angel passes the test, the extra variables t_0, z become irrelevant (they are fresh) and the game continues from the current state x that Angel chose in the first place (by selecting a duration for the evolution that Demon could not invalidate). □

Lemma 3 can eliminate all evolution domain constraints equivalently in hybrid games from now on. While evolution domain constraints are fundamental parts of standard hybrid systems [Hen96, HKPV95, ACHH92, Pla08], they turn out to be mere convenience notation for hybrid games. In that sense, hybrid games are more fundamental than hybrid systems, because they feature elementary operators.

3.3 Strategic Closure Ordinals

In order to examine whether the dG$\mathcal{L}$ semantics could be implemented directly to compute winning regions for dG$\mathcal{L}$ formulas by a reachability computation or backwards induction, this section investigates how many iterations the fixpoint for the semantics $\varsigma_{\alpha^*}(X)$ of repetition needs.

The semantics, $\varsigma_{\alpha^*}(X)$, of α^* is a least fixpoint and Knaster-Tarski's seminal fixpoint theorem entails that every least fixpoint of a monotone function on a complete lattice corresponds to some sufficiently large iteration. That is, there is some ordinal $\bar{\lambda}$ at which the $\bar{\lambda}$-th iteration, $\varsigma_\alpha^{\bar{\lambda}}(X)$, of $\varsigma_\alpha(\cdot)$ coincides with $\varsigma_{\alpha^*}(X)$, i.e. $\varsigma_{\alpha^*}(X) = \varsigma_\alpha^{\bar{\lambda}}(X)$; see Fig. 3. How big is $\bar{\lambda}$, i.e. how often does $\varsigma_\alpha(\cdot)$ need to iterate to obtain $\varsigma_{\alpha^*}(X)$?

Recall that ordinals extend natural numbers and support (non-commutative) addition, multiplication, and exponentiation, ordered as:

$$0 < 1 < 2 < \dots \omega < \omega+1 < \omega+2 < \dots \omega\cdot 2 < \omega\cdot 2+1 < \dots \omega\cdot 3 < \omega\cdot 3+1 < \dots$$
$$\omega^2 < \omega^2+1 < \dots \omega^2+\omega < \omega^2+\omega+1 < \dots \omega^\omega < \dots \omega^{\omega^\omega} < \dots \omega_1^{\mathrm{CK}} < \dots \omega_1 < \dots$$

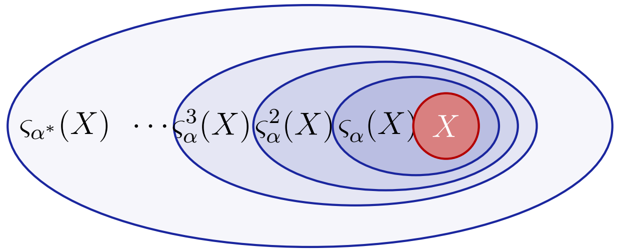

Figure 3: Least fixpoint $\varsigma_{\alpha^*}(X)$ corresponds to some higher iterate $\varsigma_\alpha^{\bar\lambda}(X)$ of $\varsigma_\alpha(\cdot)$ from winning condition X.

The first infinite ordinal is ω, the Church-Kleene ordinal ω_1^{CK}, i.e. the first nonrecursive ordinal, and ω_1 the first uncountable ordinal. Recall that every ordinal κ is either a successor ordinal, i.e. the smallest ordinal $\kappa = \iota + 1$ greater than some ordinal ι, or a limit ordinal, i.e. the supremum of all smaller ordinals. Depending on the context, 0 is considered a limit ordinal or separate.

3.3.1 Iterations and Fixpoints

For each hybrid game α, the semantics $\varsigma_\alpha(\cdot)$ is a monotone operator on the complete powerset lattice (Lemma 1). The κth iterate, $\varsigma_\alpha^\kappa(\cdot)$, of $\varsigma_\alpha(\cdot)$ is defined in line with a minor variation of Kozen's formulation of the Knaster-Tarski theorem [HKT00, Theorem 1.12], obtained by considering the sublattice with x at the bottom.

Let $\tau : L \to L$ any monotone operator on a partial order L, then defining $\tau^\lambda(x) \overset{\text{def}}{=} x \cup \bigcup_{\kappa<\lambda} \tau(\tau^\kappa(x))$ for all ordinals λ is equivalent to:

$$\begin{aligned} \tau^0(x) &\overset{\text{def}}{=} x \\ \tau^{\kappa+1}(x) &\overset{\text{def}}{=} x \cup \tau(\tau^\kappa(x)) \\ \tau^\lambda(x) &\overset{\text{def}}{=} \bigcup_{\kappa<\lambda} \tau^\kappa(x) \qquad \lambda \neq 0 \text{ a limit ordinal} \end{aligned}$$

Yet, $\bigcup$ and, thus, $\tau^\lambda(x)$ are only guaranteed to exist if L is a complete partial order.

Theorem 4 (Knaster-Tarski [HKT00, Theorem 1.12])**.** *For every complete lattice L, there is an ordinal $\bar\lambda$ of at most the cardinality of L such that, for each monotone $\tau : L \to L$, i.e. $\tau(x) \subseteq \tau(y)$ for all $x \subseteq y$, the fixpoints of τ in L are a complete lattice and for all $x \in L$ and all ordinals κ:*

$$\tau^\dagger(x) \overset{\text{def}}{=} \bigcap\{z \in L : x \subseteq z, \tau(z) \subseteq z\} = \tau^{\bar\lambda}(x) = \tau^{\bar\lambda+\kappa}(x)$$

The least ordinal $\bar\lambda$ with the property in Theorem 4 is called *closure ordinal* of τ.

The operator $\tau^\kappa(\cdot)$ enjoys useful properties. By its extensive / inflationary definition, $\tau^\kappa(x)$ is not just monotone in x but also monotone and homomorphic in κ. Since $\tau^0(x) = x$, this works for all ordinals.

Lemma 5. *τ is inductive, i.e. $\tau^\kappa(x) \subseteq \tau^\lambda(x)$ for all $\kappa \leq \lambda$ and homomorphic in κ, i.e.*

$$\tau^{\kappa+\lambda}(x) = \tau^\lambda(\tau^\kappa(x)) \text{ for all } \kappa, \lambda$$

Proof. Inductiveness, i.e. $\tau^\kappa(x) \subseteq \tau^\lambda(x)$ for $\kappa \leq \lambda$, which is monotonicity in κ, holds by definition [HKT00, Lemma 1.11]. Homomorphy in κ, i.e. $\tau^{\kappa+\lambda}(x) = \tau^\lambda(\tau^\kappa(x))$ can be proved by induction on λ, which is either 0, a successor ordinal (second line) or a limit ordinal $\neq 0$ (third line):

$$\begin{aligned}
\tau^{\kappa+0}(x) &= \tau^\kappa(x) = \tau^0(\tau^\kappa(x)) \\
\tau^{\kappa+(\lambda+1)}(x) &= x \cup \tau(\tau^{\kappa+\lambda}(x)) = x \cup \tau(\tau^\lambda(\tau^\kappa(x))) = \tau^\kappa(x) \cup \tau(\tau^\lambda(\tau^\kappa(x))) = \tau^{\lambda+1}(\tau^\kappa(x)) \\
\tau^{\kappa+\lambda}(x) &= \bigcup_{\iota<\kappa+\lambda} \tau^\iota(x) = \bigcup_{\iota<\kappa} \tau^\iota(x) \cup \bigcup_{\iota<\lambda} \tau^{\kappa+\iota}(x) \\
&= \bigcup_{\iota<\lambda} \tau^{\kappa+\iota}(x) = \bigcup_{\iota<\lambda} \tau^\iota(\tau^\kappa(x)) = \tau^\lambda(\tau^\kappa(x))
\end{aligned}$$

□

By Theorem 4, there is an ordinal $\bar{\lambda}$ of cardinality at most that of $\mathbb{R}$ such that $\varsigma_{\alpha^*}(X) = \varsigma_\alpha^{\bar{\lambda}}(X)$ for all α and all X, because the powerset lattice is complete and $\varsigma_\alpha(\cdot)$ monotone by Lemma 1. This iterative construction $\tau^{\bar{\lambda}}(X)$ corresponds to backward induction in classical game theory [vNM55, Aum95], yet it terminates at ordinal $\bar{\lambda}$ which is not necessarily finite.

3.3.2 Scott-Continuity

Repetitions in classical hybrid systems only repeat any finite number of times [Pla12a]. If the semantics of dG$\mathcal{L}$ were Scott-continuous, this would be the case for dG$\mathcal{L}$ as well, because the closure ordinal of Scott-continuous operators on a complete partial order is $\leq\omega$ by Kleene's fixpoint theorem. Dual-free α are indeed Scott-continuous, in particular, the closure ordinal for hybrid systems is ω.

Lemma 6 (Scott-continuity of ^{d}-free dG$\mathcal{L}$)**.** *The dG$\mathcal{L}$ semantics of ^{d}-free α is Scott-continuous, i.e. $\varsigma_\alpha(\bigcup_{n\in J} X_n) = \bigcup_{n\in J} \varsigma_\alpha(X_n)$ for all families $\{X_n\}_{n\in J}$ with any index set J.*

Proof. By Lemma 1, $\bigcup_{n\in J} \varsigma_\alpha(X_n) \subseteq \varsigma_\alpha(\bigcup_{n\in J} X_n)$. The converse inclusion can be shown by a simple induction on the structure of α: $\varsigma_\alpha(\bigcup_{n\in J} X_n) \subseteq \bigcup_{n\in J} \varsigma_\alpha(X_n)$. IH is short for induction hypothesis.

1. $\varsigma_{x:=\theta}(\bigcup_{n\in J} X_n) = \{s \in \mathcal{S} : s_x^{[\![\theta]\!]_s} \in \bigcup_{n\in J} X_n\} \subseteq \bigcup_{n\in J}\{s \in \mathcal{S} : s_x^{[\![\theta]\!]_s} \in X_n\} = \bigcup_{n\in J} \varsigma_{x:=\theta}(X_n)$, since $s_x^{[\![\theta]\!]_s} \in \bigcup_{n\in J} X_n$ implies $s_x^{[\![\theta]\!]_s} \in X_n$ for some n.

2. $\varsigma_{x'=\theta\,\&\,\psi}(\bigcup_{n\in J} X_n) = \{\varphi(0) \in \mathcal{S} : \frac{\mathrm{d}\,\varphi(t)(x)}{\mathrm{d}t}(\zeta) = [\![\theta]\!]_{\varphi(\zeta)}$ and $\varphi(\zeta) \in [\![\psi]\!]^I$ for all $\zeta \leq r$ for some (differentiable) $\varphi : [0, r] \to \mathcal{S}$ such that $\varphi(r) \in \bigcup_{n\in J} X_n\} \subseteq \bigcup_{n\in J} \varsigma_{x'=\theta\,\&\,\psi}(X_n) = \{\varphi(0) \in \mathcal{S} : \dots \varphi(r) \in X_n\}$, because $\varphi(r) \in \bigcup_{n\in J} X_n$ implies $\varphi(r) \in X_n$ for some n.

3. $\varsigma_{?\psi}(\bigcup_{n\in J} X_n) = [\![\psi]\!]^I \cap \bigcup_{n\in J} X_n = \bigcup_{n\in J}([\![\psi]\!]^I \cap X_n) = \bigcup_{n\in J} \varsigma_{?\psi}(X_n)$

4. $\varsigma_{\alpha\cup\beta}(\bigcup_{n\in J} X_n) = \varsigma_\alpha(\bigcup_{n\in J} X_n) \cup \varsigma_\beta(\bigcup_{n\in J} X_n) \overset{\text{IH}}{=} (\bigcup_{n\in J} \varsigma_\alpha(X_n)) \cup (\bigcup_{n\in J} \varsigma_\beta(X_n)) = \bigcup_{n\in J}(\varsigma_\alpha(X_n) \cup \varsigma_\beta(X_n)) = \bigcup_{n\in J} \varsigma_{\alpha\cup\beta}(X_n)$

5. $\varsigma_{\alpha;\beta}(\bigcup_{n\in J} X_n) = \varsigma_\alpha(\varsigma_\beta(\bigcup_{n\in J} X_n)) \overset{\text{IH}}{=} \varsigma_\alpha(\bigcup_{n\in J} \varsigma_\beta(X_n)) \overset{\text{IH}}{=} \bigcup_{n\in J} \varsigma_\alpha(\varsigma_\beta(X_n)) = \bigcup_{n\in J} \varsigma_{\alpha;\beta}(X_n)$

6. $\varsigma_{\alpha^*}(\bigcup_{n\in J} X_n) = (\bigcup_{n\in J} X_n) \cup \varsigma_\alpha(\varsigma_{\alpha^*}(\bigcup_{n\in J} X_n))$ is the least fixpoint. Prove that $\bigcup_{n\in J} \varsigma_{\alpha^*}(X_n)$ is a fixpoint, which implies $\varsigma_{\alpha^*}(\bigcup_{n\in J} X_n) \subseteq \bigcup_{n\in J} \varsigma_{\alpha^*}(X_n)$. Indeed,
$(\bigcup_{n\in J} X_n) \cup \varsigma_\alpha(\bigcup_{n\in J} \varsigma_{\alpha^*}(X_n)) \overset{\text{IH}}{=} (\bigcup_{n\in J} X_n) \cup \bigcup_{n\in J} \varsigma_\alpha(\varsigma_{\alpha^*}(X_n)) = \bigcup_{n\in J}(X_n \cup \varsigma_\alpha(\varsigma_{\alpha^*}(X_n)) = \bigcup_{n\in J} \varsigma_{\alpha^*}(X_n)$. The last equation uses that $\varsigma_{\alpha^*}(X_n)$ is a fixpoint. □

But $\varsigma_\alpha(\cdot)$ is not generally Scott-continuous, so $\bar{\lambda}$ might potentially be greater than ω for hybrid games. Games with both ^{d} and * do not generally have a Scott-continuous semantics nor an ω-chain continuous semantics, i.e. they are not even continuous for a monotonically increasing chain $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ with ω as index set:

$$\mathbb{R} = \varsigma_{y:=y+1^\times}(\bigcup_{n<\omega}(-\infty, n]) \not\subseteq \bigcup_{n<\omega} \varsigma_{y:=y+1^\times}((-\infty, n]) = \emptyset$$

$$\text{hence } \vDash \langle y := y+1^\times\rangle \exists n : \mathbb{N}\, y \le n \text{ but } \nvDash \exists n : \mathbb{N}\, \langle y := y+1^\times\rangle y \le n$$

This example shows that, even though Angel wins this game, there is no upper bound $< \omega$ on the number of iterations it takes her to win, because Demon could repeat $y := y + 1^\times$ arbitrarily often. This phenomenon is directly related to a failure of the Barcan axiom (Section 4.5). The quantifier $\exists n : \mathbb{N}$ over natural numbers is not essential here [Pla08] but mere convenience to make both lines above match directly.

If τ is countably-continuous, i.e. continuous for families with countable index sets, on a complete partial order, then its closure ordinal is $\bar{\lambda} \le \omega_1$. But this is not the case for $\varsigma_\alpha(\cdot)$ either, by the above counterexample with countable index set ω.

A function τ on sets is κ-based, for an ordinal κ, if for all X, $x \in \tau(X)$ implies $x \in \tau(Y)$ for some $Y \subseteq X$ of cardinality $<\kappa$. If τ is κ-based, then its closure ordinal is $\le\kappa$ [Acz77, Proposition 1.3.4]. The semantics $\varsigma_\alpha(\cdot)$ is not ω_1-based, however, because of Lemma 1 and removing just one state from the winning condition may lose states in the winning region:

$$[0, \infty) = \varsigma_{x'=1^d}([0, \infty))$$

$$\text{but } 0 \notin \varsigma_{x'=1^d}([0, \infty) \setminus \{a\}) = (a, \infty) \text{ for all } a > 0$$

Consequently, it is the combination of ^{d}, *, and differential equations that makes hybrid games challenging.

3.3.3 Transfinite Closure Ordinals

When will the iteration for the fixpoints in the winning region definitions stop? Hybrid games may have higher closure ordinals, because ω many repetitions of the operator (and even $<\omega_1^{\text{CK}}$ many) may not be enough to compute winning regions. In other words, $\varsigma_{\alpha^*}(X)$ will coincide with iterations $\varsigma_\alpha^\kappa(X)$ as illustrated in Fig. 3, but this may need more than ω many iterations to terminate.

Theorem 7 (Closure ordinals)**.** *The semantics of* dG$\mathcal{L}$ *has a closure ordinal* $\geq \omega_1^{CK}$*, i.e. for all* $\lambda < \omega_1^{CK}$*, there are* α *and* X *such that* $\varsigma_{\alpha^*}(X) \neq \varsigma_\alpha^\lambda(X)$*.*

Proof. The proof first shows the easier case that the closure ordinal is $\geq \omega \cdot 2$. A proof for $\geq \omega^\omega$ is shown in Appendix E. The specific dG$\mathcal{L}$ formulas considered for these increasing lower bounds show that the closure ordinal is not a simple function of the syntactic structure, because minor syntactic variations lead to vastly different closure ordinals.

To see that the closure ordinal is $> \omega$ even with just one variable, a single loop and dual, consider the semantics of the following dG$\mathcal{L}$ formula, i.e. the set of states in which it is true:

$$\langle(\underbrace{x := x+1; x' = 1^d}_{\alpha} \cup \underbrace{x := x-1}_{\beta})^*\rangle\,(0 \leq x < 1) \tag{7}$$

The winning regions for this dG$\mathcal{L}$ formula stabilize after $\omega \cdot 2$ iterations, because ω many iterations are necessary to show that *any* positive real can be reduced to $[0,1)$ by choosing β sufficiently often, whereas another ω many iterations are needed to show that choice α, which makes progress ≥ 1 but possibly more under Demon's control, can turn x into a positive real. It is easy to see that $\varsigma_{\alpha\cup\beta}^\omega([0,1)) = \bigcup_{n<\omega} \varsigma_{\alpha\cup\beta}^n([0,1)) = [0,\infty)$, because $\varsigma_{\alpha\cup\beta}^n([0,1)) = [0,n)$ holds for all $n \in \mathbb{N}$ by a simple inductive argument:

$$\begin{aligned}
&\varsigma_{\alpha\cup\beta}^1([0,1)) = [0,1)\\
&\varsigma_{\alpha\cup\beta}^{n+1}([0,1)) = [0,1) \cup \varsigma_{\alpha\cup\beta}(\varsigma_{\alpha\cup\beta}^n([0,1))) = [0,1) \cup \varsigma_{\alpha\cup\beta}([0,n))\\
&\quad = [0,1) \cup \varsigma_\alpha([0,n)) \cup \varsigma_\beta([0,n)) = [0,1) \cup \emptyset \cup [1,n+1)
\end{aligned}$$

But the iteration for the winning region does not stop at ω, as $\varsigma_{\alpha\cup\beta}^{\omega+n}([0,1)) = [-n,\infty)$ holds for all $n \in \mathbb{N}$ by another simple inductive argument:

$$\begin{aligned}
\varsigma_{\alpha\cup\beta}^{\omega+n+1}([0,1)) &= [0,1) \cup \varsigma_{\alpha\cup\beta}(\varsigma_{\alpha\cup\beta}^{\omega+n}([0,1)))\\
&= [0,1) \cup \varsigma_{\alpha\cup\beta}([-n,\infty))\\
&= [0,1) \cup \varsigma_\alpha([-n,\infty)) \cup \varsigma_\beta([-n,\infty))\\
&= [-n-1,\infty) \cup [-n,\infty)
\end{aligned}$$

Thus, $\varsigma_{\alpha\cup\beta}^{\omega\cdot 2}([0,1)) = \varsigma_{\alpha\cup\beta}^{\omega+\omega}([0,1)) = \bigcup_{n<\omega} \varsigma_{\alpha\cup\beta}^{\omega+n}([0,1)) = \mathbb{R} = \varsigma_{\alpha\cup\beta}(\mathbb{R})$. In this case, the closure ordinal is $\omega \cdot 2 > \omega$, since $\varsigma_{(\alpha\cup\beta)^*}([0,1)) = \mathbb{R} \neq \varsigma_{\alpha\cup\beta}^{\omega+n}([0,1))$ for all $n \in \mathbb{N}$.

To show that the closure ordinal is $\geq \omega_1^{\text{CK}}$, consider any ordinal $\lambda < \omega_1^{\text{CK}}$, i.e. any recursive ordinal. Let $\prec\, \subseteq M \times M$ be a corresponding recursive well-order of order type λ on a corresponding set $M \subseteq \mathbb{R}$.[6] That is, let $f_\prec$ a recursive function such that the relation $x \prec y$ given by $f_\prec(x,y) = 0$ defines a well-order on the set $M \overset{\text{def}}{=} \{x \in \mathbb{R} : f_\prec(x,y) = 0 \text{ or} f_\prec(y,x) = 0 \text{ for some } y \in \mathbb{R}\}$. Without loss of generality, assume that $0 \in M$ is the least element of M with respect to $\prec$. Since $\prec$ is recursive, denote by $?f_\prec(x,y) = 0$ the program that does not change the value of variables x, y

[6] A *well-order* is a linear order $\prec$ on M in which every non-empty subset has a least element. Two sets M, N have equal *order type* iff they have an order-isomorphism $\varphi : M \to N$, i.e. a monotone bijection with monotone inverse. More background can be found in the standard literature [Rog87].

and that implements the recursive function that terminates if $x \in M$ and either $x \prec y$ or $y \notin M$ and that otherwise fails (like $?(0=1)$ would). Consider the dG$\mathcal{L}$ formula

$$\langle\big(\underbrace{y := x; (x' = 1; x' = -1; ?f_\prec(x,y) = 0)^d}_{\alpha}\big)^*\rangle\, x = 0 \tag{8}$$

By definition of $?f_\prec(x,y) = 0$, formula (8) is valid, because x is in M after each successful run of $?f_\prec(x,y) = 0$, and $\prec$ is a well-order on M with least element 0. By construction, $\varsigma_\alpha(X) = \{a \in \mathbb{R} : b \in X \text{ for all } b \text{ with } f_\prec(b,a) = 0\}$ for $X \subseteq \mathbb{R}$. Since $\prec$ has order type λ, $\varsigma_\alpha^\kappa(\{0\}) \neq \varsigma_\alpha^\lambda(\{0\}) = M$ for all $\kappa < \lambda$, otherwise the $\varsigma_\alpha^\iota(\{0\})$ would induce a monotone injection (even order-isomorphism) from M to $\kappa < \lambda$, which is a contradiction. Indeed, $\varphi : M \to \kappa; x \mapsto \inf\{\iota : x \in \varsigma_\alpha^\iota(\{0\})\}$ would otherwise be a monotone injection as $x \prec y$ in M implies $\varphi(x) < \varphi(y)$, because $\varphi(x) \geq \varphi(y)$ implies $y \in \varsigma_\alpha(X)$ for a set $X = \varsigma_\alpha^{\varphi(y)-1}(\{0\})$ that does not contain x, contradicting $x \prec y$. Note that $\varphi(y)$ is a successor ordinal and hence $\varphi(y) - 1$ defined, since φ maps into successor ordinals and 0 by the definition of φ. Consequently, $\varsigma_\alpha^\lambda(\{0\}) = M \neq \varsigma_\alpha^{\lambda+1}(\{0\}) = \varsigma_\alpha(M) = \mathbb{R} = \varsigma_{\alpha^*}(\{0\})$, where $M \neq \mathbb{R}$ because λ is recursive hence countable and $\prec$ a linear order on M. Thus, the closure ordinal for formula (8) is $\lambda{+}1 > \lambda$. Hence, for any recursive ordinal λ, there is a hybrid game with a bigger closure ordinal. So, the closure ordinal is $\geq \omega_1^{\mathrm{CK}}$. □

By Theorem 7, the closure ordinal for dG$\mathcal{L}$ is between ω_1^{CK} and ordinals of the cardinality of the reals. In fact, the same proof works for any other well-ordering that is definable in hybrid games, not just those that are definable by classical recursive functions. The proof does not permit arbitrary well-orderings of the real numbers, however, because those may not be definable by hybrid games. Consequently the closure ordinal for dG$\mathcal{L}$ is at least ω_1^{HG}, which we define as the first ordinal λ that does not have a well-ordering of order type λ that is definable by hybrid games. This ordinal satisfies $\omega_1^{\mathrm{CK}} \leq \omega_1^{\mathrm{HG}}$ and is at most of the cardinality of the reals. These thoughts yield a more precise grasp on ω_1^{HG} in Section 5.

The fact that hybrid games require highly transfinite closure ordinals has a number of consequences. It makes reachability computations and backwards induction difficult, because they only terminate after more than ω-infinitely many steps. It requires higher bounds on the number of repetitions played in hybrid games. It causes classical arguments for relative completeness to fail (Section 4.3). And it causes semantical differences that are only visible in hybrid games, not in hybrid systems. For example, the dG$\mathcal{L}$ semantics is more general than defining $\varsigma_{\alpha^*}(X)$ to be truncated to ω-repetition $\varsigma_\alpha^\omega(X) = \bigcup_{n<\omega} \varsigma_\alpha^n(X)$, which misses out on the existence of perfectly natural winning strategies. The semantics of dG$\mathcal{L}$ is also different than advance notice semantics. For reference, both comparisons are elaborated in Appendix D.

4 Axiomatization

Section 2 has defined dG$\mathcal{L}$ so that every game play has exactly one winner. Section 3 has shown that hybrid games are determined, i.e. from every state, exactly one of the players has a winning strategy for complementary winning conditions, but how can one find out which of the players that is? In principle, one could follow the iterated winning region construction from Section 3.3

to find out, which corresponds to reachability computation or backwards induction, but that will not generally terminate in finite time, because the closure ordinal is highly transfinite. Every dGL sentence without free variables or predicate symbols is either true or false, because dGL is a classical logic. But the semantics of dGL formulas is ineffective, because computing the semantics, like classical model checking or game solving would, requires transfinite computations. This calls for other ways of proving the validity of dGL formulas.

Simple dGL formulas can be checked by a tableau procedure that expands all choices and detects loops for termination as in the game tree examples (Fig. 1 and Appendix). This principle does not extend to more general hybrid games with differential equations, inherently infinite state spaces [Hen96], and which need higher ordinals of iteration for computing winning regions by Theorem 7.

4.1 Proof Calculus

A Hilbert-type proof calculus for proving validity of dGL formulas is presented in Fig. 4.

$[\cdot]$ $[\alpha]\phi \leftrightarrow \neg\langle\alpha\rangle\neg\phi$

$\langle:=\rangle$ $\langle x := \theta\rangle\phi(x) \leftrightarrow \phi(\theta)$

$\langle'\rangle$ $\langle x' = \theta\rangle\phi \leftrightarrow \exists t{\geq}0\, \langle x := y(t)\rangle\phi \qquad (y'(t) = \theta)$

$\langle?\rangle$ $\langle ?\psi\rangle\phi \leftrightarrow (\psi \land \phi)$

$\langle\cup\rangle$ $\langle\alpha \cup \beta\rangle\phi \leftrightarrow \langle\alpha\rangle\phi \lor \langle\beta\rangle\phi$

$\langle;\rangle$ $\langle\alpha;\beta\rangle\phi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\phi$

$\langle^*\rangle$ $\phi \lor \langle\alpha\rangle\langle\alpha^*\rangle\phi \rightarrow \langle\alpha^*\rangle\phi$

$\langle^d\rangle$ $\langle\alpha^d\rangle\phi \leftrightarrow \neg\langle\alpha\rangle\neg\phi$

M $\dfrac{\phi \rightarrow \psi}{\langle\alpha\rangle\phi \rightarrow \langle\alpha\rangle\psi}$

FP $\dfrac{\phi \lor \langle\alpha\rangle\psi \rightarrow \psi}{\langle\alpha^*\rangle\phi \rightarrow \psi}$

Figure 4: Differential game logic axiomatization

The logic dGL simultaneously generalizes logics of hybrid systems and logics of discrete games and so does its proof calculus. The proof calculus of dGL shares axioms with differential dynamic logic [Pla12a] and discrete game logic [PP03]. It is based on the first-order Hilbert calculus (modus ponens, uniform substitution, and Bernays' $\forall$-generalization) with all instances of valid formulas of first-order logic as axioms, including first-order real arithmetic [Tar51]. Write $\vdash \phi$ iff dGL formula ϕ can be *proved* with the dGL proof rules from dGL axioms (Fig. 4). That is, a dGL formula is inductively defined to be *provable* in the dGL calculus if it is an instance of a dGL axiom or if it is the conclusion (below the rule bar) of an instance of one of the dGL proof rules M, FP, modus ponens, uniform substitution, or $\forall$-generalization, whose premises (above the rule bar) are all provable.

The determinacy axiom $[\cdot]$ describes the duality of winning strategies for complementary winning conditions of Angel and Demon, i.e. that Demon has a winning strategy to achieve ϕ in hybrid game α if and only if Angel does not have a counter strategy, i.e. winning strategy to achieve $\neg\phi$ in the same game α. Axiom $\langle:=\rangle$ is Hoare's assignment rule. Formula $\phi(\theta)$ is obtained from $\phi(x)$ by *substituting* θ for x at all occurrences of x, provided x does not occur in the scope of a quantifier or modality binding x or a variable of θ. A modality containing $x :=$ or x' outside the scope of tests $?\psi$ or evolution domain constraints *binds* x, because it may change the value of x. In the differential equation axiom $\langle'\rangle$, $y(\cdot)$ is the unique [Wal98, Theorem 10.VI] solution of the symbolic initial value problem $y'(t) = \theta, y(0) = x$. The duration t how long to follow solution y is for Angel to decide, hence existentially quantified. It goes without saying that variables like t are fresh in Fig. 4.

Axioms $\langle?\rangle$, $\langle\cup\rangle$, and $\langle;\rangle$ are as in dynamic logic [Pra76] and differential dynamic logic [Pla12a] except that their meaning is quite different, because they refer to winning strategies of hybrid games instead of reachability relations of systems. The challenge axiom $\langle?\rangle$ expresses that Angel has a winning strategy to achieve ϕ in the test game $?\psi$ exactly from those positions that are already in ϕ (because $?\psi$ does not change the state) and that satisfy ψ for otherwise she would fail the test and lose the game immediately. The axiom of choice $\langle\cup\rangle$ expresses that Angel has a winning strategy in a game of choice $\alpha \cup \beta$ to achieve ϕ iff she has a winning strategy in either hybrid game α or in β, because she can choose which one to play. The sequential game axiom $\langle;\rangle$ expresses that Angel has a winning strategy in a sequential game $\alpha; \beta$ to achieve ϕ iff she has a winning strategy in game α to achieve $\langle\beta\rangle\phi$, i.e. to get to a position from which she has a winning strategy in game β to achieve ϕ. The iteration axiom $\langle^*\rangle$ characterizes $\langle\alpha^*\rangle\phi$ as a pre-fixpoint. It expresses that, if the game is already in a state satisfying ϕ or if Angel has a winning strategy for game α to achieve $\langle\alpha^*\rangle\phi$, i.e. to get to a position from which she has a winning strategy for game α^* to achieve ϕ, then, either way, Angel has a winning strategy to achieve ϕ in game α^*. The converse of $\langle^*\rangle$ can be derived[7] and is also denoted by $\langle^*\rangle$. The dual axiom $\langle^d\rangle$ characterizes dual games. It says that Angel has a winning strategy to achieve ϕ in dual game α^d iff Angel does not have a winning strategy to achieve $\neg\phi$ in game α. Combining dual game axiom $\langle^d\rangle$ with the determinacy axiom $[\cdot]$ yields$\langle\alpha^d\rangle\phi \leftrightarrow [\alpha]\phi$, i.e. that Angel has a winning strategy to achieve ϕ in α^d iff Demon has a winning strategy to achieve ϕ in α. Similar reasoning derives $[\alpha^d]\phi \leftrightarrow \langle\alpha\rangle\phi$.

Monotonicity rule M is the generalization rule of monotonic modal logic **C** [Che80]. It expresses that, if the implication $\phi \to \psi$ is valid, then, from wherever Angel has a winning strategy in any hybrid game α to achieve ϕ, she also has a winning strategy to achieve ψ, because ψ holds wherever ϕ does. So rule M expresses that easier objectives are easier to win. Fixpoint rule FP characterizes $\langle\alpha^*\rangle\phi$ as a *least* pre-fixpoint. It says that, if ψ is any other formula that is a pre-fixpoint, i.e. that holds in all states that satisfy ϕ or from which Angel has a winning strategy in game α to achieve that condition ψ, then ψ also holds whereever $\langle\alpha^*\rangle\phi$ does, i.e. in all states from

[7] $\phi \vee \langle\alpha\rangle\langle\alpha^*\rangle\phi \to \langle\alpha^*\rangle\phi$ derives by $\langle^*\rangle$. Thus, $\langle\alpha\rangle(\phi \vee \langle\alpha\rangle\langle\alpha^*\rangle\phi) \to \langle\alpha\rangle\langle\alpha^*\rangle\phi$ by M. Hence, $\phi \vee \langle\alpha\rangle(\phi \vee \langle\alpha\rangle\langle\alpha^*\rangle\phi) \to \phi \vee \langle\alpha\rangle\langle\alpha^*\rangle\phi$ by propositional congruence. Consequently, $\langle\alpha^*\rangle\phi \to \phi \vee \langle\alpha\rangle\langle\alpha^*\rangle\phi$ by FP.

which Angel has a winning strategy in game α^* to achieve ϕ.

As usual, all substitutions in Fig. 4 are required to be *admissible* to avoid capture of variables, i.e. they require all variables x that are being replaced or that occur in their replacements to not occur in the scope of a quantifier or modality binding x. Recall that the uniform substitution rule from first-order logic substitutes *all* occurrences of predicate $p(\cdot)$ by a dG$\mathcal{L}$ formula $\psi(\cdot)$, i.e. it replaces all occurrences of $p(\theta)$ for any vectorial term θ by the corresponding $\psi(\theta)$ simultaneously:

$$\text{(US)}\quad \frac{\phi}{\phi_{p(\cdot)}^{\psi(\cdot)}}$$

In particular, the uniform substitution rule requires all relevant substitutions of $\psi(\theta)$ for $p(\theta)$ to be admissible and requires that no $p(\theta)$ occurs in the scope of a quantifier or modality binding a variable of $\psi(\theta)$ other than those in θ; see [Chu56, §35,40]. If admissible, the formula $\psi(\theta)$ can use variables other than those in θ, hence, the case where p is a predicate symbol without arguments enables US to generate all formula instances from the dG$\mathcal{L}$ axioms. Rule US turns axioms into axiom schemes [Chu56, §35,40].

Despite their fundamentally different semantics (reachability relations on states of hybrid system runs versus existence of winning strategies into sets of states of interactive hybrid game play) and different dynamical effects (mixed discrete, continuous, and adversarial dynamics), the axiomatization of dG$\mathcal{L}$ ends up surprisingly close to that of the logic d$\mathcal{L}$ for hybrid systems [Pla12a]. The primary difference of the axiomatization of dG$\mathcal{L}$ compared to that of d$\mathcal{L}$ is the addition of axiom $\langle^d\rangle$ for dual games, the absence of axiom K, absence of Gödel's necessitation rule (dG$\mathcal{L}$ only has the monotonic modal rule M), absence of the Barcan formula (the converse Barcan formula is still derivable[8]), and absence of the hybrid version of Harel's convergence rule [HMP77]. Due to the absence of K, the induction axiom and the convergence axiom are absent in dG$\mathcal{L}$, while corresponding proof rules are still valid; see Section 4.5 for details. The induction rule (ind) is derivable from FP.

A proof of a classical result about the interderivability of FP with the induction rule ind is included for the sake of completeness.

Lemma 8 (Invariance)**.** *Rule FP and the induction rule (ind) of dynamic logic are interderivable in the* dG$\mathcal{L}$ *calculus:*

$$\text{(ind)}\quad \frac{\psi \to [\alpha]\psi}{\psi \to [\alpha^*]\psi}$$

Proof. Rule ind derives from FP: First derive the following minor variant

$$(\text{ind}_R)\quad \frac{\psi \to [\alpha]\psi \quad \psi \to \phi}{\psi \to [\alpha^*]\phi}$$

From $\psi \to [\alpha]\psi$ and $\psi \to \phi$ propositionally derive $\psi \to \phi \land [\alpha]\psi$, from which contraposition and propositional logic yield $\neg\phi \lor \neg[\alpha]\psi \to \neg\psi$. With $[\cdot]$, this gives $\neg\phi \lor \langle\alpha\rangle\neg\psi \to \neg\psi$. Now FP

[8] From $\phi \to \exists x\, \phi$, derive $\langle\alpha\rangle\phi \to \langle\alpha\rangle\exists x\, \phi$ by M, from which first-order logic derives $\forall x\, (\langle\alpha\rangle\phi \to \langle\alpha\rangle\exists x\, \phi)$ and then derives $\exists x\, \langle\alpha\rangle\phi \to \langle\alpha\rangle\exists x\, \phi$, since converse Barcan assumes that x is not free in the succedent.

derives $\langle\alpha^*\rangle\neg\phi \to \neg\psi$, which, by $[\cdot]$, is $\neg[\alpha^*]\phi \to \neg\psi$, which gives $\psi \to [\alpha^*]\phi$ by contraposition. The classical []-induction rule ind follows by $\phi \overset{\text{def}}{\equiv} \psi$. From ind, the variant ind_R is derivable again by M on $\psi \to \phi$.

Rule FP derives from ind: From $\phi \lor \langle\alpha\rangle\psi \to \psi$, propositionally derive $\phi \to \psi$ and $\langle\alpha\rangle\psi \to \psi$. By M, the former gives $\langle\alpha^*\rangle\phi \to \langle\alpha^*\rangle\psi$. By contraposition, the latter derives $\neg\psi \to \neg\langle\alpha\rangle\psi$, which gives $\neg\psi \to [\alpha]\neg\psi$ by $[\cdot]$. Now ind derives $\neg\psi \to [\alpha^*]\neg\psi$. By contraposition $\neg[\alpha^*]\neg\psi \to \psi$, which, by $[\cdot]$, is $\langle\alpha^*\rangle\psi \to \psi$. Thus, $\langle\alpha^*\rangle\phi \to \psi$ by the formula derived above. □

In particular, the dG$\mathcal{L}$ calculus could have been equipped with rule ind instead of FP.

Example 4. The dual filibuster game formula (4) from Section 3.1 proves easily by going back and forth between players:

$$
\begin{array}{rl}
 & * \\ \hline
\mathbb{R} & x = 0 \to 0 = 0 \lor 1 = 0 \\ \hline
\langle:=\rangle & x = 0 \to \langle x := 0\rangle x = 0 \lor \langle x := 1\rangle x = 0 \\ \hline
\langle\cup\rangle & x = 0 \to \langle x := 0 \cup x := 1\rangle x = 0 \\ \hline
\langle^d\rangle & x = 0 \to \neg\langle x := 0 \cap x := 1\rangle \neg x = 0 \\ \hline
[\cdot] & x = 0 \to [x := 0 \cap x := 1] x = 0 \\ \hline
\text{ind} & x = 0 \to [(x := 0 \cap x := 1)^*] x = 0 \\ \hline
\langle^d\rangle & x = 0 \to \langle (x := 0 \cup x := 1)^\times \rangle x = 0
\end{array}
$$

A proof of a $\langle\alpha^*\rangle$ property will be considered later, because the proof technique for those properties comes from the completeness proof. More challenging hybrid games are provable in dG$\mathcal{L}$; see [QP12] for a proof of a stress-test of a highly interactive, 11-dimensional, nonlinear hybrid game in robotic factory automation.

4.2 Soundness

Soundness studies whether all provable formulas are valid. The soundness proof uses that the following congruence rule derives from two uses of monotonicity rule M:

$$
\text{(RE)} \quad \frac{\phi \leftrightarrow \psi}{\langle\alpha\rangle\phi \leftrightarrow \langle\alpha\rangle\psi}
$$

Theorem 9 (Soundness)**.** *The* dG$\mathcal{L}$ *proof calculus in Fig. 4 is sound, i.e. all provable formulas are valid.*

Proof. In order to prove soundness of an implication axiom $\phi \to \psi$, fix any interpretation I with any set of states $\mathcal{S}$ and show $[\![\phi]\!]^I \subseteq [\![\psi]\!]^I$. To prove soundness of an equivalence axiom $\phi \leftrightarrow \psi$, show $[\![\phi]\!]^I = [\![\psi]\!]^I$. To prove soundness of a proof rule

$$
\frac{\phi}{\psi}
$$

assume that ϕ is valid, i.e. $[\![\phi]\!]^I = \mathcal{S}$ in all interpretations I with any set of states $\mathcal{S}$, and prove that ψ is valid, i.e. $[\![\psi]\!]^I = \mathcal{S}$ in all I with any $\mathcal{S}$. All rules except US, satisfy the stronger condition of *local soundness*, i.e. for any interpretation I with any set of states $\mathcal{S}$: $[\![\phi]\!]^I = \mathcal{S}$ implies $[\![\psi]\!]^I = \mathcal{S}$. Recall the μ-calculus notation where $\mu Z.\Upsilon(Z)$ denotes the least fixpoint of $\Upsilon(Z)$ and $\nu Z.\Upsilon(Z)$ denotes the greatest fixpoint. Soundness of modus ponens (MP) and $\forall$-generalization (from ϕ derive $\forall x\, \phi$) is standard and not shown.

$[\cdot]$ $[\![[\alpha]\phi]\!]^I = [\![\neg\langle\alpha\rangle\neg\phi]\!]^I$ is a corollary to determinacy (Theorem 2).

$\langle :=\rangle$ $[\![\langle x := \theta\rangle\phi(x)]\!]^I = \varsigma_{x:=\theta}([\![\phi(x)]\!]^I) = \{s \in \mathcal{S} : s_x^{[\![\theta]\!]_s} \in [\![\phi(x)]\!]^I\} = \{s \in \mathcal{S} : s \in [\![\phi(\theta)]\!]^I\} = [\![\phi(\theta)]\!]^I$, where the penultimate equation holds by the substitution lemma. The classical substitution lemma is sufficient for first-order logic $\phi(\theta)$. Otherwise the proof of the substitution lemma for d$\mathcal{L}$ [Pla10b, Lemma 2.2] generalizes to dG$\mathcal{L}$.

$\langle '\rangle$ $[\![\langle x' = \theta\rangle\phi]\!]^I = \varsigma_{x'=\theta}([\![\phi]\!]^I) = \{\varphi(0) \in \mathcal{S}$: for some $\varphi{:}[0, r] \to \mathcal{S}$ so that $\varphi(r) \in [\![\phi]\!]^I$ and $\frac{\mathrm{d}\,\varphi(t)(x)}{\mathrm{d}t}(\zeta) = [\![\theta]\!]_{\varphi(\zeta)}$ for all $\zeta \le r\}$. Also, $[\![\exists t{\ge}0\, \langle x := y(t)\rangle\phi]\!]^I = \{s \in \mathcal{S} : s_t^r \in [\![\langle x := y(t)\rangle\phi]\!]^I$ for some $r \ge 0\} = \{s \in \mathcal{S} : s_t^r \in \{u \in \mathcal{S} : u_x^{[\![y(t)]\!]_u} \in [\![\phi]\!]^I\}$ for $r \ge 0\} = \{s \in \mathcal{S} : (s_t^r)_x^{[\![y(t)]\!]_{s_t^r}} \in [\![\phi]\!]^I$ for some $r \ge 0\}$. The inclusion "$\supseteq$" between both parts holds, because the function $\varphi(\zeta) := (s_t^\zeta)_x^{[\![y(t)]\!]_{s_t^\zeta}}$ solves the differential equation $x' = \theta$ by assumption. The inclusion "$\subseteq$" follows, because the solution of the (smooth) differential equation $x' = \theta$ is unique [Pla10b, Lemma 2.1].

$\langle ?\rangle$ $[\![\langle ?\psi\rangle\phi]\!]^I = \varsigma_{?\psi}([\![\phi]\!]^I) = [\![\psi]\!]^I \cap [\![\phi]\!]^I = [\![\psi \wedge \phi]\!]^I$

$\langle \cup\rangle$ $[\![\langle \alpha \cup \beta\rangle\phi]\!]^I = \varsigma_{\alpha\cup\beta}([\![\phi]\!]^I) = \varsigma_\alpha([\![\phi]\!]^I) \cup \varsigma_\beta([\![\phi]\!]^I) = [\![\langle\alpha\rangle\phi]\!]^I \cup [\![\langle\beta\rangle\phi]\!]^I = [\![\langle\alpha\rangle\phi \vee \langle\beta\rangle\phi]\!]^I$

$\langle ;\rangle$ $[\![\langle \alpha; \beta\rangle\phi]\!]^I = \varsigma_{\alpha;\beta}([\![\phi]\!]^I) = \varsigma_\alpha(\varsigma_\beta([\![\phi]\!]^I)) = \varsigma_\alpha([\![\langle\beta\rangle\phi]\!]^I) = [\![\langle\alpha\rangle\langle\beta\rangle\phi]\!]^I$.

$\langle ^*\rangle$ Since $[\![\langle\alpha^*\rangle\phi]\!]^I = \varsigma_{\alpha^*}([\![\phi]\!]^I) = \mu Z.([\![\phi]\!]^I \cup \varsigma_\alpha(Z))$ is a fixpoint, have $[\![\langle\alpha^*\rangle\phi]\!]^I = [\![\phi]\!]^I \cup \varsigma_\alpha([\![\langle\alpha^*\rangle\phi]\!]^I)$. Thus, $[\![\phi \vee \langle\alpha\rangle\langle\alpha^*\rangle\phi]\!]^I = [\![\phi]\!]^I \cup [\![\langle\alpha\rangle\langle\alpha^*\rangle\phi]\!]^I = [\![\phi]\!]^I \cup \varsigma_\alpha([\![\langle\alpha^*\rangle\phi]\!]^I) = [\![\langle\alpha^*\rangle\phi]\!]^I$. Consequently, $[\![\phi \vee \langle\alpha\rangle\langle\alpha^*\rangle\phi]\!]^I \subseteq [\![\langle\alpha^*\rangle\phi]\!]^I$.

$\langle ^d\rangle$ $[\![\langle\alpha^d\rangle\phi]\!]^I = \varsigma_{\alpha^d}([\![\phi]\!]^I) = \varsigma_\alpha(([\![\phi]\!]^I)^\complement)^\complement = \varsigma_\alpha([\![\neg\phi]\!]^I)^\complement = ([\![\langle\alpha\rangle\neg\phi]\!]^I)^\complement = [\![\neg\langle\alpha\rangle\neg\phi]\!]^I$ by Def. 4.

M Assume the premise $\phi \to \psi$ is valid in interpretation I, i.e. $[\![\phi]\!]^I \subseteq [\![\psi]\!]^I$. Then the conclusion $\langle\alpha\rangle\phi \to \langle\alpha\rangle\psi$ is valid in I, i.e. $[\![\langle\alpha\rangle\phi]\!]^I = \varsigma_\alpha([\![\phi]\!]^I) \subseteq \varsigma_\alpha([\![\psi]\!]^I) = [\![\langle\alpha\rangle\psi]\!]^I$ by monotonicity (Lemma 1).

FP Assume the premise $\phi \vee \langle\alpha\rangle\psi \to \psi$ is valid in I, i.e. $[\![\phi \vee \langle\alpha\rangle\psi]\!]^I \subseteq [\![\psi]\!]^I$. That is, $[\![\phi]\!]^I \cup \varsigma_\alpha([\![\psi]\!]^I) = [\![\phi]\!]^I \cup [\![\langle\alpha\rangle\psi]\!]^I = [\![\phi \vee \langle\alpha\rangle\psi]\!]^I \subseteq [\![\psi]\!]^I$. Thus, ψ is a pre-fixpoint of $Z = [\![\phi]\!]^I \cup \varsigma_\alpha(Z)$. Now using Lemma 1, $[\![\langle\alpha^*\rangle\phi]\!]^I = \varsigma_{\alpha^*}([\![\phi]\!]^I) = \mu Z.([\![\phi]\!]^I \cup \varsigma_\alpha(Z))$ is the least fixpoint and the least pre-fixpoint. Thus, $[\![\langle\alpha^*\rangle\phi]\!]^I \subseteq [\![\psi]\!]^I$, which implies that $\langle\alpha^*\rangle\phi \to \psi$ is valid in I.

US Standard soundness proofs for US [Chu56] generalize to $\mathsf{dG}\mathcal{L}$. A new proof based on an elegant use of the soundness of RE is shown here. Assume the premise ϕ is valid, i.e. $[\![\phi]\!]^I = \mathcal{S}$ in all interpretations I with any set of states $\mathcal{S}$. Assume that the uniform substitution is admissible, otherwise rule US is not applicable and there is nothing to show. It needs to be showsn that $\phi_{p(\cdot)}^{\psi(\cdot)}$ is valid, i.e. $[\![\phi_{p(\cdot)}^{\psi(\cdot)}]\!]^I = \mathcal{S}$ for all I with $\mathcal{S}$. Consider any particular interpretation J with set of states $\mathcal{S}$. Without loss of generality, assume p not to occur in $\psi(\cdot)$ (otherwise first replace all occurrences of p in $\psi(\cdot)$ by q and then use rule US again to replace those q by p). Thus, by uniform substitution, p does not occur in $\phi_{p(\cdot)}^{\psi(\cdot)}$ and the value of $J(p)$ is immaterial for the semantics of $\phi_{p(\cdot)}^{\psi(\cdot)}$. Therefore, pass to an interpretation I that modifies J by changing the semantics of p such that $[\![p(x)]\!]^I = [\![\psi(x)]\!]^J$ for all values of x. In particular, $[\![p(x)]\!]^I = [\![\psi(x)]\!]^I$ for all values of x, since p does not occur in $\psi(x)$. Thus, $I \models \forall x\,(p(x) \leftrightarrow \psi(x))$. Since M is locally sound, so is the congruence rule RE, which derives from M. The principle of substitution of equivalents [HC96, Chapter 13] (from $A \leftrightarrow B$ derive $\Upsilon(A) \leftrightarrow \Upsilon(B)$, where $\Upsilon(B)$ is the formula $\Upsilon(A)$ with some occurrences of A replaced by B), thus, generalizes to $\mathsf{dG}\mathcal{L}$ and is locally sound. Hence, for any particular occurrence of $p(u)$ in ϕ, have $I \models p(u) \leftrightarrow \psi(u)$, which implies $I \models \phi \leftrightarrow \phi_{p(u)}^{\psi(u)}$ for the ordinary replacement of $p(u)$ by $\psi(u)$. This process can be repeated for all occurrences of $p(u)$, leading to $I \models \phi \leftrightarrow \phi_{p(\cdot)}^{\psi(\cdot)}$. Thus, $\mathcal{S} = [\![\phi]\!]^I = [\![\phi_{p(\cdot)}^{\psi(\cdot)}]\!]^I$. Hence, $[\![\phi_{p(\cdot)}^{\psi(\cdot)}]\!]^J = \mathcal{S}$, because p no longer occurs after uniform substitution $\phi_{p(\cdot)}^{\psi(\cdot)}$, since all occurrences of p with any arguments will have been replaced at some point (since admissible). This implies that $\phi_{p(\cdot)}^{\psi(\cdot)}$ is valid since interpretation J with set of states $\mathcal{S}$ was arbitrary. □

The proof calculus in Fig. 4 does not handle differential equations $x' = \theta \,\&\, \psi$ with evolution domain constraints ψ (other than $true$). Yet, Lemma 3 from Section 3.2 eliminates all evolution domain constraints equivalently from hybrid games, so that evolution domains no longer occur.

4.3 Completeness

The converse of soundness is completeness, which is the question whether all valid formulas are provable. Completeness of $\mathsf{dG}\mathcal{L}$ is a challenging question related to a famous open problem about completeness of propositional game logic [Par83]. Based on Gödel's second incompleteness theorem [Göd31], $\mathsf{d}\mathcal{L}$ is incomplete [Pla08, Theorem 2] and so is $\mathsf{dG}\mathcal{L}$. Hence, the right question to ask is that of *relative completeness* [Coo78, HMP77], i.e. completeness relative to an oracle logic *L*. Relative completeness studies the question whether a proof calculus has all proof rules that are required for proving all valid formulas in the logic from tautologies in *L*. In a style similar to Leivant [Lei09], the question of relative completeness can be separated from that of expressivity. Relative completeness can be shown *schematically* for $\mathsf{dG}\mathcal{L}$, i.e. the $\mathsf{dG}\mathcal{L}$ calculus is complete relative to any expressive logic. This is to be contrasted with $\mathsf{d}\mathcal{L}$, whose relative completeness proof was dependent on the particular base logic and its encoding [Pla08]. In particular, the $\mathsf{dG}\mathcal{L}$ completeness result is coding-free [Mos74], which Moschovakis defines as a result that is independent of the particular encoding.

Definition 6 (Expressive)**.** A logic L is *expressive* (for dG$\mathcal{L}$) if, for each dG$\mathcal{L}$ formula ϕ there is a formula $\phi^\flat$ of L that is equivalent, i.e. $\vDash \phi \leftrightarrow \phi^\flat$. Logic L is *constructively expressive* if, in addition, the mapping $\phi \mapsto \phi^\flat$ is effective.

The classical approach for completeness proofs [Coo78, HMP77] proceeds in stages of first-order safety assertions, first-order termination assertions, and then the use of those to prove the general case. That approach does not work for dG$\mathcal{L}$, because hybrid games are so highly symmetric that they may contain operators whose proof depends on proofs about all other operators. A proof of $F \rightarrow \langle\alpha\rangle G$, for example, may require proofs of formulas of the form $A \rightarrow [\beta]B$, e.g., when α is β^d. Such an attempt of proving completeness for $\langle\alpha\rangle$ formulas would need to assume completeness for $[\beta]$ formulas and vice versa, which is a cyclic assumption. Even more involved cyclic arguments result from trying to prove completeness of $\langle\alpha^*\rangle$ and $[\alpha^*]$ formulas that way. Furthermore, the previous arguments for completeness of $\langle\alpha^*\rangle$ formulas [Coo78, HMP77, Pla08] depend on proofs about repetition counts. Those do not work in a hybrid game setting, either, because winning repetition games is more difficult within a chosen bound on the repetition count. No bound on the repetition count below the corresponding closure ordinal can be guaranteed, which can be recursively transfinite by Theorem 7. Also compare how the semantical discrepancies discussed in Appendix D relate to repetition bounds.

Instead, completeness for all dG$\mathcal{L}$ formulas of all types can be proved simultaneously, yet with a more involved well-founded partial order on formulas that ensures that the inductive argument in the completeness proof stays well-founded. This generality has beneficial side-effects, though, because the resulting proof architecture enables a result with minimal coding that makes it possible to exactly identify all complex cases.

Theorem 10 (Relative completeness)**.** *The* dG$\mathcal{L}$ *calculus is a* sound and complete axiomatization *of hybrid games relative to any expressive logic L, i.e. every valid* dG$\mathcal{L}$ *formula is provable in the* dG$\mathcal{L}$ *calculus from L tautologies.*

Proof. Write $\vdash_L \phi$ to indicate that dG$\mathcal{L}$ formula ϕ can be derived in the dG$\mathcal{L}$ proof calculus from valid L formulas. It takes a moment's thought to conclude that soundness transfers to this case from Theorem 9, so it remains to prove completeness. For every valid dG$\mathcal{L}$ formula ϕ it has to be proved that ϕ can be derived from L axioms within the dG$\mathcal{L}$ calculus: from $\vDash \phi$ prove $\vdash_L \phi$. The proof proceeds as follows: By propositional recombination, inductively identify fragments of ϕ that correspond to $\phi_1 \rightarrow \langle\alpha\rangle\phi_2$ or $\phi_1 \rightarrow [\alpha]\phi_2$ logically. Then, express subformulas ϕ_i equivalently in L by Def. 6 as needed, and derive these first-order Angel or Demon properties. Finally, prove that the original dG$\mathcal{L}$ formula can be re-derived from the subproofs in the dG$\mathcal{L}$ calculus.

By appropriate propositional derivations, assume ϕ to be given in conjunctive normal form. Assume that negations are pushed inside over modalities using the dualities $\neg[\alpha]\phi \equiv \langle\alpha\rangle\neg\phi$ and $\neg\langle\alpha\rangle\phi \equiv [\alpha]\neg\phi$ that are provable by axiom $[\cdot]$, and that negations are pushed inside over quantifiers using provable equivalences $\neg\forall x\, \phi \equiv \exists x\, \neg\phi$ and $\neg\exists x\, \phi \equiv \forall x\, \neg\phi$. The remainder of the proof follows an induction on a well-founded partial order $\prec$ induced on dG$\mathcal{L}$ formulas by the lexicographic ordering of the overall structural complexity of the hybrid games in the formula and the structural complexity of the formula itself and with L at the bottom. L is considered first-order, thus of lowest complexity, by relativity. Well-foundedness of $\prec$ is easy to see (formally from projections into concatenations of finite trees), because the overall structural complexity of hybrid games in any particular formula can only decrease finitely often at the expense of increasing the formula complexity, which can, in turn, only decrease finitely often to result in a *L* formula. The only important property for us is that, if the structure of the hybrid games in ψ is simpler than those in ϕ (somewhere simpler and nowhere worse), then $\psi \prec \phi$ even if the logical formula structure of ψ is larger than that of ϕ, e.g., when ψ has more propositional connectives, quantifiers or modalities (but of smaller overall complexity hybrid games). In the following, *IH* is short for induction hypothesis.

0. If ϕ has no hybrid games, then ϕ is a first-order formula; hence provable by assumption (even decidable [Tar51] if in first-order real arithmetic, i.e. no uninterpreted predicate symbols occur).

1. ϕ is of the form $\neg\phi_1$; then ϕ_1 is first-order, as negations are assumed to be pushed inside, so case 0 applies.

2. ϕ is of the form $\phi_1 \land \phi_2$, then $\vDash \phi_1$ and $\vDash \phi_2$, so individually deduce simpler proofs for $\vdash_L \phi_1$ and $\vdash_L \phi_2$ by IH, which combine propositionally to a proof for $\vdash_L \phi_1 \land \phi_2$.

3. The case where ϕ is of the form $\forall x\, \phi_2$, $\exists x\, \phi_2$, $[\alpha]\phi_2$ or $\langle\alpha\rangle\phi_2$ is included in case 4 with $\phi_1 \equiv \mathit{false}$.

4. ϕ is a disjunction and—without loss of generality—has one of the following forms (otherwise use provable associativity and commutativity to reorder disjunction):

$$\begin{array}{l}\phi_1 \lor [\alpha]\phi_2\\ \phi_1 \lor \langle\alpha\rangle\phi_2\\ \phi_1 \lor \exists x\, \phi_2\\ \phi_1 \lor \forall x\, \phi_2.\end{array}$$

   Let $\phi_1 \lor \langle\!\![\alpha]\!\!\rangle\phi_2$ be a unified notation for those cases. Then, $\phi_2 \prec \phi$, since ϕ_2 has less modalities or quantifiers. Likewise, $\phi_1 \prec \phi$ because $\langle\!\![\alpha]\!\!\rangle\phi_2$ contributes one modality or quantifier to ϕ that is not part of ϕ_1. By Def. 6 there are *L* formulas $\phi_1^\flat, \phi_2^\flat$ with $\vDash \phi_i \leftrightarrow \phi_i^\flat$ for $i = 1, 2$, By congruence, the validity $\vDash \phi$ yields $\vDash \phi_1^\flat \lor \langle\!\![\alpha]\!\!\rangle\phi_2^\flat$, which implies $\vDash \neg\phi_1^\flat \to \langle\!\![\alpha]\!\!\rangle\phi_2^\flat$. By induction now derive

$$\vdash_L \neg\phi_1^\flat \to \langle\!\![\alpha]\!\!\rangle \land \phi_2^\flat. \tag{9}$$

   Abbreviate the *L* formula $\neg\phi_1^\flat$ by F and the *L* formula $\phi_2^\flat$ by G, so that $\vdash_L F \to \langle\!\![\alpha]\!\!\rangle G$ remains to be proved. Observe that all subsequent proofs except for $\langle\!\![x' = \theta]\!\!\rangle$ and $\exists x$ also work without encoding when simply using ϕ_1 as F and ϕ_2 as G.

   (a) If $\langle\!\![\alpha]\!\!\rangle$ is the operator $\forall x$ then $\vDash F \to \forall x\, G$, where x can be assumed not to occur in F by renaming. Hence, $\vDash F \to G$. Since $G \prec \forall x\, G$, because it has less quantifiers, also $F \to G \prec F \to \forall x\, G$, hence $\vdash_L F \to G$ is derivable by IH. Then, $\vdash_L F \to \forall x\, G$ derives by $\forall$-generalization of first-order logic, since x does not occur in F. It is even decidable if in first-order real arithmetic [Tar51].

The remainder of the proof will conclude $(F \to \psi) \prec (F \to \phi)$ from $\psi \prec \phi$ without further notice.

(b) If $\langle\!\langle \alpha \rangle\!\rangle$ is the operator $\exists x$ then $\vDash F \to \exists x\, G$, which is first-order (i.e. in L) and, thus, provable by IH, because F, G are L formulas. It is even decidable if in first-order real arithmetic [Tar51].

(c) $\vDash F \to \langle x' = \theta \rangle G$ is an L formula and hence is provable by assumption, because F, G are L formulas. Similarly for $\vDash F \to [x' = \theta]G$.

(d) $\vDash F \to \langle x' = \theta \,\&\, \psi \rangle G$, then this formula is, by Lemma 3, equivalent to a formula without evolution domain restrictions. Using equation (6) from the proof of Lemma 3 as a definitory abbreviation concludes this case by induction hypothesis. Similarly for $\vDash F \to [x' = \theta \,\&\, \psi]G$.

(e) The cases where α is of the form $x := \theta$, $?\psi$, $\beta \cup \gamma$, or $\beta;\gamma$ are consequences of the soundness of the equivalence axioms $\langle{:=}\rangle$,$\langle ? \rangle$,$\langle \cup \rangle$,$\langle ; \rangle$ plus the duals obtained via duality axiom $[\cdot]$. Whenever their respective left-hand side is valid, their right-hand side is valid and of smaller complexity (the games get simpler), and hence derivable by IH. Thus, $F \to \langle \alpha \rangle G$ derives by applying the respective axiom. This proof shows the cases explicitly that require extra thought.

(f) $\vDash F \to \langle x := \theta \rangle G$ implies $\vDash F \land y = \theta \to G_x^y$ for a fresh variable y, where G_x^y is the result of substituting y for x. Since $F \land y = \theta \to G_x^y \prec \langle x := \theta \rangle G$, because there are less hybrid games, $\vdash_L F \land y = \theta \to G_x^y$ is derivable by IH. Hence, $\langle{:=}\rangle$ derives $\vdash_L F \land y = \theta \to \langle x := y \rangle G$. Propositional logic derives $\vdash_L F \to (y = \theta \to \langle x := y \rangle G)$, from which $\vdash_L F \to \forall y\, (y = \theta \to \langle x := y \rangle G)$ derives by $\forall$-generalization of first-order logic. Since y was fresh it does not appear in θ and G, so substitution validities of first-order logic derive $\vdash_L F \to \langle x := \theta \rangle G$. Note that direct proofs by $\langle{:=}\rangle$ are possible when the resulting substitution is admissible, but the substitution in G_x^y is always admissible, because it is a variable renaming replacing x by y.

(g) $\vDash F \to \langle \beta \cup \gamma \rangle G$ implies $\vDash F \to \langle \beta \rangle G \lor \langle \gamma \rangle G$. Since $\langle \beta \rangle G \lor \langle \gamma \rangle G \prec \langle \beta \cup \gamma \rangle G$, because, even if the propositional and modal structure increased, the structural complexity of hybrid games β and γ is smaller than that of $\beta \cup \gamma$ (formula G did not change), $\vdash_L F \to \langle \beta \rangle G \lor \langle \gamma \rangle G$ is derivable by IH. Hence, $\langle \cup \rangle$ derives $\vdash_L F \to \langle \beta \cup \gamma \rangle G$.

(h) $\vDash F \to \langle \beta;\gamma \rangle G$, which implies $\vDash F \to \langle \beta \rangle \langle \gamma \rangle G$. Since $\langle \beta \rangle \langle \gamma \rangle G \prec \langle \beta;\gamma \rangle G$, because, even if the number of modalities increased, the overall structural complexity of the hybrid games decreased because there are less sequential compositions, $\vdash_L F \to \langle \beta \rangle \langle \gamma \rangle G$ is derivable by IH. Hence, $\vdash_L F \to \langle \beta;\gamma \rangle G$ derives by $\langle ; \rangle$.

(i) $\vDash F \to \langle \beta^d \rangle G$ implies $\vDash F \to \neg\langle \beta \rangle \neg G$, which implies $\vDash F \to [\beta]G$. Since $[\beta]G \prec \langle \beta^d \rangle G$, because β^d is more complex than β, $\vdash_L F \to [\beta]G$ can be derived by IH. Axiom $[\cdot]$, thus, derives $\vdash_L F \to \neg\langle \beta \rangle \neg G$, from which axiom $\langle {}^d \rangle$ derives $\vdash_L F \to \langle \beta^d \rangle G$.

(j) $\vDash F \to [\beta^d]G$ implies $\vDash F \to \neg\langle \beta^d \rangle \neg G$, hence $\vDash F \to \langle \beta \rangle G$. Since $\langle \beta \rangle G \prec [\beta^d]G$, because β^d is more complex than β, $\vdash_L F \to \langle \beta \rangle G$ can be derived by IH. Consequently, $\vdash_L F \to \neg\neg\langle \beta \rangle \neg\neg G$ can be derived using M on $\vdash G \to \neg\neg G$. Hence, $\langle {}^d \rangle$ derives $\vdash_L F \to \neg\langle \beta^d \rangle \neg G$, from which axiom $[\cdot]$ derives $\vdash_L F \to [\beta^d]G$.

(k) $\models F \to [\beta^*]G$ can be derived by induction as follows. Formula $[\beta^*]G$, which expresses that Demon has a winning strategy in game β^* to satisfy G, is an inductive invariant of β^*, because $[\beta^*]G \to [\beta][\beta^*]G$ is valid, even provable by the variation $[\beta^*]G \to G \wedge [\beta][\beta^*]G$ of $\langle^*\rangle$ that can be obtained from axioms $\langle^*\rangle$ and $[\cdot]$. Thus, its equivalent L encoding according to Def. 6 is also an inductive invariant:

$$\varphi \equiv ([\beta^*]G)^\flat.$$

$F \to \varphi$ and $\varphi \to G$ are valid (Angel controls *), so are derivable by IH, since $(F \to \varphi) \prec \phi$ and $(\varphi \to G) \prec \phi$ hold by encoding. By M, $\langle^d\rangle$ and $[\cdot]$, the latter derivation $\vdash_L \varphi \to G$ extends to $\vdash_L [\beta^*]\varphi \to [\beta^*]G$. As above, $\varphi \to [\beta]\varphi$ is valid, and thus derivable by IH, since β has less loops. Thus, ind, which derives from FP by Lemma 8, derives $\vdash_L \varphi \to [\beta^*]\varphi$. The above derivations combine propositionally (cut with $[\beta^*]\varphi$ and φ) to $\vdash_L F \to [\beta^*]G$.

(l) $\models F \to \langle\beta^*\rangle G$. Let x the vector of free variables of $\langle\beta^*\rangle G$. Since $\langle\beta^*\rangle G$ is the least pre-fixpoint, for any d$\mathcal{GL}$ formula ψ with free variables in x:

$$\models \forall x\,(G \vee \langle\beta\rangle\psi \to \psi) \to (\langle\beta^*\rangle G \to \psi)$$

by a variation of the soundness argument for FP, which is also derivable by the (semantic) deduction theorem from FP. In particular, this holds for a fresh predicate symbol p with arguments x:

$$\models \forall x\,(G \vee \langle\beta\rangle p(x) \to p(x)) \to (\langle\beta^*\rangle G \to p(x))$$

Using $\models F \to \langle\beta^*\rangle G$, this implies

$$\models \forall x\,(G \vee \langle\beta\rangle p(x) \to p(x)) \to (F \to p(x))$$

As $\forall x\,(G \vee \langle\beta\rangle p(x) \to p(x)) \to (F \to p(x)) \prec \phi$, because, even if the formula complexity increased, the structural complexity of the hybrid games decreased, because ϕ has one more loop, so this fact is derivable by IH:

$$\vdash_L \forall x\,(G \vee \langle\beta\rangle p(x) \to p(x)) \to (F \to p(x))$$

By uniformly substituting $\langle\beta^*\rangle G$ with free variables x for $p(x)$, US derives using $p \notin F, G$:

$$\vdash_L \forall x\,(G \vee \langle\beta\rangle\langle\beta^*\rangle G \to \langle\beta^*\rangle G) \to (F \to \langle\beta^*\rangle G) \tag{10}$$

Yet, $\langle^*\rangle$ derives $\vdash G \vee \langle\beta\rangle\langle\beta^*\rangle G \to \langle\beta^*\rangle G$, from which $\vdash \forall x\,(G \vee \langle\beta\rangle\langle\beta^*\rangle G \to \langle\beta^*\rangle G)$ derives by $\forall$-generalization. Now modus ponens derives $\vdash_L F \to \langle\beta^*\rangle G$ using (10).

This concludes the derivation of (9). Further $\models \phi_1 \leftrightarrow \phi_1^\flat$ implies $\models \neg\phi_1 \to \neg\phi_1^\flat$, which is derivable by IH, because $\phi_1 \prec \phi$. Combine $\vdash_L \neg\phi_1 \to \neg\phi_1^\flat$ with (9) (cut with $\neg\phi_1^\flat$) to derive

$$\vdash_L \neg\phi_1 \to \langle\!\langle\alpha\rangle\!\rangle\phi_2^\flat. \tag{11}$$

Likewise $\vDash \phi_2 \leftrightarrow \phi_2^\flat$ implies $\vDash \phi_2^\flat \rightarrow \phi_2$, which is derivable by IH, as $\phi_2 \prec \phi$. From $\vdash_L \phi_2^\flat \rightarrow \phi_2$ derive $\vdash_L \langle\!\lbrack\alpha\rbrack\!\rangle\phi_2^\flat \rightarrow \langle\!\lbrack\alpha\rbrack\!\rangle\phi_2$ by M if $\langle\!\lbrack\alpha\rbrack\!\rangle$ is $\langle\alpha\rangle$, by M and $\langle^d\rangle$ if $\langle\!\lbrack\alpha\rbrack\!\rangle$ is $[\alpha]$, by $\forall$-generalization if $\langle\!\lbrack\alpha\rbrack\!\rangle$ is $\forall x$, and by $\forall$-generalization and duality if $\langle\!\lbrack\alpha\rbrack\!\rangle$ is $\exists x$. Finally combine the latter derivation propositionally with (11) by a cut with $\langle\!\lbrack\alpha\rbrack\!\rangle\phi_2^\flat$ to derive $\vdash_L \neg\phi_1 \rightarrow \langle\!\lbrack\alpha\rbrack\!\rangle\phi_2$, from which $\vdash_L \phi_1 \vee \langle\!\lbrack\alpha\rbrack\!\rangle\phi_2$ derives propositionally.

This completes the proof of completeness (Theorem 10). □

The proof of Theorem 10 is constructive, so Theorem 10 is constructive if L is constructively expressive. To highlight, the proof works without coding, except for $x' = \theta$, $\exists$ and $[\beta^*]$. The result is even coding-free in the sense of Moschovakis [Mos74]. Using US, the case for $\langle\beta^*\rangle G$ in the proof of Theorem 10 reveals an explicit $^\flat$-free reduction to a dG$\mathcal{L}$ formula with less loops, which can be considered a modal analogue of characterizations in the Calculus of Constructions [CH88]. These observations easily reprove a classical result of Meyer and Halpern [MH82] about the semidecidability of termination assertions (logical formulas $F \rightarrow \langle\alpha\rangle G$ of uninterpreted dynamic logic with first-order F, G and regular programs α without differential equations). In fact, this proves a stronger result about semidecidability of dynamic logic without any $[\alpha]\cdot$ with loops [Sch84]. Theorem 10 shows that this result continues to hold for uninterpreted game logic in the fragment where * only occurs with even ^{d}-polarity in $\langle\alpha\rangle$ and only of odd ^{d}-polarity in $[\alpha]$ (the conditions on tests in α are accordingly).

The constructive nature of Theorem 10 characterizes exactly which part of hybrid games proving is difficult: finding computationally succinct weaker invariants for $[\beta^*]G$ and finding succinct differential (in)variants [Pla10a] for $[x' = \theta]$ and $\langle x' = \theta\rangle$, of which a solution is a special case [Pla12c]. The case $\exists x\, G$ is interesting in that a closer inspection of Theorem 10 reveals that its complexity depends on whether that quantifier supports Herbrand disjunctions. That is the case for uninterpreted first-order logic and first-order real arithmetic [Tar51], but not for $G \equiv [\beta^*]\psi$, which already gives $\exists x\, G$ the full Π^1_1-complete complexity even for classical dynamic logic [HKT00, Theorems 13.1,13.2]. Herbrand disjunctions for $\exists x\, G$ justify how Theorem 10 implies the result of Schmitt [Sch84].

The proof of Theorem 10 uses *minimal* coding. The case $[\beta^*]$ needs encoding, because $F \rightarrow [\beta^*]G$ validity is already Π^0_2-complete for classical dynamic logic [HKT00, Theorem 13.5]. The case $\exists$ needs encoding in the presence of $[\beta^*]$, because $\exists x\, [\beta^*]G$ validity is Π^1_1-complete for classical dynamic logic [HKT00, Theorems 13.1]. The case $x' = \theta$ leads to classical Δ^1_1-hardness over $\mathbb{N}$ [Pla08, Lemma 4].

The completeness proof indicates a coding-free way of proving Angel properties $\langle\beta^*\rangle G$ that is similar to characterizations in the Calculus of Constructions and works efficiently in practice. Examples are shown in Appendix A. In particular, dG$\mathcal{L}$ does not need Harel's convergence rule [HMP77] for completeness and, thus, neither does logic for hybrid systems, even though it was previously based on it [Pla12a]. These results correspond to a hybrid game reading of influential views of understanding program invariants as fixpoints [CC77, Cla79].

4.4 Expressibility

The dG$\mathcal{L}$ calculus is complete relative to any expressive logic L. One natural choice for an oracle logic is $L_{\mu\mathrm{D}}$, the modal μ-calculus of differential equations (*fixpoint logic of differential equations*):

$$\phi ::= X(\theta) \mid p(\theta) \mid \theta_1 \geq \theta_2 \mid \neg\phi \mid \phi \wedge \psi \mid \langle x' = \theta\rangle\phi \mid \mu X.\phi$$

where $\mu X.\phi$ requires all occurrences of X in ϕ to be positive. The semantics is the usual, e.g., $\mu X.\phi$ binds set variable X and real variable (vector) x and is interpreted as the least fixpoint X of ϕ, i.e. the smallest denotation of X such that $X(x) \leftrightarrow \phi$ holds for all x [Koz83, Lub89]. A more careful inspection of the proofs in this article reveals that the two-variable fragment of $L_{\mu\mathrm{D}}$ is enough, which gives a stronger statement as long as the variable hierarchy for $L_{\mu\mathrm{D}}$ does not collapse [BGL07]. The logic $L_{\mu\mathrm{D}}$ is considered in this context, because it exposes the most natural interactivity on top of differential equations and makes the constructions most apparent and minimally coding themselves.

Lemma 11 (Continuous expressibility)**.** *$L_{\mu D}$ is constructively* expressive *for* dG$\mathcal{L}$.

Proof. Of course, $(p(\theta))^\flat = p(\theta)$ etc. The inductive cases are shown in Fig. 5. It is easy to check

$$\begin{aligned}
(\neg\phi)^\flat &\equiv \neg(\phi^\flat)\\
(\phi \wedge \psi)^\flat &\equiv \phi^\flat \wedge \psi^\flat\\
(\exists x\,\phi)^\flat &\equiv \exists x\,(\phi^\flat)\\
(\langle x := \theta\rangle\phi)^\flat &\equiv \forall y\,(y = \theta \rightarrow (\phi_x^y)^\flat)\\
(\langle x' = \theta\rangle\phi)^\flat &\equiv \langle x' = \theta\rangle(\phi^\flat)\\
(\langle ?\psi\rangle\phi)^\flat &\equiv (\psi \wedge \phi)^\flat\\
(\langle \alpha \cup \beta\rangle\phi)^\flat &\equiv (\langle\alpha\rangle\phi \vee \langle\beta\rangle\phi)^\flat\\
(\langle \alpha; \beta\rangle\phi)^\flat &\equiv (\langle\alpha\rangle\langle\beta\rangle\phi)^\flat\\
(\langle \alpha^*\rangle\phi)^\flat &\equiv \mu X.(\phi \vee \langle\alpha\rangle X(x))^\flat\\
(\langle \alpha^d\rangle\phi)^\flat &\equiv (\neg\langle\alpha\rangle\neg\phi)^\flat\\
([\alpha]\phi)^\flat &\equiv (\langle\alpha^d\rangle\phi)^\flat
\end{aligned}$$

Figure 5: Inductive cases for constructive *expressivity* of $L_{\mu\mathrm{D}}$.

that $\phi^\flat$ is equivalent to ϕ, e.g. based on the soundness of the dG$\mathcal{L}$ axioms. Note that $(\phi \vee \psi)^\flat \equiv \phi^\flat \vee \psi^\flat$ is a consequence of the above definitions and the abbreviation $\phi \vee \psi \equiv \neg(\neg\phi \wedge \neg\psi)$. The quantifier in the definition of $(\langle x := \theta\rangle\phi)^\flat$ is not necessary if the substitution of θ for x is admissible. The variable renaming of fresh variable y for x in ϕ with the result ϕ_x^y is always admissible. Note that quantifiers are expressible in $L_{\mu\mathrm{D}}$ via $\exists x\,\phi \equiv \langle x' = 1\rangle\phi \vee \langle x' = -1\rangle\phi$. Recall that $x' = \theta\,\&\,\psi$ is expressible by Lemma 3. The case $(\langle\alpha^*\rangle\phi)^\flat$ is defined as the least fixpoint of the reduction of $\phi \vee \langle\alpha\rangle X(x)$, where x are the variables of α using classical short notation [Lub89].

In particular, $(\langle\alpha^*\rangle\phi)^\flat$ satisfies $\phi \lor \langle\alpha\rangle(\langle\alpha^*\rangle\phi)^\flat \leftrightarrow (\langle\alpha^*\rangle\phi)^\flat$ and $(\langle\alpha^*\rangle\phi)^\flat$ is the formula with the smallest such interpretation, which is all that these proofs depend on. □

A discrete analog of Lemma 11 follows from a (constructive) equi-expressibility result [Pla12a, Theorem 9].

Corollary 12 (Discrete expressibility)**.** *The (first-order) discrete μ-calculus over $\mathbb{R}$ is constructively expressive for* $\mathsf{dG}\mathcal{L}$*.*

This aligns the discrete and the continuous side of hybrid games in a constructive provably equivalent way similar to corresponding results about hybrid systems [Pla12a]. Yet, the interactivity of two-variable fixpoints stays, which turns out to be necessary (Section 5).

Corollary 13 (Relative completeness)**.** *The* $\mathsf{dG}\mathcal{L}$ *calculus is a sound and complete axiomatization of* $\mathsf{dG}\mathcal{L}$ *relative to $L_{\mu D}$. With the Euler axiom [Pla12a], the* $\mathsf{dG}\mathcal{L}$ *calculus is a sound and complete axiomatization of* $\mathsf{dG}\mathcal{L}$ *relative to the discrete μ-calculus over $\mathbb{R}$.*

Proof. Follows from Theorem 10, Lemma 11, and Corollary 12. □

An interesting question is whether fragments of $\mathsf{dG}\mathcal{L}$ are complete relative to smaller logics, which Theorem 10 and Lemma 11 reduce to a study of expressing (two-variable) $L_{\mu\mathrm{D}}$. This yields the following hybrid versions of Parikh's completeness results for fragments of game logic [Par83].

Corollary 14 (Relative completeness of *-free $\mathsf{dG}\mathcal{L}$)**.** *The* $\mathsf{dG}\mathcal{L}$ *calculus is a* sound and complete axiomatization *of *-free hybrid games relative to* $\mathsf{d}\mathcal{L}$*.*

Proof. Lemma 11 reduces to $\mathsf{d}\mathcal{L}$, even the first-order logic of differential equations [Pla12a], for *-free hybrid games. □

Corollary 15 (Relative completeness of ^{d}-free $\mathsf{dG}\mathcal{L}$)**.** *The* $\mathsf{dG}\mathcal{L}$ *calculus is a* sound and complete axiomatization *of ^{d}-free hybrid games relative to* $\mathsf{d}\mathcal{L}$*.*

Proof. ^{d}-free loops are Scott-continuous by Lemma 6, so have closure ordinal ω and are, thus, equivalent to their $\mathsf{d}\mathcal{L}$ form, and even expressible in the first-order logic of differential equations by [Pla12a, Theorem 9]. □

By Corollary 15, $\mathsf{d}\mathcal{L}$ is relatively complete without the convergence rule that had been used before [Pla08]. In combination with the first and second relative completeness theorems of $\mathsf{d}\mathcal{L}$ [Pla12a], it follows that the $\mathsf{dG}\mathcal{L}$ calculus is a sound and complete axiomatization of *-free hybrid games and of ^{d}-free hybrid games relative to the first-order logic of differential equations. When adding the Euler axiom [Pla12a], both are sound and complete axiomatizations of those classes of hybrid games relative to discrete dynamic logic [Pla12a]. Similar completeness results for $\mathsf{dG}\mathcal{L}$ relative to $\mathsf{d}\mathcal{L}$, and, thus, relative to the first-order logic of differential equations, follow from Theorem 10 with some thought, e.g., for the case of hybrid games with winning regions that are finite rank Borel sets.

4.5 Separating Axioms

In order to illustrate how and why reasoning about hybrid games differs from reasoning about hybrid systems, separating axioms can be identified, that is, axioms of d$\mathcal{L}$ [Pla08, Pla12a] that do not hold in dG$\mathcal{L}$. This article investigates the difference in terms of important classes of modal logics; recall [HC96] or Appendix B.

Theorem 16. dG$\mathcal{L}$ *is a subregular, sub-Barcan, monotonic modal logic without the induction axiom of dynamic logic.*

The proof of Theorem 16 is in Appendix B, where a simple counterexample for each separating axiom illustrates what makes hybrid games different than hybrid systems. The difference in axioms is summarized in Fig. 6, where $\text{Cl}_\forall$ is the universal closure with respect to all variables bound in hybrid game α.

~~K~~ $[\alpha](\phi \to \psi) \to ([\alpha]\phi \to [\alpha]\psi)$

M $\langle\alpha\rangle\phi \lor \langle\alpha\rangle\psi \to \langle\alpha\rangle(\phi \lor \psi)$

~~G~~ $\dfrac{\phi}{[\alpha]\phi}$

$\text{M}_{[\cdot]}$ $\dfrac{\phi \to \psi}{[\beta]\phi \to [\beta]\psi}$

~~R~~ $\dfrac{\phi_1 \land \phi_2 \to \psi}{[\alpha]\phi_1 \land [\alpha]\phi_2 \to [\alpha]\psi}$

~~B~~ $\langle\alpha\rangle\exists x\, \phi \to \exists x\, \langle\alpha\rangle\phi \qquad (x \notin \alpha)$

$\overleftarrow{\text{B}}$ $\exists x\, \langle\alpha\rangle\phi \to \langle\alpha\rangle\exists x\, \phi \qquad (x \notin \alpha)$

~~I~~ $[\alpha^*](\phi \to [\alpha]\phi) \to (\phi \to [\alpha^*]\phi)$

$\forall$I $\text{Cl}_\forall\,(\phi \to [\alpha]\phi) \to (\phi \to [\alpha^*]\phi)$

~~FA~~ $\langle\alpha^*\rangle\phi \to \phi \lor \langle\alpha^*\rangle(\neg\phi \land \langle\alpha\rangle\phi)$

Figure 6: Separating axioms: The axioms and rules on the left are sound for hybrid systems but not for hybrid games. The related axioms on the right are sound for hybrid games.

Harel's convergence rule [HMP77] is not a separating axiom, because it is sound for dG$\mathcal{L}$, just unnecessary. In light of Theorem 7, it is questionable whether the convergence rule would be relatively complete for hybrid games, because it is based on the existence of bounds on the repetition count. The hybrid version of Harel's convergence rule [Pla08] reads as follows (it assumes that v does not occur in α):

$$\frac{\varphi(v+1) \land v+1 > 0 \to \langle\alpha\rangle\varphi(v)}{\exists v\, \varphi(v) \to \langle\alpha^*\rangle\exists v{\leq}0\, \varphi(v)}$$

If the convergence rule could prove, e.g., dG$\mathcal{L}$ formula (7) from Theorem 7, then $\varphi(\cdot)$ would yield a bound on the number of repetitions, which, by the proof of Theorem 7 does not exist below closure ordinal $\omega \cdot 2$. The premise of a use of the convergence rule makes the bound induced by $\varphi(v)$ progress by 1 in each iteration. The postcondition in the conclusion makes it terminate for $v \leq 0$. And the conclusion's antecedent requires a real number for the initial bound. Thus, the convergence rule only permits bounds below ω, not the required transfinite ordinal $\omega \cdot 2$.

These thoughts further suggest a transfinite version of the convergence rule with an extra inductive premise for limit ordinals. That would be interesting, but is technically more involved than the dG$\mathcal{L}$ axiomatization, because it would require multi-sorted quantifiers and proof rules for ordinal arithmetic.

5 Expressiveness

Differential game logic dG$\mathcal{L}$ is a logic for hybrid games. How does it compare to differential dynamic logic d$\mathcal{L}$ [Pla08, Pla12a], which is the corresponding logic for hybrid systems? Hybrid systems are expected to be single-player hybrid games where one of the players never gets to decide. And, d$\mathcal{L}$ is expected to be a sublogic of dG$\mathcal{L}$. But what about the converse? How the expressiveness of dG$\mathcal{L}$ relates to that of d$\mathcal{L}$ is related to classical long-standing questions for the propositional case [Par85, BGL07]. Note that even known classical results about expressiveness for the propositional case do not transfer to dG$\mathcal{L}$ [Par85].

The notation $L_1 \leq L_2$ signifies that logic L_2 is expressive for logic L_1 (Def. 6). Likewise, $L_1 \equiv L_2$ signifies equivalent expressiveness, i.e. $L_1 \leq L_2$ and $L_2 \leq L_1$. Further, $L_1 < L_2$ means that L_1 is strictly less expressive than L_2, i.e. $L_1 \leq L_2$ but not $L_2 \leq L_1$.

Lemma 17 (Single-player hybrid games)**.** d$\mathcal{L}$ $\leq$ dG$\mathcal{L}$ *by syntactic embedding.*

Proof. Hybrid systems form single-player hybrid games, i.e. ^{d}-free hybrid games. So, identity is a syntactic embedding of d$\mathcal{L}$ into dG$\mathcal{L}$, which preserves the semantics as follows. With Lemma 6, Kleene's fixpoint theorem implies that ω is the closure ordinal for ^{d}-free hybrid games α. Hence, for ^{d}-free α, a simple induction shows

$$\varsigma_{\alpha^*}(X) = \varsigma_\alpha^\omega(X) = \bigcup_{n<\omega} \varsigma_\alpha^n(X) = \bigcup_{n<\omega} \varsigma_{\alpha^n}(X) \tag{12}$$

where α^n is the n-fold sequential composition of α given by $\alpha^0 \equiv ?true$ and $\alpha^{n+1} \equiv \alpha; \alpha^n$. The semantics of ^{d}-free dG$\mathcal{L}$ agrees with that defined for d$\mathcal{L}$ originally [Pla08, Pla12a] by a simple comparison using (12) for the crucial case α^*. ☐

What about the converse? Is the logic dG$\mathcal{L}$ truly new or could it have been expressed in d$\mathcal{L}$? Without any doubt, unlike d$\mathcal{L}$, dG$\mathcal{L}$ is meant for hybrid games and makes it more convenient to refer directly to questions about hybrid games.[9] Does dG$\mathcal{L}$ provide features strictly necessary for hybrid games that d$\mathcal{L}$ is missing? Finitely bounded hybrid games are expressible in d$\mathcal{L}$ by Theorem 14. What about other hybrid games? Both possible outcomes are interesting. If d$\mathcal{L}$ $\equiv$ dG$\mathcal{L}$, then Theorem 10 implies that dG$\mathcal{L}$ is complete relative to d$\mathcal{L}$ and relative to the smaller logics that d$\mathcal{L}$ is complete for [Pla12a]. If d$\mathcal{L}$ $<$ dG$\mathcal{L}$, instead, then dG$\mathcal{L}$ is a provably more expressive logic with features that are strictly necessary for hybrid games. The answer takes some preparations.

[9] Even if a logic is not strictly more expressive but "only" more convenient, it is still often strongly preferable. Program logics and their cousins, for example, are used widely, even though first-order integer arithmetic would theoretically suffice [HMP77, HK84, HKT00].

Let $(\!|y_0, \dots, y_n|\!)$ denote a $\mathbb{R}$-Gödel encoding, i.e. a bijective function pairing any n-tuple of real numbers $y_0, y_1, \dots, y_n$ into a single real that, along with its inverse, is definable in FOD [Pla08, Lemma 4]. FOD is the *first-order logic of differential equations* [Pla08], i.e., the first-order fragment of d$\mathcal{L}$ where all hybrid games α are of the form $x' = \theta$. By Lemma 17, FOD is a sublogic of dG$\mathcal{L}$ and, thus, $\mathbb{R}$-Gödel encodings are definable in dG$\mathcal{L}$. *(Rich-test) regular dynamic logic* (DL) [HKT00, Pla12a] over $\mathbb{R}$ is the fragment of d$\mathcal{L}$ (and by Lemma 17 of dG$\mathcal{L}$) without ^{d} and without differential equations. So both FOD [Pla08, Lemma 4] and DL [Pla12a, Theorem 9] can define $\mathbb{R}$-Gödel encodings. *Acceptable structures* are structures in which elementary $\mathbb{R}$-Gödel encodings are definable [Mos74].

The open *recursive game quantifier* $\Game$ of length ω applied to formula $\varphi(x, y)$ is

$$\Game y\, \varphi(x, y) \overset{\text{def}}{\equiv} \forall y_0\, \exists y_1\, \forall y_2\, \exists y_3\, \dots \bigvee_{n<\omega} \varphi(x, (\!|y_0, \dots, y_n|\!)) \tag{13}$$

which has a semantics as a two player Gale-Stewart game [GS53] in which two players alternate in choosing values for the ω many variables y_{2i} and y_{2i+1}. Player $\exists$ wins if $\varphi(x, (\!|y_0, \dots, y_n|\!))$ holds for some $n < \omega$ so that the infinitary disjunction $\bigvee_{n<\omega} \varphi(x, (\!|y_0, \dots, y_n|\!))$ is satisfied; see [Mos74, Vää11] for details.

Lemma 18 (Game quantifier)**.** *Recursive game quantifier $\Game$ is definable in* dG$\mathcal{L}$*.*

Proof. Let $\varphi(x, y)$ a dG$\mathcal{L}$ formula, which, to simplify notation, is assumed to check the sequence that y encodes only at even indices n. Then $\Game y\, \varphi(x, y)$ is definable in dG$\mathcal{L}$:

$$\langle y := (\!|\,|\!);\, \big(z' = 1^d;\, z' = -1^d;\, y := (\!|y, z|\!);\, z' = 1;\, z' = -1;\, y := (\!|y, z|\!)\big)^*\rangle\, \varphi(x, y) \tag{14}$$

This dG$\mathcal{L}$ formula uses $(\!|y_0, y_1, y_2, \dots, y_n|\!)$ reordered as $(\!|\dots(\!|(\!|(\!|(\!|\,|\!), y_0|\!), y_1|\!), y_2|\!), \dots y_n|\!)$ for simplicity, which is a recursive permutation. Angel and Demon alternate differential equations for z in (14) that get successively paired into y by $y := (\!|y, z|\!)$, which is definable [Pla08, Lemma 4]. This alternation of differential equations corresponds to the alternation of quantifiers in $\Game$. The number of actual alternations played can be exactly any $n < \omega$, because the semantics of $\langle \alpha^* \rangle$ is a least fixpoint, so well-founded. □

Note that (14) equivalently defines (13), even though the former is a finite dG$\mathcal{L}$ formula while the latter is an infinite formula in an infinitary logic augmented with the game quantifier [Vää11], so (14) is infinitely more concise. The closed recursive game quantifier $\neg \Game y\, \neg \phi(x, y)$ is definable in dG$\mathcal{L}$ by duality as well, noting that open as well as closed Gale-Stewart games are determined [GS53]. Finally observe that (14) would not define (13) in the (weaker) advance notice semantics (Appendix D), as that corresponds to swapping the quantifier alternation with $\bigvee_{n<\omega}$:

$$\bigvee_{n<\omega} \forall y_0\, \exists y_1\, \forall y_2\, \exists y_3\, \dots \varphi(x, (\!|y_0, \dots, y_n|\!))$$

With this preparation, dG$\mathcal{L}$ can be proved to be strictly more expressive than d$\mathcal{L}$, which means that hybrid games are fundamentally more than hybrid systems.

Theorem 19 (Expressive power)**.** $\mathsf{d}\mathcal{L} < \mathsf{dG}\mathcal{L}$.

Proof. By Lemma 17, it only remains to refute $\mathsf{dG}\mathcal{L} \leq \mathsf{d}\mathcal{L}$. $\mathbb{R}$-Gödel encodings etc. are elementarily definable in FOD [Pla08, Lemma 4], thus, also in DL over $\mathbb{R}$ [Pla12a, Theorem 9]. This makes $\mathbb{R}$ an acceptable structure [Mos74] when augmented with the corresponding definitions from FOD or DL over $\mathbb{R}$. Further, $\mathsf{d}\mathcal{L} \equiv$ FOD [Pla08] and $\mathsf{d}\mathcal{L} \equiv$ DL over $\mathbb{R}$ [Pla12a, Theorem 9]. On any acceptable structure, DL defines exactly all first-order definable relations [HK84, Theorems 3 and 4]. For acceptable structures, the open recursive game quantifier $\Game y\, \varphi(x, y)$ for first-order formulas $\varphi(x, y)$ exactly defines all (positive first-order) inductively definable relations [Mos72, Mos74, Theorem 5C.2].[10] The logic $\mathsf{dG}\mathcal{L}$ can define $\Game$ by Lemma 18, so all inductive relations. In acceptable structures, not all inductively definable relations are first-order definable [Mos74, Theorem 5B.2]. Thus, $\mathsf{dG}\mathcal{L}$ can define an inductive relation that DL cannot define over $\mathbb{R}$, so neither can $\mathsf{d}\mathcal{L}$. Hence, $\mathsf{d}\mathcal{L} \equiv \text{DL} < \mathsf{dG}\mathcal{L}$ over $\mathbb{R}$. □

Thus, hybrid games can characterize relations that hybrid systems cannot. The proof of Theorem 7 implies that ω_1^{HG} exceeds all order types of all inductive well-orders, because all inductive relations can be characterized in $\mathsf{dG}\mathcal{L}$. All closure ordinals of inductive relations occur as order types of some inductive well-order, because the staging order of inductive definitions is well-founded [Mos74, Theorems 3A.3, 3C.1]. Thus, ω_1^{HG} equals the closure ordinal of the underlying structure.

The game quantifier and its characterization in the proof of Lemma 18 along with the differential equation characterization of Gödel encodings [Pla08, Lemma 4] implies the existence of a smaller syntactic fragment of $\mathsf{dG}\mathcal{L}$ that is expressive, so that $\mathsf{dG}\mathcal{L}$ is complete relative to this fragment of $\mathsf{dG}\mathcal{L}$ by Theorem 10. By (13), alternating differential equations in a single loop are the dominant feature of this fragment. The only modification to the proof of Lemma 11 is the case of $(\langle\alpha^*\rangle\phi)^\flat$ which then uses (13) with a (definable) formula $\varphi(x, (\!|y_0, \dots, y_n|\!))$ that simply checks whether the decision sequence $y_0, \dots, y_n$ gives a valid play of hybrid game α^* in which Angel wins. The fact that $\Game$ assumes strict alternation of the players is easily overcome by choosing φ to be independent of y_i when the player for its quantifier does not get to choose in α^* at step i. The actions can be chosen, e.g., as discussed in Appendix C.

6 Related Work

Games and logic have been shown to interact fruitfully in many ways [GS53, Ehr61, Par83, Par85, Aum95, HS97, Sti01, AHK02, PP03, CHP07, AG11, Vää11]. The present article focuses on using logic to specify and verify properties of hybrid games, inspired by Parikh's propositional game logic for finite-state discrete games [Par83, Par85, PP03]. Game logic generalizes (propositional discrete) dynamic logic to discrete games played on a finite state space. Game logic is elegant but challenging. Its expressiveness has only begun to be understood after two decades [Ber03, BGL07].

Discrete games and the interaction of games and logic for various purposes have been studied with much success [vNM55, Par85, Aum95, HS97, Sti01, AHK02, PP03, Ber03, CHP07, BGL07,

[10] The game quantifier in [Mos72] starts with $\exists y_1$, which is a difference easily overcome.

BP09, AG11, Vää11]. Propositional game logic [PP03] subsumes ΔPDL and CTL*. After more than two decades, it has been shown that the alternation hierarchy in propositional game logic is strict and encodes parity games that span the full alternation hierarchy of the (propositional) modal μ-calculus [Ber03] and that, being in the two variable fragment, it is less expressive than the (propositional) modal μ-calculus [BGL07]. Another influential propositional modal logic, ATL* has been used for model checking finite-state systems [AHK02] and is related to propositional game logic [BP09]. Applications and relations of game logic, ATL* [AHK02], and strategy logic [CHP07] have been discussed in the literature [AHK02, PP03, CHP07, BP09]. These logics for the propositional case of finite-state discrete games are interesting, but it is not clear how their decision procedures should be generalized to the highly undecidable domain of hybrid games with differential equations, uncountable choices, and higher closure ordinals. The logic $\mathsf{dG}\mathcal{L}$ shows how such hybrid games can be proved, enjoys completeness, and comes with a rich theory

Differential games have been studied with many different notions of solutions [Isa67, Fri71, Pet93, Bre10]. They are of interest when actions are solely in continuous time. The present article considers the complementary model of hybrid games where the underlying system is that of a hybrid system with interacting discrete and continuous dynamics, but the game actions are chosen at discrete instants of time, even if they take effect in continuous time.

Hybrid games provide a complementary perspective on differential games, just like hybrid systems provide a complementary perspective on continuous dynamical systems. Differential games formalize various notions of adversarial control on variables for a single differential equation [Isa67, Fri71, Pet93], including solutions based on a non-anticipatory measurable input to an integral interpretation of the differential equations [Fri71], joint limits of lower and upper limits of δ-anticipatory or δ-delayed strategies for $\delta \to 0$ [Pet93], and Pareto-optimal, Nash, or Stackelberg equilibria, whose computation requires solving PDEs that quickly become ill-posed (e.g., for feedback Nash equilibria except in dimension one or for linear-quadratic games); see Bressan [Bre10] for an overview. Hybrid games, instead, distinguish discrete versus continuous parts of the dynamics, which simplifies the concepts, because easier pieces are involved, and, simultaneously, have been argued to make other aspects like delays in decisions and the integration of computer-decision into continuous physics more realistic [TPS98, TLS00, BBC10, VPVD11, PHP01, QP12]. The situation is similar to hybrid systems, which provide a complementary perspective on (continuous) dynamical systems [Hen96, Pla12a] that can model more complicated systems as a combination of simpler concepts and can model computational effects more realistically.

Some reachability aspects of games for hybrid systems have been studied before. A game view on hybrid systems verification has been proposed following a Hamilton-Jacobi-Bellman PDE formulation [TMBO03, MBT05], with subsequent extensions by Gao et al. [GLQ07]. Their primary focus is on adversarial choices in the continuous dynamics, which is very interesting, but not what is considered here. The PDE formulation is related to an approach with viability theory for hybrid games applications in finance [SP04]. WCTL properties of STORMED hybrid games, which are restricted to evolve linearly in one "direction" all the time, have been shown to be decidable using bisimulation quotients [VPVD11]. STORMED hybrid games generalize o-minimal hybrid games which have been shown to be decidable earlier [BBC10]. The case of rectangular hybrid games is known to be decidable [HHM99] as well as the special case of timed games [CHP11]. Many applications do not fall into these decidable classes [QP12], so that a study of more general hybrid games is called for. The results in this article have implications for reachability analysis. They show, for example, that reachability computations and backwards induction for hybrid games require highly transfinite closure ordinals $\geq \omega_1^{\mathrm{CK}}$. Completeness further characterizes the challenging cases in hybrid games verification.

This article takes a complementary view and studies logics and proofs for hybrid games instead of searching for decidable fragments using bisimulation quotients [HHM99, BBC10, VPVD11], which do not generally exist. It provides a proof-based verification technique for more general hybrid games with nonlinear dynamics. This article's notion of hybrid games is more flexible, because it allows arbitrary nested hybrid game choices rather than the fixed controller-plant interaction considered in related work. This results in more general logical formulas with nested modal game operators. This article does not consider concurrent games [BBC10], though, only sequential games.

There is more than one way how logic can help to understand games of hybrid systems. In concurrent work, it has been shown that games can be added as separate constructs on top of unmodified differential dynamic logic [QP12], focusing on the special case of advance notice semantics (Appendix D). The present article follows a different principle. Instead of leaving differential dynamic logic untouched and adding several separate game constructs on top of full hybrid systems reachability operators as in [QP12], the logic $\mathsf{dG}\mathcal{L}$ becomes a proper game logic by adding a single operator ^{d} for adversariality into the system dynamics. The logic $\mathsf{dG}\mathcal{L}$ results in a much simplified but nevertheless more general logic with a simpler and more general semantics (and not restricted to advance notice) and simpler and more general calculus. The present article studies a Hilbert calculus and focuses on fundamental logical properties and theory instead. See [QP12] for practical aspects like sequent calculus automation and a very challenging robotic factory automation case study that translates to $\mathsf{dG}\mathcal{L}$. What is more difficult in $\mathsf{dG}\mathcal{L}$ in comparison to that fragment [QP12], however, is the need to carefully identify which axioms are no longer sound for games, which is what is pursued in Section 4.5.

The logic $\mathsf{dG}\mathcal{L}$ presented here has similarities with stochastic differential dynamic logic (S$\mathsf{d}\mathcal{L}$) [Pla11], because both may be used to verify properties of the hybrid system dynamics with partially uncertain behavior. Both approaches do, however, address uncertainty in fundamentally different ways. S$\mathsf{d}\mathcal{L}$ takes a probabilistic perspective on uncertainty in the system dynamics. The $\mathsf{dG}\mathcal{L}$ approach put forth in this paper, instead, takes an adversarial perspective on uncertainty. Both views on how to handle uncertain behavior are useful but serve quite different purposes, depending on the nature of the system analysis question at hand. A probabilistic understanding of uncertainty can be superior whenever good information is available about the distribution of choices made by the environment and other agents. Whenever that is not possible, adversarial views may be more appropriate, since they do not lead to the inadequate biases that arbitrary probabilistic assumptions would impose. Adversarial dynamics is also called for in cases of true competition, like in RoboCup.

7 Conclusions and Future Work

This article has introduced differential game logic (dGL) for hybrid games that combine discrete, continuous, and adversarial dynamics. Just like hybrid games unify hybrid systems with discrete games, the logic dGL unifies logic of hybrid systems with Parikh's propositional game logic of finite-state discrete games. Hybrid games are challenging, because their winning regions require closure ordinals $\geq \omega_1^{\text{HG}}$. The logic dGL for hybrid games is, furthermore, more expressive than the corresponding logic dL for hybrid systems. Nevertheless, dGL has a simple modal semantics and a simple proof calculus, which is proved to be a sound and complete axiomatization of hybrid games relative to any expressive logic.

The completeness proof is constructive with minimal coding, thereby exactly characterizing the difficult parts of hybrid games proving. The proof identifies an efficient fixpoint-style proof technique, which can be considered a modal analogue of characterizations in the Calculus of Constructions [CH88], and relates to hybrid game versions of influential views of understanding program invariants as fixpoints [CC77, Cla79]. Relative completeness shows that dGL has all axioms and proof rules for dealing with hybrid games and only games of differential equations themselves are difficult. The study of (fragments of) dGL which are complete for smaller logics is interesting future work. By the schematic completeness result, this reduces to questions of expressiveness that give rise to interesting problems in descriptive set theory.

It is intriguing to observe the overwhelming impact of the innocent addition of a dual operator. Yet, it is reassuring to find that logical robustness makes logical foundations continue to work despite the formidable extra challenges of hybrid games.

The dGL calculus is strikingly similar to the calculus for stochastic differential dynamic logic SdL [Pla11], despite their fundamentally different semantical presuppositions (adversarial nondeterminism versus stochasticity), which indicates the existence of a deeper logical connection relating stochastic and adversarial uncertainty. Because of the axiomatic similarity, the rich theory of dGL may shed light on the logical theory of stochastic hybrid systems, which so far remained elusive.

The logic of hybrid games opens up many directions for future work, including the study of computationally bounded winning strategies, e.g., only polynomial strategies, strategies that are constructible with small closure ordinals, or with finite rank Borel winning regions, as well as an explicit study of constructive dGL to retain the winning strategies as part of the proof. Yet, challenges abound, given the ability of dGL to define closed elementary games won by a player for whom no hyperelementary quasiwinning strategies exist, which follows from Theorem 19 [Mos74, Chapter 7].

Draws, coalitions, rewards, and payoffs different from ± 1 can be expressed in dGL, but it may be useful to include direct syntactical support. Imperfect information games and equivalent concurrent games are interesting but nontrivial extensions that remain challenging even in the discrete case, because imperfect information leads to Henkin quantifiers. The logic dGL can be augmented with differential games as a new kind of atomic games. Thanks to its compositional semantics, this results in a modular construction, yet there are many ways to do that, because there are different notions of differential games. Combining dGL with axioms for differential equations [Pla10a, Pla12a] already provides a way of handling hybrid games with nonlinear differential equations, differential-algebraic inequalities and differential equations with input.

A Example dG$\mathcal{L}$ Proofs

The completeness proof suggests the use of iteration axiom $\langle^*\rangle$ and US to prove $\langle\alpha^*\rangle$ properties. The following examples illustrate how this works in practice. Observe how logic programming saturation with widening quickly proves the resulting arithmetic.

Example 5 (Non-game system). The simple non-game dG$\mathcal{L}$ formula

$$x \geq 0 \rightarrow \langle(x := x-1)^*\rangle 0 \leq x < 1$$

is provable, shown in Fig. 7, where $\langle\alpha^*\rangle 0 \leq x < 1$ is short for $\langle(x := x-1)^*\rangle(0 \leq x < 1)$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\forall x\,(0 \leq x < 1 \lor p(x-1) \rightarrow p(x)) \rightarrow (x \geq 0 \rightarrow p(x))}\mathbb{R}}{\forall x\,(0 \leq x < 1 \lor \langle x := x-1\rangle p(x) \rightarrow p(x)) \rightarrow (x \geq 0 \rightarrow p(x))}\langle:=\rangle}{\forall x\,(0 \leq x < 1 \lor \langle x := x-1\rangle\langle\alpha^*\rangle 0 \leq x < 1 \rightarrow \langle\alpha^*\rangle 0 \leq x < 1) \rightarrow (x \geq 0 \rightarrow \langle\alpha^*\rangle 0 \leq x < 1)}\text{US}}{\forall x\,(0 \leq x < 1 \lor \langle x := x-1\rangle\langle\alpha^*\rangle 0 \leq x < 1 \rightarrow \langle\alpha^*\rangle 0 \leq x < 1)}\langle^*\rangle,\forall}{x \geq 0 \rightarrow \langle\alpha^*\rangle 0 \leq x < 1}\text{MP}
$$

Figure 7: dG$\mathcal{L}$ Angel proof for Example 5 using technique from completeness proof

Example 6 (Choice game). The dG$\mathcal{L}$ formula

$$x = 1 \land a = 1 \rightarrow \langle(x := a; a := 0 \cap x := 0)^*\rangle x \neq 1$$

which comes from (17) on p. 51 is provable as shown in Fig. 8, where $\beta \cap \gamma$ is short for $x := a; a := 0 \cap x := 0$ and $\langle(\beta \cap \gamma)^*\rangle x \neq 1$ short for $\langle(x := a; a := 0 \cap x := 0)^*\rangle x \neq 1$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\forall x\,(x \neq 1 \lor p(a,0) \land p(0,a) \rightarrow p(x,a)) \rightarrow (\mathit{true} \rightarrow p(x,a))}\mathbb{R}}{\forall x\,(x \neq 1 \lor \langle\beta\rangle p(x,a) \land \langle\gamma\rangle p(x,a) \rightarrow p(x,a)) \rightarrow (\mathit{true} \rightarrow p(x,a))}\langle;\rangle,\langle:=\rangle}{\forall x\,(x \neq 1 \lor \langle\beta \cap \gamma\rangle p(x,a) \rightarrow p(x,a)) \rightarrow (\mathit{true} \rightarrow p(x,a))}\langle\cup\rangle,\langle^d\rangle}{\forall x\,(x \neq 1 \lor \langle\beta \cap \gamma\rangle\langle(\beta \cap \gamma)^*\rangle x \neq 1 \rightarrow \langle(\beta \cap \gamma)^*\rangle x \neq 1) \rightarrow (\mathit{true} \rightarrow \langle(\beta \cap \gamma)^*\rangle x \neq 1)}\text{US}}{\mathit{true} \rightarrow \langle(\beta \cap \gamma)^*\rangle x \neq 1}\langle^*\rangle,\forall,\text{MP}}{x = 1 \land a = 1 \rightarrow \langle(\beta \cap \gamma)^*\rangle x \neq 1}\mathbb{R}
$$

Figure 8: dG$\mathcal{L}$ Angel proof for Example 6 using technique from completeness proof

Example 7 (Hybrid game). The dG$\mathcal{L}$ formula

$$\langle(x := 1; x' = 1^d \cup x := x-1)^*\rangle 0 \leq x < 1$$

$$
\begin{array}{ll}
 & * \\
\hline \mathbb{R} & \forall x\,(0 \le x < 1 \lor \forall t{\ge}0\, p(1+t) \lor p(x-1) \to p(x)) \to (\mathit{true} \to p(x)) \\
\hline \langle{:=}\rangle & \forall x\,(0 \le x < 1 \lor \langle x := 1\rangle\neg\exists t{\ge}0\, \langle x := x+t\rangle\neg p(x) \lor p(x-1) \to p(x)) \to (\mathit{true} \to p(x)) \\
\hline \langle{'}\rangle & \forall x\,(0 \le x < 1 \lor \langle x := 1\rangle\neg\langle x' = 1\rangle\neg p(x) \lor p(x-1) \to p(x)) \to (\mathit{true} \to p(x)) \\
\hline \langle;\rangle,\langle^{d}\rangle & \forall x\,(0 \le x < 1 \lor \langle\beta\rangle p(x) \lor \langle\gamma\rangle p(x) \to p(x)) \to (\mathit{true} \to p(x)) \\
\hline \langle\cup\rangle & \forall x\,(0 \le x < 1 \lor \langle\beta\cup\gamma\rangle p(x) \to p(x)) \to (\mathit{true} \to p(x)) \\
\hline \text{US} & \forall x\,(0{\le}x{<}1 \lor \langle\beta\cup\gamma\rangle\langle(\beta\cup\gamma)^*\rangle 0 \le x < 1 \to \langle(\beta\cup\gamma)^*\rangle 0 \le x < 1) \to (\mathit{true} \to \langle(\beta\cup\gamma)^*\rangle 0 \le x < 1) \\
\hline \langle^*\rangle,\forall,\text{MP} & \mathit{true} \to \langle(\beta\cup\gamma)^*\rangle 0 \le x < 1
\end{array}
$$

Figure 9: dGL Angel proof for Example 7 using technique from completeness proof

which comes from (18) on p. 53 is provable as shown in Fig. 9, where the notation $\langle(\beta\cup\gamma)^*\rangle 0 \le x < 1$ is short for $\langle(x := 1; x' = 1^d \cup x := x-1)^*\rangle(0 \le x < 1)$: The proof steps for β use in $\langle{'}\rangle$ that $t \mapsto x+t$ is the solution of the differential equation, so the subsequent use of $\langle{:=}\rangle$ substitutes 1 in for x to obtain $t \mapsto 1+t$. Recall that the winning regions for formula (18) need $>\omega$ iterations to converge. It is still provable easily. A variation of this proof shows dGL formula (2) from p. 9, where the handling of the nonlinear differential equation is a bit more complicated.

A variation of Example 7 proves dGL formula (7) from the proof of Theorem 7, whose closure ordinal is $\omega \cdot 2$.

B Proof of Separating Axioms

This section proves Theorem 16 with an emphasis on simple counterexamples for each separating axiom.

B.0.1 Subnormal Modal Logic

Unlike dL, dGL is not a normal modal logic [HC96]. Axiom K, the modal modus ponens from normal modal logic [HC96], dynamic logic [Pra76], and differential dynamic logic [Pla12a], i.e.

$$[\alpha](\phi \to \psi) \to ([\alpha]\phi \to [\alpha]\psi)$$

is not sound for dGL as witnessed using the choice $\alpha \equiv (x := 1 \cap x := 0); y := 0$ and $\phi \equiv x = 1$, $\psi \equiv y = 1$; see Fig. 10. The global version of K, i.e. the implicative version of Gödel's generalization rule is still sound and derives with $\langle^d\rangle$ and $[\cdot]$ from M using $\alpha \equiv \beta^d$

$$\frac{\phi \to \psi}{[\beta]\phi \to [\beta]\psi}$$

The normal Gödel generalization rule G, i.e.

$$\frac{\phi}{[\alpha]\phi}$$

however, is not sound for dGL as witnessed by the choice $\alpha \equiv (?\mathit{false})^d$, $\phi \equiv \mathit{true}$.

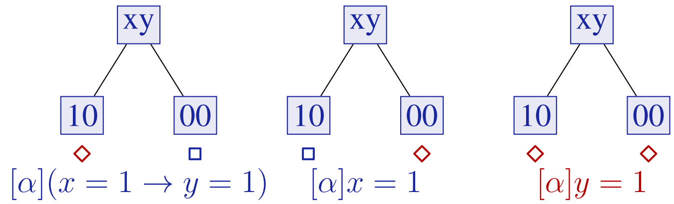

Figure 10: Game trees for counterexample to axiom K using $\alpha \equiv (x := 1 \cap x := 0); y := 0$.

B.0.2 Subregular Modal Logic

Regular modal logics are monotonic modal logics [Che80] that are weaker than normal modal logics. But the regular modal generalization rule [Che80], i.e.

$$\frac{\phi_1 \wedge \phi_2 \to \psi}{[\alpha]\phi_1 \wedge [\alpha]\phi_2 \to [\alpha]\psi}$$

is not sound for dG$\mathcal{L}$ either as witnessed by the choice $\alpha \equiv (x := 1 \cap x := 0); y := 0$, $\phi_1 \equiv x = 1, \phi_2 \equiv x = y, \psi \equiv x = 1 \wedge x = y$; see Fig. 11.

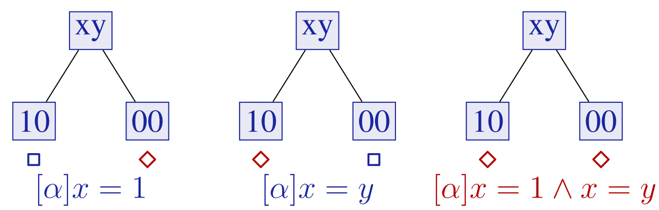

Figure 11: Game trees for counterexample to regular modal rule using $\alpha \equiv (x := 1 \cap x := 0); y := 0$.

B.0.3 Monotonic Modal Logic

The axiom that is closest to K but still sound for dG$\mathcal{L}$ is a monotonicity axiom. This axiom is sound for dG$\mathcal{L}$, yet already included in the monotonicity rule M:

Lemma 20 ([Che80, Theorem 8.13])**.** *In the presence of rule RE from p. 23, rule M is interderivable with axiom M:*

$$\langle\alpha\rangle\phi \vee \langle\alpha\rangle\psi \to \langle\alpha\rangle(\phi \vee \psi)$$

Proof. Axiom M derives from rule M: From $\phi \to \phi \vee \psi$, M derives $\langle\alpha\rangle\phi \to \langle\alpha\rangle(\phi \vee \psi)$. From $\psi \to \phi \vee \psi$, M derives $\langle\alpha\rangle\psi \to \langle\alpha\rangle(\phi \vee \psi)$, from which propositional logic yields $\langle\alpha\rangle\phi \vee \langle\alpha\rangle\psi \to \langle\alpha\rangle(\phi \vee \psi)$.

Conversely, rule M derives from axiom M and rule RE: From $\phi \to \psi$ propositional logic derives $\phi \vee \psi \leftrightarrow \psi$, from which RE derives $\langle\alpha\rangle(\phi \vee \psi) \leftrightarrow \langle\alpha\rangle\psi$. From axiom M, propositional logic, thus, derives $\langle\alpha\rangle\phi \to \langle\alpha\rangle\psi$. □

The converse of axiom M is sound for dL but not for dGL, however, as witnessed by $\alpha \equiv x := 1 \cap x := 0$, $\phi \equiv x = 1$, $\psi \equiv x = 0$; see Fig. 12:

$$\langle\alpha\rangle(\phi \lor \psi) \to \langle\alpha\rangle\phi \lor \langle\alpha\rangle\psi$$

The presence of the regular congruence rule RE and the fact that $[\alpha]\phi \leftrightarrow \neg\langle\alpha\rangle\neg\phi$ still make dGL

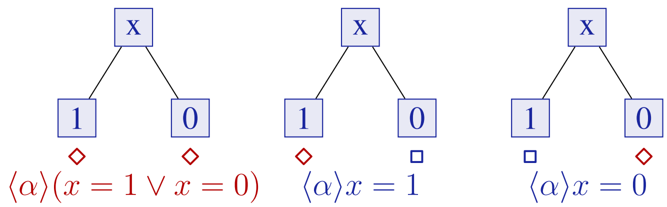

Figure 12: Game trees for counterexample to converse monotone axiom using $\alpha \equiv x := 1 \cap x := 0$.

a classical modal logic [Che80]. Rule M even makes dGL a monotone modal logic [Che80].

B.0.4 Sub-Barcan

The most important axioms about the interaction of quantifiers and modalities in first-order modal logic are the Barcan and converse Barcan axioms [Bar46], which, together, characterize constant domain in normal first-order modal logics [HC96]. The Barcan axiom B, which characterizes anti-monotonic domains in first-order modal logic [HC96], is sound for constant-domain first-order dynamic logic and for differential dynamic logic dL when x does not occur in α [Pla12a]:

$$\langle\alpha\rangle\exists x\, \phi \to \exists x\, \langle\alpha\rangle\phi \qquad (x \notin \alpha)$$

but the Barcan axiom is not sound for dGL as witnessed by the choice $\alpha \equiv y := y + 1^\times$ or $\alpha \equiv y' = 1^d$ and $\phi \equiv x \geq y$. The equivalent Barcan formula

$$\forall x\, [\alpha]\phi \to [\alpha]\forall x\, \phi \qquad (x \notin \alpha)$$

is not sound for dGL as witnessed by the choice $\alpha \equiv y := y + 1^\times$ or $\alpha \equiv y' = 1^d$ and $\phi \equiv y \geq x$. The converse Barcan formula of first-order modal logic, which characterizes monotonic domains [HC96], is sound for dGL and can be derived when x does not occur in α (see 8 on p. 22):

$$(\overleftarrow{\mathrm{B}}) \quad \exists x\, \langle\alpha\rangle\phi \to \langle\alpha\rangle\exists x\, \phi \quad \text{where } x \notin \alpha$$

B.0.5 No Induction Axiom

The induction axiom

$$[\alpha^*](\phi \to [\alpha]\phi) \to (\phi \to [\alpha^*]\phi) \tag{15}$$

holds for dL, but, unlike induction rule ind, does not hold for dGL as witnessed by $\alpha^* \equiv ((x := a; a := 0) \cap x := 0)^*$ and $\phi \equiv x = 1$; see Fig. 13. Note that the failure of the induction

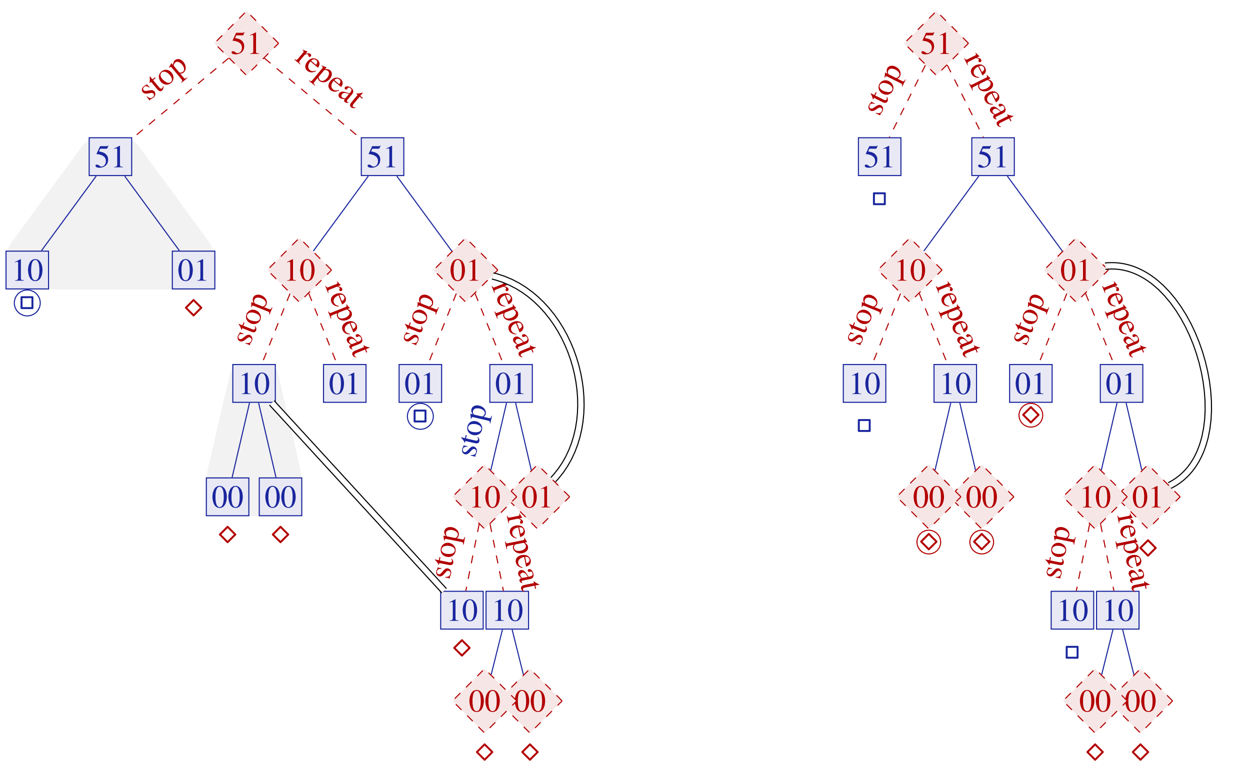

Figure 13: Game trees for counterexample to induction axiom (notation: x, a) with game $\alpha \equiv (x := a; a := 0) \cap x := 0$. **(left)** $[\alpha^*](x = 1 \to [\alpha]x = 1)$ is true by the strategy "if Angel chose stop, choose $x := a; a := 0$, otherwise always choose $x := 0$" **(right)** $[\alpha^*]x = 1$ is false by strategy "repeat once and repeat once more if $x = 1$, then stop." If a winning state can be reached by a winning strategy, the mark is enclosed in a circle ◈ or ▣, respectively.

axiom in the counterexample for (15) hinges on the fact that Angel is free to decide whether or not to repeat α after each round depending on the state. This would be different if dG$\mathcal{L}$ had had an advance notice semantics for α^*; see Appendix D. By a variation of the soundness argument for FP, it can be shown, however, that a variation of the induction axiom is still sound if the induction rule ind is translated into an axiom using the universal closure, denoted $\mathrm{Cl}_\forall$, with respect to all variables bound in α:

$$\mathrm{Cl}_\forall(\phi \to [\alpha]\phi) \to (\phi \to [\alpha^*]\phi)$$

This trick with universal closures does not work for the dual of the induction axiom, the first arrival axiom $\langle\alpha^*\rangle\phi \to \phi \vee \langle\alpha^*\rangle(\neg\phi \wedge \langle\alpha\rangle\phi)$. This axiom holds for d$\mathcal{L}$. It expresses that, if ϕ holds after a repetition of α, then it either holds right away or α can be repeated so that ϕ does not hold yet but can hold after one more repetition [PP03]. This axiom does not hold, however, for dG$\mathcal{L}$ as witnessed by $\alpha^* \equiv ((x := x - y \cap x := 0); y := x)^*$ and $\phi \equiv x = 0$, since two iterations surely yield $x = 0$, but one iteration may or may not yield $x = 0$, depending on Demon's choice; see Fig. 14. Observe how the failure of the first arrival axiom in dG$\mathcal{L}$ relates to the impossibility of predicting precise enough repetition counts in hybrid games (recall corresponding discussions for Theorem 7, Section 4.3, and Appendix D).

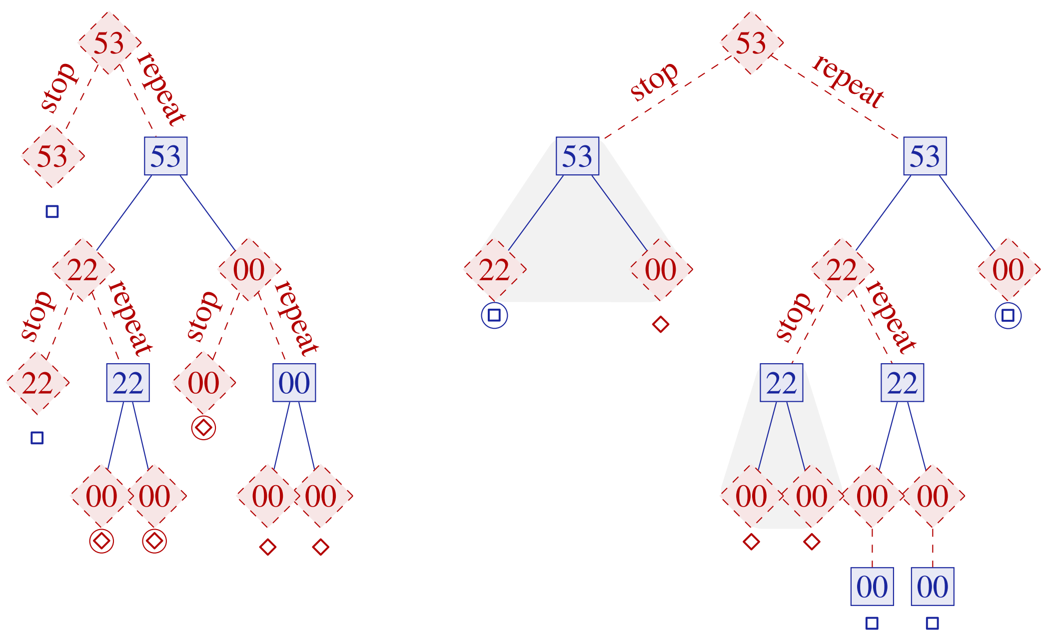

Figure 14: Game trees for counterexample to first arrival axiom with game $\alpha \equiv (x := x - y \cap x := 0); y := x$ (notation: x, y). **(left)** $\langle\alpha^*\rangle x = 0$ is true no matter what Demon chooses **(right)** $\langle\alpha^*\rangle(x \neq 0 \wedge \langle\alpha\rangle x = 0)$ is false, because stop can be defeated by $x := x - y$ and repeat can be defeated by $x := 0$.

C Operational Game Semantics

In order to relate the intuition of interactive game play to the denotational semantics of hybrid games, this section shows an operational semantics for hybrid games that is more complicated than the modal semantics from Section 2.2 but makes strategies explicit and more directly reflects the intuition how hybrid games are played successively. The modal semantics is beneficial, because it is simpler. The results in this section are not needed in the rest of the paper and play an informative role. The operational semantics formalizes the intuition behind the game tree in Fig. 1 and relates to standard notions in game theory and descriptive set theory. Theorem 21 below proves that the operational game semantics is equivalent to the modal semantics from Section 2.2. The (denotational) modal semantics is much simpler but the operational semantics makes winning strategies explicit. As the set of actions A for a hybrid game choose:

$$\{\mathfrak{l}, \mathfrak{r}, \mathfrak{s}, \mathfrak{g}, \mathfrak{d}\} \cup \{(x := \theta) : x \text{ variable, } \theta \text{ term}\} \\ \cup \{(x' = \theta \,\&\, \psi @ r) : x \text{ variable, } \theta \text{ term}, \psi \text{ formula}, r \in \mathbb{R}_{\geq 0}\} \\ \cup \{?\psi : \psi \text{ formula}\}$$

For game $\alpha \cup \beta$, action $\mathfrak{l}$ decides to descend left into α, $\mathfrak{r}$ is the action of descending right into β. In game α^*, action $\mathfrak{s}$ decides to stop repeating, action $\mathfrak{g}$ decides to go back and repeat. Action $\mathfrak{d}$ starts and ends a dual game for α^d. The other actions represent the actions for atomic games: assignment actions, continuous evolution actions (in which time r is the critical decision), and test actions.

The operational game semantics uses standard notions from descriptive set theory [Kec94]. The set of finite sequences of actions is denoted by $A^{(\mathbb{N})}$, the set of countably infinite sequences by $A^{\mathbb{N}}$. The empty sequence of actions is $()$. The concatenation, $s\hat{\ }t$, of sequences $s, t \in A^{(\mathbb{N})}$ is defined as $(s_1, \ldots, s_n, t_1, \ldots, t_m)$ if $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_m)$. For an $a \in A$, write $a\hat{\ }t$ for $(a)\hat{\ }t$ and write $t\hat{\ }a$ for $t\hat{\ }(a)$. For a set $S \subseteq A^{(\mathbb{N})}$, write $S\hat{\ }t$ for $\{s\hat{\ }t : s \in S\}$ and $t\hat{\ }S$ for $\{t\hat{\ }s : s \in S\}$. The state $\lfloor t \rfloor_s$ reached by *playing* a sequence of actions $t \in A^{(\mathbb{N})}$ from a state s in interpretation I is inductively defined by applying the actions sequentially, i.e. as follows:

1. $\lfloor x := \theta \rfloor_s = s_x^{[\![\theta]\!]_s}$
2. $\lfloor x' = \theta \,\&\, \psi @ r \rfloor_s = \varphi(r)$ for the unique $\varphi : [0, r] \to \mathcal{S}$ differentiable, $\varphi(0) = s$, $\frac{\mathrm{d}\,\varphi(t)(x)}{\mathrm{d}t}(\zeta) = [\![\theta]\!]_{\varphi(\zeta)}$ and $\varphi(\zeta) \in [\![\psi]\!]^I$ for all $\zeta \leq r$. Note that $\lfloor x' = \theta \,\&\, \psi @ r \rfloor_s$ is not defined if no such φ exists.
3. $\lfloor ?\psi \rfloor_s = \begin{cases} s & \text{if } s \in [\![\psi]\!]^I \\ \text{not defined} & \text{otherwise} \end{cases}$
4. $\lfloor \mathfrak{l} \rfloor_s = \lfloor \mathfrak{r} \rfloor_s = \lfloor \mathfrak{s} \rfloor_s = \lfloor \mathfrak{g} \rfloor_s = \lfloor \mathfrak{d} \rfloor_s = \lfloor () \rfloor_s = s$
5. $\lfloor a\hat{\ }t \rfloor_s = \lfloor t \rfloor_{(\lfloor a \rfloor_s)}$ for $a \in A$ and $t \in A^{(\mathbb{N})}$

A *tree* is a set $T \subseteq A^{(\mathbb{N})}$ that is closed under prefixes, that is, whenever $t \in T$ and s is a prefix of t (i.e. $t = s\hat{\ }r$ for some $r \in A^{(\mathbb{N})}$), then $s \in T$. A node $t \in T$ is a successor of node $s \in T$ iff

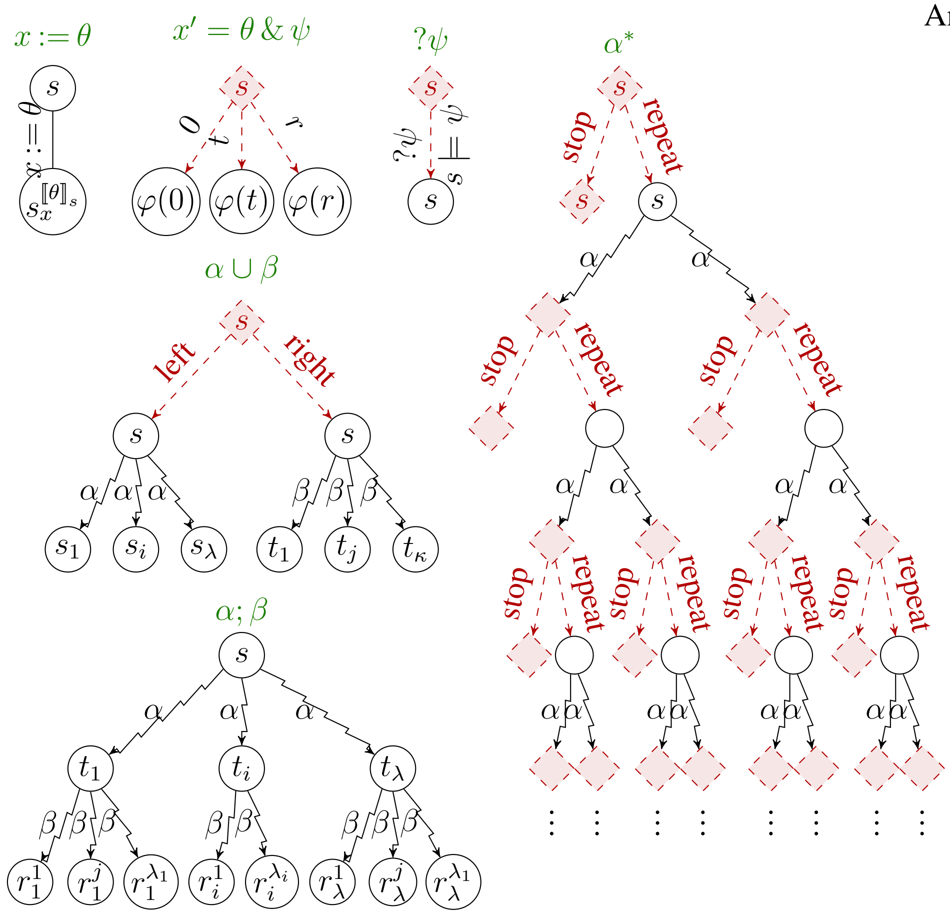

Figure 15: Operational game semantics for hybrid games of dG$\mathcal{L}$

$t = s\hat{\ }a$ for some $a \in A$. Denote by $\mathrm{leaf}(T)$ the set of all leaves of T, i.e. nodes $t \in T$ that have no successor in T.

Definition 7 (Operational game semantics)**.** The *operational game semantics* of hybrid game α is, for each state s of each interpretation I, a tree $\mathsf{g}(\alpha)(s) \subseteq A^{(\mathbb{N})}$ defined as follows (see Fig. 15 for a schematic illustration):

1. $\mathsf{g}(x := \theta)(s) = \{(x := \theta)\}$
2. $\mathsf{g}(x' = \theta \,\&\, \psi)(s) = \{(x' = \theta \,\&\, \psi @ r) : r \in \mathbb{R}, r \geq 0, \varphi(0) = s$ for some (differentiable) $\varphi : [0, r] \to \mathcal{S}$ such that $\frac{\mathsf{d}\,\varphi(t)(x)}{\mathsf{d}t}(\zeta) = [\![\theta]\!]_{\varphi(\zeta)}$ and $\varphi(\zeta) \in [\![\psi]\!]^I$ for all $\zeta \leq r\}$
3. $\mathsf{g}(?\psi)(s) = \{(?\psi)\}$
4. $\mathsf{g}(\alpha \cup \beta)(s) = \mathfrak{l}\hat{\ }\mathsf{g}(\alpha)(s) \cup \mathfrak{r}\hat{\ }\mathsf{g}(\beta)(s)$
5. $\mathsf{g}(\alpha; \beta)(s) = \mathsf{g}(\alpha)(s) \cup \bigcup_{t \in \mathrm{leaf}(\mathsf{g}(\alpha)(s))} \mathsf{g}(\beta)(\lfloor t \rfloor_s)$

6. $\mathrm{g}(\alpha^*)(s) = \bigcup_{n<\omega} f^n(\{(\mathfrak{s}),(\mathfrak{g})\})$
   where f^n is the n-fold composition of the function
   $f(Z) \stackrel{\text{def}}{=} Z \cup \bigcup_{t\hat{}\mathfrak{g}\in\mathrm{leaf}(Z)} t\hat{}\mathfrak{g}\hat{}\mathrm{g}(\alpha)(\lfloor t\hat{}\mathfrak{g}\rfloor_s)\hat{}\{(\mathfrak{s}),(\mathfrak{g})\}$

7. $\mathrm{g}(\alpha^d)(s) = \mathfrak{d}\hat{}\mathrm{g}(\alpha)(s)\hat{}\mathfrak{d}$

Note the implicit closure under prefixes in the definition of $\mathrm{g}(\alpha)(s)$ for readability reasons. For example, $\mathrm{g}(\alpha^d)(s) = \mathfrak{d}\hat{}\mathrm{g}(\alpha)(s)\hat{}\mathfrak{d}$ means $\mathrm{g}(\alpha^d)(s) = \{(),(\mathfrak{d})\} \cup \mathfrak{d}\hat{}\mathrm{g}(\alpha)(s) \cup \mathfrak{d}\hat{}\mathrm{g}(\alpha)(s)\hat{}\mathfrak{d}$.

Angel gets to choose which action to take at node $t \in \mathrm{g}(\alpha)(s)$ if t has an even number of occurrences of $\mathfrak{d}$, otherwise Demon gets to choose. In the former case *Angel acts at* t, in the latter *Demon acts at* t. Thus, at every t, exactly one of the players acts at t. If the player who acts at t is deadlocked, then that player loses immediately. A player who acts at $t \in \mathrm{g}(\alpha)(s)$ is *deadlocked* at t if $t \notin \mathrm{leaf}(\mathrm{g}(\alpha)(s))$ and no successor s is enabled, i.e. $\lfloor s\rfloor_s$ is not defined. This can happen if the last action in s has a condition that is not satisfied like $?x \geq 0$ or $x' = \theta \,\&\, x \geq 0$ at a state where $x < 0$. Note that the player who acts at $t \in \mathrm{g}(\alpha^*)(s)$ cannot choose $\mathfrak{g}$ infinitely often for that loop.

A *strategy for Angel* from initial state s is a nonempty subtree $\sigma \subseteq \mathrm{g}(\alpha)(s)$ such that

1. for all $t \in \sigma$ at which Demon acts, $t\hat{}a \in \sigma$ for all $a \in A$ such that $t\hat{}a \in \mathrm{g}(\alpha)(s)$.
2. for all $t \in \sigma$ at which Angel acts, if $t \notin \mathrm{leaf}(\mathrm{g}(\alpha)(s))$, then there is a unique $a \in A$ with $t\hat{}a \in \sigma$.

Strategies for Demon are defined accordingly, with "Angel" and "Demon" swapped. The action sequence $\sigma \oplus \tau$ played from state s in interpretation I when Angel plays strategy σ and Demon plays strategy τ from s is defined as the sequence $(a_1,\dots,a_n) \in A^{(\mathbb{N})}$ of maximal length such that

$$a_{n+1} := \begin{cases} a & \text{if Angel acts at } (a_1,\dots,a_n) \\ & \text{and } (a_1,\dots,a_n)\hat{}a \in \sigma \\ a & \text{if Demon acts at } (a_1,\dots,a_n) \\ & \text{and } (a_1,\dots,a_n)\hat{}a \in \tau \\ \text{not defined} & \text{otherwise} \end{cases}$$

By definition of a strategy for Angel/Demon, the a is unique. A *winning strategy for Angel* for winning condition $X \subseteq \mathcal{S}$ from state s in interpretation I is a strategy $\sigma \subseteq \mathrm{g}(\alpha)(s)$ for Angel from s such that, for all strategies $\tau \subseteq \mathrm{g}(\alpha)(s)$ for Demon from s: Demon deadlocks or $\lfloor \sigma \oplus \tau\rfloor_s \in X$. A *winning strategy for Demon* for (Demon's) winning condition $X \subseteq \mathcal{S}$ from state s in interpretation I is a strategy $\tau \subseteq \mathrm{g}(\alpha)(s)$ for Demon from s such that, for all strategies $\sigma \subseteq \mathrm{g}(\alpha)(s)$ for Angel from s: Angel deadlocks or $\lfloor \sigma \oplus \tau\rfloor_s \in X$.

The denotational modal semantics from Section 2.2 is equivalent to the operational semantics:

Theorem 21 (Equivalent semantics)**.** *The modal semantics of* dGL *is equivalent to the game tree operational semantics of* dGL*, i.e. for each hybrid game α, each initial state s in each interpretation I, and each winning condition $X \subseteq \mathcal{S}$:*

$$s \in \varsigma_\alpha(X) \iff \textit{there is a winning strategy } \sigma \subseteq \mathsf{g}(\alpha)(s) \textit{ for Angel to achieve } X \textit{ from } s$$
$$s \in \delta_\alpha(X^\complement) \iff \textit{there is a winning strategy } \tau \subseteq \mathsf{g}(\alpha)(s) \textit{ for Demon to achieve } X^\complement \textit{ from } s$$

Proof. Proceed by simultaneous induction on the structure of α and prove equivalence. As part of the equivalence proof, construct a winning strategy σ achieving X using that $s \in \varsigma_\alpha(X)$. The simultaneous induction steps for $\delta_\alpha(X^\complement)$ are simple dualities, except for the case of α^*. It is easy to see that Angel and Demon cannot both have a winning strategy from the same state s for complementary winning conditions X and $X^\complement$ in the same game $\mathsf{g}(\alpha)(s)$. Theorem 2 implies $\delta_\alpha(X^\complement) = \varsigma_\alpha(X)^\complement$.

1. $s \in \varsigma_{x:=\theta}(X) \iff s_x^{[\![\theta]\!]_s} \in X \iff \lfloor \sigma \oplus \tau \rfloor_s = \lfloor x := \theta \rfloor_s = s_x^{[\![\theta]\!]_s} \in X$, using $\sigma \stackrel{\text{def}}{=} \{(x := \theta)\} = \mathsf{g}(x := \theta)(s)$. The converse direction follows, because the strategy σ follows the only permitted strategy.

2. $s \in \varsigma_{x'=\theta \,\&\, \psi}(X) \iff s = \varphi(0), \varphi(r) \in X$ for some $r \in \mathbb{R}$ and some (differentiable) $\varphi : [0, r] \to \mathcal{S}$ such that $\frac{\mathsf{d}\,\varphi(t)(x)}{\mathsf{d}t}(\zeta) = [\![\theta]\!]_{\varphi(\zeta)}$ and $\varphi(\zeta) \in [\![\psi]\!]^I$ for all $\zeta \leq r \iff \lfloor \sigma \oplus \tau \rfloor_s = \lfloor x' = \theta \,\&\, \psi @ r \rfloor_s = \varphi(r) \in X$, using $\sigma \stackrel{\text{def}}{=} \{(x' = \theta \,\&\, \psi @ r)\} \subseteq \mathsf{g}(x' = \theta \,\&\, \psi)(s)$. The converse direction follows, because this σ has the only permitted form for a strategy where different values of r that lead to X are equivalently useful.

3. $s \in \varsigma_{?\psi}(X) = [\![\psi]\!]^I \cap X \iff \lfloor \sigma \oplus \tau \rfloor_s = \lfloor ?\psi \rfloor_s = s \in X$, with $s \in [\![\psi]\!]^I$ using $\sigma \stackrel{\text{def}}{=} \{(?\psi)\} = \mathsf{g}(?\psi)(s)$. The converse direction uses that this σ is the only permitted strategy and it deadlocks exactly if $s \notin [\![\psi]\!]^I$.

4. $s \in \varsigma_{\alpha \cup \beta}(X) = \varsigma_\alpha(X) \cup \varsigma_\beta(X) \iff s \in \varsigma_\alpha(X)$ or $s \in \varsigma_\beta(X)$. By induction hypothesis, this is equivalent to: there is a winning strategy $\sigma_\alpha \subseteq \mathsf{g}(\alpha)(s)$ for Angel for X from s or there is a winning strategy $\sigma_\beta \subseteq \mathsf{g}(\beta)(s)$ for Angel for X from s. This is equivalent to $\sigma \subseteq \mathsf{g}(\alpha \cup \beta)(s)$ being a winning strategy for Angel for X from s, using either $\sigma \stackrel{\text{def}}{=} \mathfrak{l}\hat{\ }\sigma_\alpha$ or $\sigma \stackrel{\text{def}}{=} \mathfrak{r}\hat{\ }\sigma_\beta$.

5. $s \in \varsigma_{\alpha;\beta}(X) = \varsigma_\alpha(\varsigma_\beta(X))$. By induction hypothesis, this is equivalent to the existence of a strategy $\sigma_\alpha \subseteq \mathsf{g}(\alpha)(s)$ for Angel such that for all strategies $\tau \subseteq \mathsf{g}(\alpha)(s)$ for Demon: $\lfloor \sigma_\alpha \oplus \tau \rfloor_s \in \varsigma_\beta(X)$. By induction hypothesis, $\lfloor \sigma_\alpha \oplus \tau \rfloor_s \in \varsigma_\beta(X)$ is equivalent to the existence of a winning strategy σ_τ for Angel (which depends on the state $\lfloor \sigma_\alpha \oplus \tau \rfloor_s$ that the previous α game led to) with winning condition X from $\lfloor \sigma_\alpha \oplus \tau \rfloor_s$. This is equivalent to $\sigma \subseteq \mathsf{g}(\alpha;\beta)(s)$ being a winning strategy for Angel for X from s, using

$$\sigma \stackrel{\text{def}}{=} \sigma_\alpha \cup \bigcup (\sigma_\alpha \oplus \tau)\hat{\ }\sigma_\tau \tag{16}$$

The union is over all leaves $\sigma_\alpha \oplus \tau \in \mathrm{leaf}(\mathtt{g}(\alpha)(s))$ for which the game is not won by a player yet. Note that σ is a winning strategy for X, because, for all plays for which the game is decided during α, the strategy σ_α already wins the game. For the others, σ_τ wins the game from the respective state $\lfloor \sigma_\alpha \oplus \tau \rfloor_s$ that was reached by the actions $\sigma_\alpha \oplus \tau$ according to the strategy τ that Demon was observed (when α terminates) to have played during α. The converse direction uses that strategies do not depend on moves that have not been played yet and that any strategy can be factorized by prefixes of what has actually been played to be coerced into the form (16).

6. Both inclusions of the case α^* are proved separately. If W denotes the set of states from which Angel has a winning strategy in $\mathtt{g}(\alpha^*)(s)$ to achieve X, then need to show that $\varsigma_{\alpha^*}(X) = W$. For $\varsigma_{\alpha^*}(X) \subseteq W$, it is enough to show that W is a pre-fixpoint, i.e. $X \cup \varsigma_\alpha(W) \subseteq W$, because $\varsigma_{\alpha^*}(X)$ is the least (pre-)fixpoint. Consider any $s \in X \cup \varsigma_\alpha(W) \subseteq W$. If $s \in X$ then $s \in W$ with the winning strategy $\sigma \stackrel{\text{def}}{=} \{(\mathfrak{s})\}$ for Angel to achieve X in α^* from s. Otherwise, $s \in \varsigma_\alpha(W) \subseteq W$ implies, by induction hypothesis, that there is a winning strategy $\sigma_\alpha \subseteq \mathtt{g}(\alpha)(s)$ for Angel in α to achieve W from s. By definition of W, Angel has a winning strategy in $\mathtt{g}(\alpha^*)(s)$ to achieve X from all states reached after playing α from s according to σ_α, i.e. $\lfloor \sigma_\alpha \oplus \tau \rfloor_s \in W$ for all strategies τ of Demon. Thus, by composing σ_α with the respective (state-dependent) winning strategies σ_τ for all possible resulting states (which are all in W) corresponding to the respective possible strategies τ that Demon could play during the first α, a winning strategy is obtained of the form

$$\sigma \stackrel{\text{def}}{=} \mathfrak{g}\hat{}\sigma_\alpha \cup \bigcup \mathfrak{g}\hat{}(\sigma_\alpha \oplus \tau)\hat{}\sigma_\tau$$

for Angel to achieve X in α^* from s, where the union is over all leaves $\sigma_\alpha \oplus \tau \in \mathrm{leaf}(\mathtt{g}(\alpha)(s))$ in any strategy τ of Demon for which the game is not won by a player yet during the first α.

The converse inclusion $\varsigma_{\alpha^*}(X) \supseteq W$ is equivalent to $\varsigma_{\alpha^*}(X)^\complement \subseteq W^\complement$. For this, recall $\varsigma_{\alpha^*}(X)^\complement = \delta_{\alpha^*}(X^\complement) = \bigcup\{Z \subseteq \mathcal{S} : Z \subseteq X^\complement \cap \delta_\alpha(Z)\}$ by Theorem 2. Thus, since $\varsigma_{\alpha^*}(X)^\complement$ is a greatest (post-)fixpoint, it is enough to show $Z \subseteq W^\complement$ for all Z with $Z \subseteq X^\complement \cap \delta_\alpha(Z)$. Since, $Z \subseteq \delta_\alpha(Z)$, Demon has a winning strategy in α to achieve Z from all $s \in Z$, by induction hypothesis. By composing the respective winning strategies for Demon, obtain a winning strategy τ for Demon to achieve Z in α^* for *any* number of repetitions that Angel chooses (recall that Angel cannot choose to repeat α^* infinitely often to win). Since $Z \subseteq X^\complement$, Angel cannot have a winning strategy to achieve X in α^* from any $s \in Z$ by Theorem 2. Thus, $Z \subseteq W^\complement$.

7. $s \in \varsigma_{\alpha^d}(X) = \varsigma_\alpha(X^\complement)^\complement. \iff s \notin \varsigma_\alpha(X^\complement)$. By induction hypothesis, this is equivalent to: there is no winning strategy $\sigma \subseteq \mathtt{g}(\alpha)(s)$ for Angel winning $X^\complement$ in α from s. Since $\varsigma_{\alpha^d}(X) = \delta_\alpha(X)$ by Theorem 2, this is equivalent to: there is a winning strategy $\tau \subseteq \mathtt{g}(\alpha)(s)$ for Demon winning X in α from s. Since the nodes where Angel acts swap with the nodes where Demon acts when moving from α to α^d, this is equivalent to: there is a winning strategy $\sigma \subseteq \mathtt{g}(\alpha^d)(s)$ for Angel winning X in α^d from s using $\sigma \stackrel{\text{def}}{=} \mathfrak{d}\hat{}\tau\hat{}\mathfrak{d}$. The converse direction uses that all strategies permitted for α^d begin and end with $\mathfrak{d}$. □

D Alternative Semantics

To see why the dGL semantics is both natural and general, briefly consider alternative choices for the semantics, focusing on the role of repetition in the context of hybrid games. It turns out that alternative semantics require prior bounds of repetitions of $<\omega$ (Appendix D.1) and ω (Appendix D.2), respectively.

D.1 Advance Notice Semantics

One alternative semantics is the *advance notice semantics* for α^*, which requires the players to announce the number of times that game α will be repeated when the loop begins. The advance notice semantics defines $\varsigma_{\alpha^*}(X)$ as $\bigcup_{n<\omega} \varsigma_{\alpha^n}(X)$ where $\alpha^{n+1} \equiv \alpha^n; \alpha$ and $\alpha^0 \equiv ?true$ and defines $\delta_{\alpha^*}(X)$ as $\bigcap_{n<\omega} \delta_{\alpha^n}(X)$. When playing α^*, Angel, thus, announces to Demon how many repetitions n are going to be played when the game α^* begins and Demon announces how often to repeat $\alpha^\times$. This advance notice makes it easier for Demon to win loops α^* and easier for Angel to win loops $\alpha^\times$, because the opponent announces an important feature of their strategy immediately as opposed to revealing whether or not to repeat the game once more one iteration at a time as in Def. 4. Angel announces the number $n < \omega$ of repetitions when α^* starts.

In hybrid systems, the advance notice semantics and the least fixpoint semantics are equivalent (Lemma 6), but the advance notice semantics and dGL's least fixpoint semantics are different for hybrid games. The following formula is valid in dGL (see Fig. 16), but would not be valid in the advance notice semantics:

$$x = 1 \land a = 1 \to \langle((x := a; a := 0) \cap x := 0)^*\rangle x \neq 1 \tag{17}$$

If, in the advance notice semantics, Angel announces that she has chosen n repetitions of the game, then Demon wins (for $a \neq 0$) by choosing the $x := 0$ option $n - 1$ times followed by one choice of $x := a; a := 0$ in the last repetition. This strategy would not work in the dGL semantics, because Angel is free to decide whether to repeat α^* after each repetition based on the resulting state of the game.

Conversely, the dual formula would be valid in the advance notice semantics but is not valid in dGL:

$$x = 1 \land a = 1 \to [((x := a; a := 0) \cap x := 0)^*]x = 1$$

The dGL semantics is more general, because it gives the player in charge of repetition more control as the state can be inspected before deciding on whether to repeat again. Advance notice semantics, instead, only allows the choice of a fixed number of repetitions. The advance notice games can be expressed easily in dGL by having the players choose a counter c before the loop that decreases to 0 during the repetition. The advance notice semantics can be expressed in dGL, e.g., for (17) as

$$x = 1 \land a = 1 \to \langle c := 0; c := c + 1^*; (((x := a; a := 0) \cap x := 0); c := c - 1)^*; ?c = 0\rangle x \neq 1$$

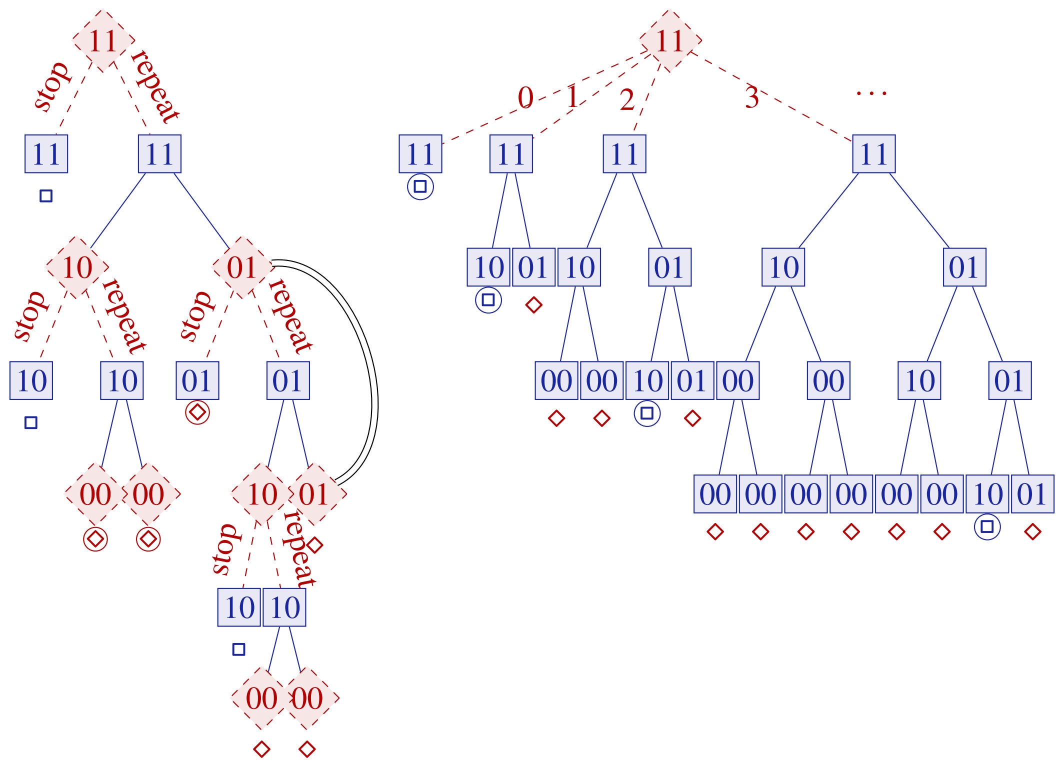

Figure 16: Game trees for $x = 1 \wedge a = 1 \rightarrow \langle \alpha^* \rangle x \neq 1$ with game $\alpha \equiv (x := a; a := 0) \cap x := 0$ (notation: x, a). **(left)** valid in dG$\mathcal{L}$ by strategy "repeat once and repeat once more if $x = 1$, then stop" **(right)** false in advance notice semantics by the strategy "$n - 1$ choices of $x := 0$ followed by $x := a; a := 0$ once", where n is the number of repetitions Angel announced

D.2 ω-Strategic Semantics

Another alternative choice for the semantics would have been to allow only arbitrary finite iterations of the strategy function for computing the winning region by using the *ω-strategic semantics*, which defines $\varsigma_{\alpha^*}(X)$ as $\varsigma_\alpha^\omega(X) = \bigcup_{n<\omega} \varsigma_\alpha^n(X)$ along with a corresponding definition for $\delta_{\alpha^*}(X)$. Like the dG$\mathcal{L}$ semantics, but quite unlike the advance notice semantics, the ω-strategic semantics does not require Angel to disclose how often she is going to repeat when playing α^*. Similarly, Demon does not have to announce how often to repeat when playing $\alpha^\times$. Nevertheless, the semantics are different. The ω-strategic semantics would make the following valid dG$\mathcal{L}$ formula invalid:

$$\langle (x := 1; x' = 1^d \cup x := x - 1)^* \rangle \, (0 \le x < 1) \tag{18}$$

By a simple variation of the argument in the proof of Theorem 7, $\varsigma_\alpha^\omega([0,1)) = [0,\infty)$, because $\varsigma_\alpha^n([0,1)) = [0,n)$ for all $n \in \mathbb{N}$. Yet, this ω-level of iteration of the strategy function for winning regions misses out on the perfectly reasonable winning strategy "first choose $x := 1; x' = 1^d$ and then always choose $x := x - 1$ until stopping at $0 \le x < 1$". The existence of this winning strategy is only found at the level $\varsigma_\alpha^{\omega+1}([0,1)) = \varsigma_\alpha([0,\infty)) = \mathbb{R}$. Even though any particular use of the winning strategy in any game play uses only some finite number of repetitions of the loop, the argument why it will always work requires $> \omega$ many iterations of $\varsigma_\alpha(\cdot)$, because Demon can change x to an arbitrarily big value, so that ω many iterations of $\varsigma_\alpha(\cdot)$ are needed to conclude that Angel has a winning strategy for any positive value of x. There is no smaller upper bound on the number of iterations it takes Angel to win, in particular Angel cannot promise ω as a bound on the repetition count, which is what the ω-semantics would effectively require her to do. Bu strategies do converge after $\omega + 1$ iterations. According to Theorem 7, the same shortcomings of the ω-semantics apply at higher transfinite closure ordinals.

The dG$\mathcal{L}$ semantics is also more general, because, by Theorem 7, its closure ordinal is $\geq \omega_1^{\text{CK}}$, in contrast to the ω-semantics, which has closure ordinal ω by construction. The same observation shows a fundamental difference between the dG$\mathcal{L}$ semantics and the advance notice semantics, which has closure ordinal $\le \omega$.

E Proof of Higher Closure Ordinals

of Theorem 7. In this proof, proceed in stages of increasing difficulty. That the closure ordinal is $\geq \omega \cdot 2$ has already been shown on p. 18. Now prove the bounds $\geq \omega^2$ and finally $\geq \omega^\omega$. In order to see that the closure ordinal is at least ω^2 even for a single nesting layer of dual and loop, follow a similar argument using more variables. Consider the family of formulas (for some $N \in \mathbb{N}$) of the form

$$\langle \big(\underbrace{x_N := x_N - 1; x'_{N-1} = 1^d \cup \ldots \cup x_2 := x_2 - 1; x'_1 = 1^d \cup x_1 := x_1 - 1}_{\alpha} \big)^* \rangle \bigwedge_{i=1}^{N} x_i < 0$$

The winning regions for this dG$\mathcal{L}$ formula stabilizes after $\omega \cdot N$ iterations, because ω many iterations are necessary to show that *any* x_1 can be reduced to $(-\infty, 0)$ by choosing the last action sufficiently often, whereas another ω many iterations are needed to show that x_2 can then be reduced to $(-\infty, 0)$ by choosing the second-to-last action sufficiently often, increasing x_1 arbitrarily under Demon's control, which can still be won because this adversarial increase in x_1 can be compensated for by the first part of the winning strategy. The vector space of variables $(x_N, \dots, x_1)$ is used in that order. It is easy to see that $\varsigma^\omega(\alpha)(-\infty,0)^N = \bigcup_{n<\omega} \varsigma^n(\alpha)(-\infty,0)^N = (-\infty,0)^{N-1} \times \mathbb{R}$, because $\varsigma^{n+1}(\alpha)(-\infty,0) = (-\infty,0)^{N-1} \times (-\infty,n)$ holds for all $n \in \mathbb{N}, n$ by a simple inductive argument:

$$\begin{aligned}
\varsigma^1(\alpha)(-\infty,0)^N &= (-\infty,0)^N \\
\varsigma^{n+1}(\alpha)(-\infty,0)^N &= (-\infty,0)^N \cup \varsigma_\alpha(\varsigma^n(\alpha)(-\infty,0)^N) = (-\infty,0)^N \cup \varsigma_\alpha((-\infty,0)^{N-1} \times (-\infty,n-1)) \\
&= (-\infty,0)^{N-1} \times (-\infty,n)
\end{aligned}$$

Inductively, $\varsigma^{\omega\cdot(k+1)}(\alpha)(-\infty,0)^N = \bigcup_{n<\omega} \varsigma^{\omega\cdot k+n}(\alpha)(-\infty,0)^N = (-\infty,0)^{N-k-1} \times \mathbb{R}^{k+1}$, because $\varsigma^{\omega\cdot k+n+1}(\alpha)(-\infty,0) = (-\infty,0)^{N-k-1} \times (-\infty,n) \times \mathbb{R}^k$ holds for all $n \in \mathbb{N}$ by a simple inductive argument:

$$\begin{aligned}
\varsigma^{\omega\cdot k+n+1}(\alpha)(-\infty,0)^N &= (-\infty,0)^N \cup \varsigma_\alpha(\varsigma^{\omega\cdot k+n}(\alpha)(-\infty,0)^N) \\
&= (-\infty,0)^N \cup \varsigma_\alpha((-\infty,0)^{N-k-1} \times (-\infty,n-1) \times \mathbb{R}^k) \\
&= (-\infty,0)^{N-k-1} \times (-\infty,n) \times \mathbb{R}^k
\end{aligned}$$

Consequently, $\varsigma_{\alpha^*}((-\infty,0)^N) = \varsigma^{\omega\cdot N}(\alpha)(-\infty,0)^N \neq \varsigma^{\omega\cdot(N-1)+n}(\alpha)(-\infty,0)^N$, which makes $\omega \cdot N$ the closure ordinal for α. Since hybrid games α of the above form can be considered with arbitrarily big $N \in \mathbb{N}$, the common closure ordinal has to be $\geq \omega \cdot N$ for all $N \in \mathbb{N}$, i.e. it has to be $\geq \omega^2$.

In order to see that the closure ordinal is at least ω^ω, follow an argument expanding on the previous case. Consider the family of formulas (for some $N \in \mathbb{N}$) of the form

$$\langle(\underbrace{?x_{N-1} < 0; x'_{N-1} = 1^d; x_N := x_N - 1 \cup \ldots \cup ?x_1 < 0; x'_1 = 1^d; x_2 := x_2 - 1 \cup x_1 := x_1 - 1}_{\alpha})^*\rangle \bigwedge_{i=1}^{N} x_i < 0$$

The winning regions for this "clockwork ω" formula stabilizes after ω^N iterations, ω many iterations are necessary to show that *any* x_1 can be reduced to $(-\infty, 0)$ by choosing the last action sufficiently often, whereas another ω many iterations are needed to show that x_2 can then be reduced to $(-\infty, 0)$ by choosing the second-to-last action sufficiently often in case x_1 has already been reduced to $(-\infty, 0)$. Every time the second-to-last action is chosen, however, Demon increases x_1 arbitrarily, which again takes ω many steps of the last action to understand how x_1 can again be reduced to $(-\infty, 0)$ before the second-to-last action can be chosen again to decrease x_2 further. This phenomenon that ω many actions on x_{i-1} are needed before x_i can be decreased by 1 holds for all i recursively. Note that in any particular game play, Demon can only increase x_i by some finite amount. But Angel does not have a finite bound on that increment, so she will first have to convince herself that she has a winning strategy that could tolerate any change in x_i, which takes ω many iterations of the previous argument.

The vector space of variables $(x_N, \dots, x_1)$ is used in that order. For $b_N, \dots, b_1 \in \mathbb{N} \cup \{\infty\}$, use the short hand notation

$$b_N \dots b_2 b_1 \overset{\text{def}}{=} (-\infty, b_N) \times \dots \times (-\infty, b_2) \times (-\infty, b_1)$$

and write b_i^n for $(-\infty, b_i)^n$ in that context. Let $\vec{b} = (b_N, \dots, b_1)$. Prove that $\forall n \in \mathbb{N}\ \forall j \in \mathbb{N}, j > 0$

$$\begin{aligned}
\varsigma^{\omega^j(n+1)}(\alpha) b_N \dots b_j \dots b_1 &= b_N \dots (b_{j+1} + n)\infty^j && \text{if ① } b_N, \dots, b_j < \infty, j > 0\\
\varsigma^{\omega^j(n+1)}(\alpha) b_N \dots b_{j+1}\infty^j &= b_N \dots (b_{j+1} + n + 1)\infty^j && \text{if ② } b_N, \dots, b_{j+1} < \infty, b_j = \infty = \dots b_1\\
\varsigma^{\omega^j(n+1)}(\alpha) b_N \dots b_{k+1}\infty^{k-j}\infty^j &= b_N \dots (b_{k+1} + 1)1^{k-j-1}(n+1)\infty^j \cup \vec{b} && \text{if ③ } b_N, \dots, b_{k+1} < \infty, b_k = \infty, k > j
\end{aligned}$$

by induction on the lexicographical order of j and n. Note that, in the case ③, there are some subordinate cases which do not need to be tracked in this analysis, because they are strategic dead ends. IH is short for induction hypothesis.

The base case $j = 0, n = 0$ is vacuous for ① and can be checked easily for ②.

$$\begin{aligned}
\varsigma^{\omega^0 1}(\alpha) b_N \dots b_1 \infty^0 &= \varsigma^1(\alpha) b_N \dots b_1 = b_N \dots (b_1 + 1) = b_N \dots (b_1 + 1)\infty^0\\
\varsigma^{\omega^0(n+1)}(\alpha) b_N \dots b_1 \infty^0 &= \vec{b} \cup \varsigma(\alpha)\varsigma^n(\alpha) b_N \dots b_1 = \vec{b} \cup \varsigma(\alpha) b_N \dots (b_1 + n) = b_N \dots (b_1 + n + 1)
\end{aligned}$$

For ③, the case $j = 0$ holds only after an extra offset k, however:

$$\begin{aligned}
\varsigma^1(\alpha) b_N \dots b_{k+1}\infty^k &= \vec{b} \cup b_N \dots (b_{k+1} + 1)0\infty^{k-1}\\
\varsigma^{n+1}(\alpha) b_N \dots b_{k+1}\infty^k &= \varsigma^n(\alpha) b_N \dots b_{k+1}\infty^k \cup b_N \dots (b_{k+1} + 1)1^n 0\infty^{k-n-1} && \text{for } n < k\\
\varsigma^{k+n+1}(\alpha) b_N \dots b_{k+1}\infty^k &= \varsigma^{k+n}(\alpha) b_N \dots b_{k+1}\infty^k \cup b_N \dots (b_{k+1} + 1)1^{k-1}(n+1)
\end{aligned}$$

So instead, prove base case $j = 1, n = 0$, because the finite extra offset k has been overcome at ω:

$$\begin{aligned}
\varsigma^{\omega^1 1}(\alpha) b_N \dots b_1 &= \bigcup_{n<\omega} \varsigma^{\omega^0(n+1)}(\alpha) b_N \dots b_1\infty^0 = \bigcup_{n<\omega} b_N \dots (b_1 + n + 1) = b_N \dots b_2\infty && \text{if ①}\\
\varsigma^{\omega^1 1}(\alpha) b_N \dots b_2\infty &= \bigcup_{n<\omega} \varsigma^{\omega^0(n+1)}(\alpha) b_N \dots b_2\infty^1 = b_N \dots (b_2 + 1)\infty && \text{if ②}\\
\varsigma^{\omega^1 1}(\alpha) b_N \dots b_{k+1}\infty^k &= \bigcup_{n<\omega} \varsigma^{\omega^0(n+1)}(\alpha) b_N \dots b_{k+1}\infty^k = \bigcup_{n<\omega} b_N \dots (b_{k+1} + 1)1^{k-1}(n+1) \cup \vec{b}\\
&= b_N \dots (b_{k+1} + 1)1^{k-1}\infty \cup \vec{b} && \text{if ③}
\end{aligned}$$

In case ③, there are some subordinate cases $\cup\vec{b}$ coming from mixed occurrences $b_N \dots (b_{k+1} + 1)^i 0\infty^{k-i-1}$, but do not need to be tracked, because they are strategic dead ends. By construction of α, no counter can be changed without resetting all smaller variables to 0 first as indicated.

$j \curvearrowright j+1, n=0$: For the step from j to $j+1$ prove the case $n=0$ as follows.

$$\varsigma^{\omega^{j+1}\cdot(0+1)}(\alpha)b_N\dots b_j\dots b_1 = \varsigma^{\omega^j\cdot\omega}(\alpha)b_N\dots b_j\dots b_1 = \bigcup_{n<\omega}\varsigma^{\omega^j\cdot(n+1)}(\alpha)b_N\dots b_j\dots b_1$$

$$\stackrel{IH}{=}\begin{cases}\bigcup_{n<\omega} b_N\dots(b_{j+1}+n)\infty^j & \text{if ①}\\ \bigcup_{n<\omega} b_N\dots(b_{j+1}+n+1)\infty^j & \text{if ②}\\ \bigcup_{n<\omega} b_N\dots(b_{k+1}+1)1^{k-j-1}(n+1)\infty^j\cup\vec{b} & \text{if ③}\end{cases}$$

$$\stackrel{IH}{=}\begin{cases}b_N\dots b_{j+2}\infty^{j+1} & \text{if } b_N,\dots,b_j<\infty\\ b_N\dots b_{j+2}\infty^{j+1} & \text{if } b_N,\dots,b_{j+1}<\infty\\ b_N\dots(b_{j+2}+1)\infty^{j+1} & \text{if } b_N,\dots,b_{j+2}<\infty, b_{j+1}=\infty, k=j+1\\ b_N\dots(b_{k+1}+1)1^{k-j-2}1\infty^{j+1}\cup\vec{b} & \text{if } b_N,\dots,b_{k+1}<\infty, b_k=\infty, k>j+1\end{cases}$$

$n \curvearrowright n+1$: Within any level j, prove the step from n to $n+1$ as follows. If $n=0$, then $\varsigma^{\omega^j(n+1)}(\alpha)b_N\dots b_j\dots b_1 = \varsigma^{\omega^j}(\alpha)b_N\dots b_j\dots b_1$ already has the property by induction hypothesis. Otherwise $n>0$, which allows us to conclude:

$$\varsigma^{\omega^j(n+1)}(\alpha)b_N\dots b_j\dots b_1 = \varsigma^{\omega^j n+\omega^j}(\alpha)b_N\dots b_j\dots b_1 \stackrel{Lemma\,5}{=} \varsigma^{\omega^j}(\alpha)\varsigma^{\omega^j n}(\alpha)b_N\dots b_j\dots b_1$$

$$\stackrel{IH}{=}\begin{cases}\varsigma^{\omega^j}(\alpha)b_N\dots(b_{j+1}+n-1)\infty^j & \text{if ①}\\ \varsigma^{\omega^j}(\alpha)b_N\dots(b_{j+1}+n)\infty^j & \text{if ②}\\ \varsigma^{\omega^j}(\alpha)b_N\dots(b_{k+1}+1)1^{k-j-1}n\infty^j\cup\vec{b} & \text{if ③}\end{cases}$$

$$\stackrel{IH}{=}\begin{cases}b_N\dots(b_j+n)\infty^j & \text{if ①}\\ b_N\dots(b_j+n+1)\infty^j & \text{if ②}\\ b_N\dots(b_{k+1}+1)1^{k-j-1}(n+1)\infty^j\cup\vec{b} & \text{if ③}\end{cases}$$

Consequently, $\varsigma_{\alpha^*}((-\infty,0)^N) = \varsigma^{\omega^N}(\alpha)(-\infty,0)^N = \mathbb{R}^N \neq \varsigma^{\omega^{N-1}\cdot n}(\alpha)(-\infty,0)^N$ for all $n\in\mathbb{N}$, which makes ω^N the closure ordinal for α. Since hybrid games α of the above form can be considered with arbitrarily big $N\in\mathbb{N}$, the common closure ordinal has to be $\geq\omega^N$ for all $N\in\mathbb{N}$, i.e. it has to be $\geq\omega^\omega$. □

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