Differential Hybrid Games

André Platzer

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School of Computer Science
Carnegie Mellon University
Pittsburgh, PA 15213

Abstract

This paper introduces *differential hybrid games*, which combine differential games with hybrid games. In both kinds of games, two players interact with continuous dynamics. The difference is that hybrid games also provide all the features of hybrid systems and discrete games, but only deterministic differential equations. Differential games, instead, provide differential equations with input by both players, but not the luxury of hybrid games, such as mode switches and discrete or alternating interaction. This paper augments *differential game logic* with modalities for the combined dynamics of differential hybrid games. It shows how hybrid games subsume differential games and introduces *differential game invariants* and *differential game variants* for proving properties of differential games inductively.

1 Introduction

Differential games [Isa67, Fri71, Háj75, KS88, Pet93, BRP99, EK74, ES84, Sou85] support adversarial interaction and game play during the continuous dynamics of a differential equation. They allow the two players to control inputs to the differential equation during its evolution by measurable functions. This is to be contrasted with hybrid systems [Hen96] and hybrid games, where differential equations are deterministic and the only decision is how long to evolve. Differential games are useful, e.g., for studying pursuit-evasion in aircraft if both players can react continuously to each other. They are a good match for tight-loop, analog, or rapid interaction.

Hybrid games [NRY96, HHM99, TLS00, DR06, BBC10, VPVD11, Pla14] are games of two players on a hybrid system's discrete and continuous dynamics where the players have control over some discrete-time choices during the evolution of the system, but the continuous dynamics stays deterministic. Hybrid games can model discrete aspects like decision delays and games with different dynamics or different controls in different modes of the system. They are a good match for sporadic or discrete-time interaction with discrete sensors or reaction delays. There is even a game aspect in sporadic interactions such as requesting flight plan changes.

The primary purpose of this paper is to show that both game principles are not in conflict but can be integrated seamlessly to complement each other. This paper introduces *differential hybrid games* that combine the aspects of hybrid games with those of differential games resulting in a model where discrete, continuous, and adversarial dynamics mix freely. This makes it possible to model games that combine continuous-time interactions (e.g. auto-evasion curves for aircraft) with discrete interactions (e.g. whether to call air traffic control for a collision avoidance advisory or whether to disengage the autopilot and follow a nonstandard manual flight maneuver).

The key insight behind hybrid systems is that it is helpful to understand each aspect of a system on its natural level [Pla12b]. Discrete dynamics is a good fit for some aspects. Continuous dynamics is more natural for others. Differential hybrid games enable the same flexibility for games rather than systems, so that each adversarial aspect in a cyber-physical system can be understood on its most natural level. Which level that is depends on modeling/analysis tradeoffs. Differential hybrid games provide a unifying framework in which both game aspects coexist and combine freely to enable such tradeoffs.

This paper introduces an adaptation of *differential game logic* dG$\mathcal{L}$ [Pla14] for *differential hybrid games*, extending differential game logic for *hybrid games* [Pla14]. Since this yields a compositional logic and proof technique, the primary attention in this paper is on how differential games combine seamlessly with hybrid games and how properties of differential games can be proved. Proof techniques for the resulting differential hybrid games then follow from compositionality principles.

In addition to presenting the first logical language that can model differential hybrid game dynamics, this paper presents inductive proof rules for differential games to obtain a sound and compositional proof calculus for differential hybrid games. *Differential game invariants* and their companions (*differential game variants* and *differential game refinements*) give a logical approach for differential games, complementing geometric viability theory and other approaches based on numerical integration of partial differential equations (PDEs). The advantage is that differential game (in)variants provide simple and sound witnesses for the existence of winning strategies for

differential games, even in unbounded time, and without building a formally verified numerical solver for PDEs with formally verified error bounds to obtain sound formal proofs.

Soundness is a big issue in differential games due to their surprising subtleties. It took 30 years to correctly relate Isaacs' PDEs to the differential games they were intended for [Isa67, BRP99]. After a long period of gradual progress, differential games are now handled primarily by solving the PDEs they induce or corresponding geometric equivalents from viability theory for the same PDEs [CQSP07]. Soundness issues with a number of approaches for differential games were reported [MBT05].[1]

This raises the challenge how to prove properties of differential games with the correctness demands of a proof system. This paper advocates for dedicated proof rules for differential games alongside proof rules for hybrid games. Logic is good at combining sound proof rules soundly with each other in a modular way and at adding them to a theorem prover.

The paper concludes with a theoretical insight. Hybrid systems have been shown to be equivalently reducible proof-theoretically to differential equations [Pla12a] and even to equivalently reduce to discrete systems [Pla12a]. This trend reverses for hybrid games, which do *not* reduce to differential games.

While the results are elegant and all background for the proofs is given, these proofs draw from many areas, including logic, proof theory, Carathéodory solutions, viscosity solutions of partial differential equations, real algebraic geometry, and real analysis. Byproducts of the soundness proof yield results of independent interest. All new proofs, which are the ones for results without citations, are included.

2 Differential Game Logic

This section introduces the *differential game logic* dG$\mathcal{L}$ *of differential hybrid games*, which adds differential games to differential game logic of (non-differential) hybrid games from previous work [Pla14]. The difference between hybrid games and differential hybrid games is that only the latter allow differential games, while the former allow only differential equations instead. The respective differential game logics are built in the same way around the respective game models.

Differential hybrid games are games of two players, Angel and Demon. Differential game logic uses modalities: $[\alpha]\phi$ refers to the existence of winning strategies for Demon for the objective specified by formula ϕ in differential hybrid game α and $\langle\alpha\rangle\phi$ refers to the existence of winning strategies for Angel for objective ϕ in differential hybrid game α. So $[\alpha]\phi$ and $\langle\alpha\rangle\neg\phi$ refer to complementary winning conditions (ϕ for Demon $\neg\phi$ for Angel) in the same differential hybrid game α.

2.1 Syntax

The terms θ of dG$\mathcal{L}$ are polynomial terms (more general terms are possible but do not guarantee decidable arithmetic). In applications, it may be convenient to use $\min, \max$ terms as well, which

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[1]The results presented here are of independent interest, because they provide a fix for an incorrect cyclic quantifier dependency in a proof in said paper [MBT05, Lem. 8] that was confirmed by the authors.

are definable as semialgebraic functions [Tar51].

Definition 1 (Differential hybrid games). The *differential hybrid games of differential game logic* dG$\mathcal{L}$ are defined by the following grammar (α, β are differential hybrid games, x a variable, θ a term, C a dG$\mathcal{L}$ formula, $y \in Y$ and $z \in Z$ formulas in free variable y or z, respectively, $f(x, y, z)$ a term in the free variables x, y, z):[2]

$$\alpha, \beta ::= x' = f(x, y, z) \,\&\, y \in Y^d \,\&\, z \in Z \mid x := \theta \mid ?C \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^* \mid \alpha^d$$

Definition 2 (dG$\mathcal{L}$ formulas). The *formulas of differential game logic* dG$\mathcal{L}$ are defined by the following grammar (ϕ, ψ are dG$\mathcal{L}$ formulas, θ_i are (polynomial) terms, x a variable, and α is a differential hybrid game):

$$\phi, \psi ::= \theta_1 \geq \theta_2 \mid \neg\phi \mid \phi \wedge \psi \mid \exists x\, \phi \mid \langle\alpha\rangle\phi \mid [\alpha]\phi$$

Other operators $>, =, \leq, <, \vee, \rightarrow, \leftrightarrow, \forall x$ can be defined as usual, e.g., $\forall x\, \phi \equiv \neg\exists x\, \neg\phi$. The modal formula $\langle\alpha\rangle\phi$ expresses that Angel has a winning strategy[3] to achieve ϕ in differential hybrid game α, i.e. Angel has a strategy to reach any of the states satisfying dG$\mathcal{L}$ formula ϕ when playing differential hybrid game α, no matter what Demon does. The modal formula $[\alpha]\phi$ expresses that Demon has a winning strategy to achieve ϕ in differential hybrid game α, i.e. a strategy to reach any of the states satisfying ϕ, no matter what Angel does.

As usual, $x := \theta$ is an *assignment* and $?C$ the *test game* or challenge that Angel only passes if C holds true in the current state. Otherwise she loses immediately, because she failed a test. Further, $\alpha \cup \beta$ is a *game of choice* where Angel gets to choose to play α or β. The *sequential game* $\alpha; \beta$ plays α followed by β (unless a player lost a challenge during α). The *repeated game* α^* plays α repeatedly and permits Angel to decide after each play of α whether she wants to play another iteration. The *dual game* α^d is the same as game α except all choices that Angel has in α are resolved by Demon in α^d and all choices that Demon has in α are resolved by Angel in α^d. During a differential hybrid game, players can only lose by violating the rules of the game, expressed in the tests. The proper winning condition is specified in the postcondition ϕ.

The (important) addition compared to prior work [Pla14] is the inclusion of differential games in the syntax for hybrid games. Predicate symbols have been removed, because they are of no immediate concern for the core focus of this paper: adding differential games to hybrid games. All occurrences of y, z in $x' = f(x, y, z) \,\&\, y \in Y^d \,\&\, z \in Z$ are bound. Finally, x, y, z can be vectorial provided that $f(x, y, z)$ is a vectorial term of the same dimension as x.

Observe the use of suggestive notation that is adopted in the interest of traceability with mathematical results throughout this paper: $\forall y{\in}Y\, \phi$ stands for $\forall y\, (y \in Y \rightarrow \phi)$ and $\exists y{\in}Y\, \phi$ for $\exists y\, (y \in Y \wedge \phi)$, in which $y \in Y$ is understood as convenient notation for a logical formula of one free variable (vector) y. The Euclidean scalar product of vectors v, w is denoted by $v \cdot w$. The Euclidean norm of a vector v by $|v| \overset{\text{def}}{=} \sqrt{v \cdot v}$.

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[2]The ^{d} in differential game $x' = f(x, y, z) \,\&\, y \in Y^d \,\&\, z \in Z$ is a mere mnemonic reminder that Demon controls *d*ual input y and Angel controls z.

[3]The mnemonic is that, just like the diamond operator $\langle\cdot\rangle$ Angel has wings.

2.2 Differential Games

This section recalls the definition of differential games using the approach of nonanticipative strategies [EK74, ES84, BRP99].

Definition 3 (Differential game). Let $Y \subset \mathbb{R}^k$, $Z \subset \mathbb{R}^l$ compact sets of controls for the respective players. And let $f : [\eta, T] \times \mathbb{R}^n \times Y \times Z \to \mathbb{R}^n$ bounded, uniformly continuous[4], and in x uniformly Lipschitz[5]. For time horizon $T \geq 0$, initial time $\eta < T$ and state $\xi \in \mathbb{R}^n$, a *differential game* has the form

$$\begin{cases} x'(s) = f(s, x(s), y(s), z(s)) & \eta \leq s \leq T \\ x(\eta) = \xi \end{cases} \tag{1}$$

where the *controls* $y : [\eta, T] \to Y$ and $z : [\eta, T] \to Z$ are measurable functions for the respective players. The *terminal*[6] *payoff* is defined by a bounded and Lipschitz $g : \mathbb{R}^n \to \mathbb{R}$.

Let $\mathcal{M}_Y$ and $\mathcal{M}_Z$ denote the set of (measurable) controls.

Lemma 1 (Response). *For each controls $y \in \mathcal{M}_Y$, $z \in \mathcal{M}_Z$ and initial data η, ξ, the differential equation* (1) *has a unique Carathédory solution*[7] $x : [\eta, T] \to \mathbb{R}^n$, *called* response *and denoted by* $x(s; \xi, y, z) \stackrel{\text{def}}{=} x(s)$ *as a function of time s. Finally, $x(s; \xi, y, z) = x(s; \xi, \hat{y}, \hat{z})$ if $y = \hat{y}$ a.e. and $z = \hat{z}$ a.e.*

Proof. $f(s, x, y(s), z(s))$ is continuous in x, because f is (even uniformly) continuous, and measurable in s, because it is a composition of measurable functions. Let $l(s)$ denote the maximum of the bound of f and its Lipschitz constant L from Footnote 5. Then l is measurable and integrable on $[\eta, T]$ (it is constant) and satisfies $|f(s, 0, y(s), z(s)| \leq l(s)$ and the (generalized) Lipschitz condition for all s, x, a:

$$|f(s, x, y(s), z(s)) - f(s, a, y(s), z(s))| \leq l(s)|x - a|$$

Thus, Carathéodory's existence and uniqueness theorem [Wal00, §10.XX] shows the existence of a unique solution that can be continued to the boundary of the domain, hence is global since f is bounded. The fundamental theorem of calculus for Lebesgue-integrals [Wal95, Thm. 9.23] implies that changing y and z on a set of measure zero does not change the response $x(s; \xi, y, z)$. $\square$

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[4]i.e. for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(t, x, y, z) - f(s, a, b, c)| < \varepsilon$ for all $|(t, x, y, z) - (s, a, b, c)| < \delta$. The usual relations help simplify: $f \in C^1$ on a compact set $\Rightarrow$ differentiable with bounded derivatives $\Rightarrow$ Lipschitz $\Longleftrightarrow$ 1-Hölder continuous $\Rightarrow$ Hölder continuous $\Rightarrow$ uniformly continuous $\Rightarrow$ continuous. Where f is λ-*Hölder continuous* iff for some C $|f(x) - f(y)| \leq C|x - y|^\lambda$ for all x, y. So 0-Hölder $\Longleftrightarrow$ bounded. Continuous functions on compact sets are bounded and uniformly continuous.

[5]i.e. there is an L such that f is uniformly L-*Lipschitz* in x, that is $|f(t, x, y, z) - f(t, a, y, z)| \leq L|x - a|$ for all t, x, a, y, z.

[6]Differential games with a running payoff h can be converted to terminal payoff by adding a differential equation $r'(s) = h(s, x(s), y(s), z(s))$ with terminal payoff $\tilde{g}(x, r) = g(x) + r$.

[7]Carathéodory solutions are absolutely continuous functions satisfying the differential equation *a.e.* (*almost everywhere*, which means except on a subset of a measure 0 set).

The (terminal) *payoff* for controls $y(\cdot)$ and $z(\cdot)$ of (1) is

$$g(x(T)) = g(x(T; \xi, y, z))$$

Let $\mathcal{S}_{Y \to Z}$ be the set of *causal* or *nonanticipative strategies* for Z, i.e. the set of $\beta : \mathcal{M}_Y \to \mathcal{M}_Z$ such that for all $\eta \leq s \leq T$ and all $y, \hat{y} \in \mathcal{M}_Y$:

$$\text{if } y = \hat{y} \text{ a.e. on } [\eta, s] \quad (\text{i.e. } y(\tau) = \hat{y}(\tau) \text{ for a.e. } \eta \leq \tau \leq s)$$
$$\text{then } \beta(y) = \beta(\hat{y}) \text{ a.e. on } [\eta, s]$$

That is, nonanticipative strategies for Z give the player for Z the current state, history (which is irrelevant because the games are Markovian), and the opponent's current action. Unlike mere state feedback strategies, time-dependent controls ensure the response exists and is unique [Háj75, §2.2]. The reaction $\beta(y)$ of a nonanticipative strategy to y cannot depend on the opponent's future input beyond the current time. The set of *nonanticipative strategies* for Y is $\mathcal{S}_{Z \to Y}$, i.e. of $\alpha : \mathcal{M}_Z \to \mathcal{M}_Y$ such that for all $\eta \leq s \leq T$ and all $z, \hat{z} \in \mathcal{M}_Z$:

$$\text{if } z = \hat{z} \text{ a.e. on } [\eta, s]$$
$$\text{then } \alpha(z) = \alpha(\hat{z}) \text{ a.e. on } [\eta, s]$$

2.3 Semantics

The semantics extends the meaning of hybrid games seamlessly to differential hybrid games by adding differential game winning regions. The modular design of $\mathsf{dG\mathcal{L}}$ makes this integration of differential games with hybrid games simple. Since hybrid games have been described before [Pla14], the focus will be on elaborating the new case of differential games.

A *state* ξ is a mapping from variables to $\mathbb{R}$. Let $\mathcal{S}$ the set of states, which, for n variables, is isomorphic to Euclidean space $\mathbb{R}^n$. For a subset $X \subseteq \mathcal{S}$ the complement $\mathcal{S} \setminus X$ is denoted $X^{\complement}$. Let ξ_x^d denote the state that agrees with state ξ except for the interpretation of variable x, which is changed to $d \in \mathbb{R}$. The value of term θ in state ξ is denoted by $\llbracket \theta \rrbracket_\xi$. The denotational semantics of $\mathsf{dG\mathcal{L}}$ formulas will be defined in Def. 4 by simultaneous induction along with the denotational semantics, $\varsigma_\alpha(\cdot)$ and $\delta_\alpha(\cdot)$, of differential hybrid games, defined in Def. 5, because $\mathsf{dG\mathcal{L}}$ formulas are defined by simultaneous induction with differential hybrid games.

Unlike the $\mathsf{dG\mathcal{L}}$ quantifiers $\exists x, \forall x$, the notation for quantifiers in the mathematical metalanguage is $\exists \xi$ and $\forall \xi$.

Definition 4 ($\mathsf{dG\mathcal{L}}$ semantics). The *semantics of a $\mathsf{dG\mathcal{L}}$ formula* ϕ is the subset $\llbracket \phi \rrbracket \subseteq \mathcal{S}$ of states in which ϕ is true. It is defined inductively as follows

1. $\llbracket \theta_1 \geq \theta_2 \rrbracket = \{\xi \in \mathcal{S} : \llbracket \theta_1 \rrbracket_\xi \geq \llbracket \theta_2 \rrbracket_\xi\}$
2. $\llbracket \neg \phi \rrbracket = (\llbracket \phi \rrbracket)^{\complement}$
3. $\llbracket \phi \wedge \psi \rrbracket = \llbracket \phi \rrbracket \cap \llbracket \psi \rrbracket$
4. $\llbracket \exists x\, \phi \rrbracket = \{\xi \in \mathcal{S} : \xi_x^\kappa \in \llbracket \phi \rrbracket \text{ for some } \kappa \in \mathbb{R}\}$

5. $[\![\langle\alpha\rangle\phi]\!] = \varsigma_\alpha([\![\phi]\!])$
6. $[\![[\alpha]\phi]\!] = \delta_\alpha([\![\phi]\!])$

Formula ϕ is *valid*, written $\vDash \phi$, iff $[\![\phi]\!] = \mathcal{S}$.

Definition 5 (Semantics of differential hybrid games)**.** The *semantics of a differential hybrid game* α is a function $\varsigma_\alpha(X)$, that, for each of Angel's winning states $X \subset \mathcal{S}$, gives the *winning region* of Angel, i.e. the set of states $\varsigma_\alpha(\cdot)$ from which Angel has a winning strategy to achieve X (whatever strategy Demon chooses). It is defined inductively as

1. $\varsigma_{x'=f(x,y,z)\,\&\,y\in Y^d\&z\in Z}(X) = \{\xi \in \mathcal{S} :$
   $\exists T{\geq}0 \,\exists\beta \in \mathcal{S}_{Y\to Z} \,\forall y \in \mathcal{M}_Y \,\exists 0{\leq}\zeta{\leq}T$
   $x(\zeta; \xi, y, \beta(y)) \in X\}$
2. $\varsigma_{x:=\theta}(X) = \{\xi \in \mathcal{S} : \xi_x^{[\![\theta]\!]_\xi} \in X\}$
3. $\varsigma_{?C}(X) = [\![C]\!] \cap X$
4. $\varsigma_{\alpha\cup\beta}(X) = \varsigma_\alpha(X) \cup \varsigma_\beta(X)$
5. $\varsigma_{\alpha;\beta}(X) = \varsigma_\alpha(\varsigma_\beta(X))$
6. $\varsigma_{\alpha^*}(X) = \bigcap\{Z \subseteq \mathcal{S} : X \cup \varsigma_\alpha(Z) \subseteq Z\}$
7. $\varsigma_{\alpha^d}(X) = (\varsigma_\alpha(X^\complement))^\complement$

The *winning region* of Demon, i.e. the set of states $\delta_\alpha(\cdot)$ from which Demon has a winning strategy to achieve X (whatever strategy Angel chooses) is defined inductively as

1. $\delta_{x'=f(x,y,z)\,\&\,y\in Y^d\&z\in Z}(X) = \{\xi \in \mathcal{S} :$
   $\forall T{\geq}0 \,\forall\beta \in \mathcal{S}_{Y\to Z} \,\exists y \in \mathcal{M}_Y \,\forall 0{\leq}\zeta{\leq}T$
   $x(\zeta; \xi, y, \beta(y)) \in X\}$
2. $\delta_{x:=\theta}(X) = \{\xi \in \mathcal{S} : \xi_x^{[\![\theta]\!]_\xi} \in X\}$
3. $\delta_{?C}(X) = ([\![C]\!])^\complement \cup X$
4. $\delta_{\alpha\cup\beta}(X) = \delta_\alpha(X) \cap \delta_\beta(X)$
5. $\delta_{\alpha;\beta}(X) = \delta_\alpha(\delta_\beta(X))$
6. $\delta_{\alpha^*}(X) = \bigcup\{Z \subseteq \mathcal{S} : Z \subseteq X \cap \delta_\alpha(Z)\}$
7. $\delta_{\alpha^d}(X) = (\delta_\alpha(X^\complement))^\complement$

Time horizon T, nonanticipative strategy β, and stopping times ζ of differential games are Angel's choice while control of y is Demon's choice. In particular, Angel needs to choose a finite time horizon T but the corresponding control $\beta(y)$ from her nonanticipative strategy β gives her a chance to observe Demon's current action from Demon's control y. Angel ultimately gets to inspect the resulting state and decide whether she wants to stop playing the differential game. This is the continuous counterpart of α^*, where Angel gets to inspect the state and decides whether she wants to repeat the loop again or not, which follows from the fixpoint semantics of α^* as elaborated in prior work [Pla14]. The fact that Angel has to choose some arbitrarily large but finite horizon T first corresponds to her not being allowed to play the differential game indefinitely, just like she

is not allowed to repeat playing α^* forever, which again results from its least fixpoint semantics [Pla14]. Demon has a winning strategy in the differential game $x' = f(x,y,z) \,\&\, y \in Y^d \,\&\, z \in Z$ to achieve X if for all of Angel's time horizons T and all of Angel's nonanticipative strategies β for Demon's controls there is a control y for Demon such that for all of Angel's stopping times ζ the game ends in one of Demon's winning states (i.e. in X). Demon knows $\beta \in \mathcal{S}_{Y \to Z}$ when choosing $y \in \mathcal{M}_Y$, so he can predict the states over time by solving (1) via Lemma 1. But Angel's nonanticipative β allows $\beta(y)(s)$ to depend on $y(s)$, which gives her an information advantage. The (dual) quantifier order for $\langle \cdot \rangle$ is the same, so that Angel finds some β that works for any y since she cannot predict what Demon will play. Hence, the informational advantage of the opponent's current action as well as the advantage of controlling time in a differential game consistently goes to Angel, whether in $\langle \cdot \rangle$ or in $[\cdot]$.

It might appear that the last quantifier ζ is not important, because, if Angel wins, there is a maximum time horizon T within which she wins so that it seems like it would be enough for her to choose that maximum time horizon T and check for the terminal state at time T. However, Demon might then still let Angel "win" earlier by playing suboptimally if that gives him the possibility of moving outside Angel's winning condition before time T again when the winning condition is checked. It is, thus, important for Angel to be able to stop the differential game at any time based on the state she observes. She will want to stop when the game reached her target. Consider, e.g., the *race car game*, where Demon is in control of a car toward a goal $x^2 < 1$ and Angel is in control of time:

$$x = -9 \land t = 0 \to [x' = y, t' = 1 \,\&\, y \in [1,2]^d](4 < t < 8 \to x^2 < 1)$$

If Angel were to declare $\zeta = T$ upfront by advance notice, Demon could compute an optimal velocity $y = \frac{8}{\zeta} \in [1,2]$ to win, since Angel has to choose $4 < \zeta < 8$ to stand a chance. Since Angel declares a time bound T and chooses ζ only after Demon revealed $y \in \mathcal{M}_Y$, she can choose $T = 7$ and, if Demon has not moved x to $x^2 < 1$ at time 4.1, end the game at $\zeta = 4.1$ to win. If Demon has moved x to $x^2 < 1$, so $x > -1$, then Angel still wins by waiting the full time $\zeta = T$, which is at least 2 more seconds, during which Demon must have moved at least 2 and left $x^2 < 1$ to lose. This example hinges on a postcondition that is neither open nor closed.

Since differential hybrid games have the same information structure for $\langle \alpha \rangle$ and $[\alpha]$, just referring to another player's winning strategy, the determinacy theorem [Pla14, Thm. 2] extends to differential hybrid games. In each state, exactly one player has a winning strategy. The proof is easier than (partial) determinacy results for other scenarios and information patterns [Car96].

Theorem 2 (Determinacy). *Differential hybrid games are determined, i.e.* $\vDash \neg\langle\alpha\rangle\neg\phi \leftrightarrow [\alpha]\phi$.

Proof. The proof shows by induction on the structure of α that $\varsigma_\alpha(X^\complement)^\complement = \delta_\alpha(X)$ for all $X \subseteq \mathcal{S}$ with some set of states $\mathcal{S}$, which implies the validity of $\neg\langle\alpha\rangle\neg\phi \leftrightarrow [\alpha]\phi$ using $X \overset{\text{def}}{=} \llbracket\phi\rrbracket$ by Def. 4. The only difference to a corresponding proof for hybrid games [Pla14] is the additional case of differential games:

1. $\varsigma_{x'=f(x,y,z) \,\&\, y \in Y^d \,\&\, z \in Z}(X^\complement)^\complement =$
   $\{\xi \in \mathcal{S} : \exists T \geq 0 \, \exists \beta \in \mathcal{S}_{Y \to Z} \, \forall y \in \mathcal{M}_Y \, \exists 0 \leq \zeta \leq T \ x(\zeta; \xi, y, \beta(y))) \notin X\}^\complement =$
   $\{\xi \in \mathcal{S} : \text{ not } \exists T \geq 0 \, \exists \beta \in \mathcal{S}_{Y \to Z} \, \forall y \in \mathcal{M}_Y \, \exists 0 \leq \zeta \leq T \ x(\zeta; \xi, y, \beta(y))) \notin X\} =$
   $\{\xi \in \mathcal{S} : \forall T \geq 0 \, \forall \beta \in \mathcal{S}_{Y \to Z} \, \exists y \in \mathcal{M}_Y \, \forall 0 \leq \zeta \leq T \ x(\zeta; \xi, y, \beta(y))) \in X\} = \delta_{x'=f(x,y,z) \,\&\, y \in Y^d \,\&\, z \in Z}(X)$

2. $\varsigma_{x:=\theta}(X^\complement)^\complement = \{\xi \in \mathcal{S} : \xi_x^{[\![\theta]\!]_\xi} \notin X\}^\complement = \varsigma_{x:=\theta}(X) = \delta_{x:=\theta}(X)$

3. $\varsigma_{?C}(X^\complement)^\complement = ([\![C]\!] \cap X^\complement)^\complement = ([\![C]\!])^\complement \cup (X^\complement)^\complement = \delta_{?C}(X)$

4. $\varsigma_{\alpha \cup \beta}(X^\complement)^\complement = (\varsigma_\alpha(X^\complement) \cup \varsigma_\beta(X^\complement))^\complement = \varsigma_\alpha(X^\complement)^\complement \cap \varsigma_\beta(X^\complement)^\complement = \delta_\alpha(X) \cap \delta_\beta(X) = \delta_{\alpha \cup \beta}(X)$

5. $\varsigma_{\alpha;\beta}(X^\complement)^\complement = \varsigma_\alpha(\varsigma_\beta(X^\complement))^\complement = \varsigma_\alpha(\delta_\beta(X)^\complement)^\complement = \delta_\alpha(\delta_\beta(X)) = \delta_{\alpha;\beta}(X)$

6. $\varsigma_{\alpha^*}(X^\complement)^\complement = \left(\bigcap\{Z \subseteq \mathcal{S} : X^\complement \cup \varsigma_\alpha(Z) \subseteq Z\}\right)^\complement = \left(\bigcap\{Z \subseteq \mathcal{S} : (X \cap \varsigma_\alpha(Z)^\complement)^\complement \subseteq Z\}\right)^\complement$

   $= \left(\bigcap\{Z \subseteq \mathcal{S} : (X \cap \delta_\alpha(Z^\complement))^\complement \subseteq Z\}\right)^\complement = \bigcup\{Z \subseteq \mathcal{S} : Z \subseteq X \cap \delta_\alpha(Z)\} = \delta_{\alpha^*}(X).$ [8]

7. $\varsigma_{\alpha^d}(X^\complement)^\complement = (\varsigma_\alpha((X^\complement)^\complement)^\complement)^\complement = \delta_\alpha(X^\complement)^\complement = \delta_{\alpha^d}(X)$ $\qquad\qquad\square$

A formula is called *atomically open* if its negation normal form is built from $\wedge, \vee, >, <$. Atomically open formulas define topologically open sets. The converse is not true, because there can be spurious extra subformulas: $0 < x \wedge x < 5 \vee x = 2$ is topologically open but has an irrelevant topologically closed subformula $x = 2$. A formula is *atomically closed* if its negation normal form is built from $\wedge, \vee, \geq, \leq, =$. Atomically closed formulas define topologically closed sets. Both converses can always be made true by transforming it to avoid superfluous subformulas [BCR98, 2.7.2]. The primary focus in this paper is on postconditions that are open or closed.

Lemma 3 ($\mathbb{R}$ arithmetization). *There is an effective mapping $(\cdot)^\Re$ from first-order formulas to (continuous) terms of mixed polynomials, min, and max. If F is atomically open, then $\vDash F \leftrightarrow (F^\Re > 0)$. If F is atomically closed, $\vDash F \leftrightarrow (F^\Re \geq 0)$.*

Proof. By quantifier elimination, F can be assumed to be quantifier-free and in negation normal form. The term $F^\Re$ of mixed polynomials, min, max for F is defined inductively:

$$(a > b)^\Re \equiv a - b$$
$$(a < b)^\Re \equiv (b > a)^\Re$$
$$(a \geq b)^\Re \equiv a - b$$
$$(a \leq b)^\Re \equiv (b \geq a)^\Re$$
$$(a = b)^\Re \equiv (a \geq b \wedge b \geq a)^\Re$$
$$(F \wedge G)^\Re \equiv \min(F^\Re, G^\Re)$$
$$(F \vee G)^\Re \equiv \max(F^\Re, G^\Re) \qquad\qquad\square$$

Even if not all are necessary, the assumptions in Def. 3 will be required to hold when playing differential games from Def. 1. They can be checked using the remarks in Footnote 4, which are decidable for the relevant terms in first-order real arithmetic [Tar51]. The easiest criterion is the following:

---

[8] The penultimate equation follows from the μ-calculus equivalence $\nu Z.\Upsilon(Z) \equiv \neg\mu Z.\neg\Upsilon(\neg Z)$ and the fact that least pre-fixpoints are fixpoints and that greatest post-fixpoints are fixpoints for monotone functions.

Lemma 4 (Well-definedness). *If f is bounded for compact $[\![y \in Y]\!], [\![z \in Z]\!]$ and F is open or closed, all differential games for $[x' = f(x, y, z) \,\&\, y \in Y^d \,\&\, z \in Z]F$ and $\langle x' = f(x, y, z) \,\&\, y \in Y^d \,\&\, z \in Z \rangle F$ are well-defined.*

Proof. Let b a bound on the norm of f. For any initial state ξ and any time horizon $T \geq 0$ (from Def. 5) any response $x(\zeta; \xi, y, \beta(y))$ of (1) remains on the compact ball of radius bT around ξ. Without changing the differential game, f can, thus, be replaced by a $\hat{f}$ that agrees with f on this compact ball and accordingly for the payoff. On that compact set, the dG$\mathcal{L}$ term f and the arithmetization $F^\Re$ define Lipschitz continuous functions (even when using $\min, \max$ terms) as follows. Polynomials are smooth and, thus, Lipschitz on compact sets. The absolute value function is Lipschitz. The composition $\min(x, y) = (x + y)/2 - |x - y|/2$ of Lipschitz functions is Lipschitz[9] and so is $\max(x, y) = -\min(-x, -y)$. By Tietze [Wal95, 2.19], there are Lipschitz continuous extensions $\hat{f}$ of f and $\hat{g}$ of $F^\Re$ that agree on the compact ball and remain bounded. The differential game $x' = \hat{f}(x, y, z) \,\&\, y \in Y^d \,\&\, z \in Z$ with payoff $\hat{g}$ is equivalent by Lemma 1 and Lemma 3 and it meets the requirements in Def. 3. $\qquad\square$

For any horizon T and initial state ξ, f can be replaced in similar ways by a bounded function without changing the game [GS11], since f is continuous by Def. 1. Unlike semantic differential games (Def. 3), differential games in the logic (Def. 1) have no implicit time-dependency but need an explicit extra clock variable t with differential equation $t' = 1$ to express time-dependencies. An explicit time-dependency for semantic differential games (Def. 3) is helpful for the proofs.

3 Differential Game Proofs

This section presents new inductive proof rules for differential games with differential game invariants and differential game variants as well as with differential game refinements.

Differential equations are already hard to solve and it is challenging to use their solutions for proofs [Pla12a]. It is even more difficult, however, to solve differential games, because their Carathéodory solution depends on the control choices adopted by the two players, which can be arbitrary measurable functions. A faithful representation of this would, thus, require not just alternating quantification over arbitrary measurable functions but also the ability to solve all resulting ordinary differential equations and to prove properties about all their respective behaviors – a truly daunting enterprise.

Differential game invariants, instead, define a simple induction principle for differential games. The proof rule of *differential game invariants* and its counterpart for *differential game variants* are shown in Figure 1. Differential game invariants have a simple intuition. If, in each state x ($\forall x$ is implicit in the premise of DGI and follows from the definition of validity in Def. 4), there is a choice of control action y for Demon such that for all choices of Angel's control action z the derivative $F'^{f(x,y,z)}_{x'}$ of F holds when substituting the right-hand side $f(x, y, z)$ of the differential game for the left-hand side x'. Intuitively, this means there is always a way for Demon to make F

---

[9] $|f(g(x)) - f(g(y))| \leq L|g(x) - g(y)| \leq LK|x - y|$ makes $f \circ g$ Lipschitz when L, K are the Lipschitz constants of f and g, respectively.

$$\text{(DGI)} \quad \frac{\exists y \in Y \, \forall z \in Z \; F'^{f(x,y,z)}_{x'}}{F \to [x' = f(x,y,z) \,\&\, y \in Y^d \,\&\, z \in Z]F}$$

$$\text{(DGV)} \quad \frac{\exists \varepsilon{>}0 \, \forall x \, \exists z \in Z \, \forall y \in Y \, (g \leq 0 \to g'^{f(x,y,z)}_{x'} \geq \varepsilon)}{\langle x' = f(x,y,z) \,\&\, y \in Y^d \,\&\, z \in Z \rangle g \geq 0}$$

$$\text{(DGR)} \quad \frac{\forall u \in U \, \exists y \in Y \, \forall z \in Z \, \exists v \in V \, \forall x \, (f(x,y,z) = g(x,u,v))}{[x' = g(x,u,v) \,\&\, u \in U^d \,\&\, v \in V]F \to [x' = f(x,y,z) \,\&\, y \in Y^d \,\&\, z \in Z]F}$$

Figure 1: Differential game proof rules.

"more true" with y, whatever Angel is trying with z. So, Demon has a winning strategy no matter how long Angel decides to evolve. Recall that Angel gets to inspect Demon's current y action in her nonanticipative strategy before choosing z, which explains the order of quantifiers in DGI.

Differential game variants (rule DGV) also have a simple intuition. Angel can reach the post-condition if, from any state where she has not won yet, there is a progress of at least ε towards the goal that, uniformly at all x, she can realize for some z control choice of hers, no matter what y action Demon chose. The quantifier order in DGV is conservative to simplify the proofs. Other postconditions are possible but DGV becomes notationally more involved then.

Differential game refinements (rule DGR) relate differential games whose equations can be made to agree when matching the U control that Demon sought in its antecedent with some of Demon's Y control in the succedent if any control Z that Angel has in the succedent can be matched by a control V that Angel already had in the antecedent. Via the induced identification, Demon's winning strategy for the differential game in the antecedent carries over to a winning strategy for the differential game in the succedent if Demon has more corresponding control power in the succedent while Angel has less. A dual of DGR for $\langle \cdot \rangle$ is induced by Theorem 2.

As with invariants, it may sometimes be difficult to find good differential game invariants or differential game variants for the proof of a property. Once found, they are computationally attractive, since easy to check by decidable arithmetic.

Differential game invariants use syntactic total derivations to compute differential game derivatives syntactically.

Definition 6 (Derivation). The operator $\nabla(\cdot)$ that is defined as follows on terms is called *syntactic (total) derivation* from (polynomial) terms to differential terms, i.e. terms in which differential symbols x' for variables x are allowed:

$$\nabla(r) = 0 \qquad \text{for numbers } r \in \mathbb{R} \tag{2a}$$

$$\nabla(x) = x' \qquad \text{for variables } x \tag{2b}$$

$$\nabla(a + b) = \nabla(a) + \nabla(b) \tag{2c}$$

$$\nabla(ab) = \nabla(a)b + a\nabla(b) \tag{2d}$$

It extends to (quantifier-free) first-order real-arithmetic formulas F as follows:

$$\nabla(F \wedge G) \equiv \nabla(F) \wedge \nabla(G) \tag{3a}$$

$$\nabla(F \vee G) \equiv \nabla(F) \wedge \nabla(G) \tag{3b}$$
$$\nabla(a \geq b) \equiv \nabla(a) \geq \nabla(b) \tag{3c}$$
$$\nabla(a \leq b) \equiv \nabla(a) \leq \nabla(b) \tag{3d}$$
$$\nabla(a = b) \equiv \nabla(a) = \nabla(b) \tag{3e}$$
$$\nabla(a > b) \equiv \nabla(a) \geq \nabla(b) \tag{3f}$$
$$\nabla(a < b) \equiv \nabla(a) \leq \nabla(b) \tag{3g}$$

Define $F'^{\theta}_{x'}$ to be the result of substituting θ for x' in $\nabla(F)$ and substituting 0 for all other differential symbols c' that have no differential equation / differential game.

The relation of the syntactic derivation $\nabla(e)$ to analytic differentiation is identified in the following result, which further identifies the syntactic term $\nabla(e)^{\theta}_{x'}$ with a Lie-derivative.

Lemma 5 (Derivation). *Let θ a (vectorial) term of the same dimension as x and e a term, then $[\![\nabla(e)^{\theta}_{x'}]\!]_{\xi} = [\![\theta]\!]_{\xi} \cdot D_x[\![e]\!]_{\xi}$, where $D_x[\![e]\!]_{\xi}$ is the gradient at ξ with respect to x of the value of e.*

Proof. By a minor variation of [Pla12c, Lem. 3.3]. □

The rules in Figure 1 assume the well-definedness condition from Lemma 4. For reference, the complete axiomatization for the other hybrid game operators of $\mathsf{dG\mathcal{L}}$ [Pla14] are shown in Appendix B. They play no further role for the purposes of this paper, though, except to manifest how seamlessly differential games proving integrates with hybrid games proving in $\mathsf{dG\mathcal{L}}$.

While a strong point of $\mathsf{dG\mathcal{L}}$ is that it enables such a seamless integration of differential games and hybrid games in modeling and analysis, the examples shown only explore simple differential games for space reasons. An example for DGR is in the proof of Lemma 24. Consider the strength game with $-1 \leq y \leq 1$ for $y \in I$, which proves easily with DGI:

$$\mathbb{R} \frac{\dfrac{*}{\exists y \in I \, \forall z \in I \, 0 \leq 3x^2(-1 + 2y + z)}}{\dfrac{\exists y \in I \, \forall z \in I \, (0 \leq 3x^2 x')^{-1+2y+z}_{x'}}{{}^{\mathrm{DGI}}1 \leq x^3 \rightarrow [x' = -1 + 2y + z \, \& \, y \in I^d \& z \in I]1 \leq x^3}}$$

Using vectorial notation, let $y \in B$ be $y_1^2 + y_2^2 \leq 1$. Let terms $L \leq M$ denote the maximum speeds of l and m. The simple pursuit [Isa67], that vector m can escape the vector l, proves easily:

$$\mathbb{R} \frac{\dfrac{*}{\exists y \in B \, \forall z \in B \, (2(l-m) \cdot (Lz - My) \geq 0)}}{\dfrac{\exists y \in B \, \forall z \in B \, (2(l-m) \cdot (l' - m') \geq 0)^{My}_{m'} {}^{Lz}_{l'}}{{}^{\mathrm{DGI}}|l-m|^2 > 0 \rightarrow [m' = My, l' = Lz \, \& \, y \in B^d \& z \in B]|l-m|^2 > 0}}$$

Non-convex $y \in Y$ defined as $y^2 = 1$ or games with input by just one player work as well (similar for higher dimensions):

$$\frac{\dfrac{*}{\exists y^2 = 1 \, (3/2x^2 x^3 y \geq 2xx^3 y)}}{\dfrac{\exists y^2 = 1 \, (3/2x^2 x' \geq 2xx')^{x^3 y}_{x'}}{{}^{\mathrm{DGI}}x^3/2 > x^2 - 1 \rightarrow [x' = x^3 y \, \& \, y^2 = 1^d]x^3/2 > x^2 - 1}}$$

To fit to the simple well-definedness condition (Lemma 4), the differential equation

$$x' = \max(\min(x^3 y, k), -k)$$

could be used instead, which still proves for all bounds $k \geq 0$. Alternatively, the global bounding with $x' = x^3 y/(1 + \sqrt{(x^3 y)^2})$, which does not change the game outcome [GS11], proves as well. These simple proofs entail for all nonanticipative strategies, the existence of measurable control functions to win the game.

4 Soundness Proof

The differential game invariant proof rule DGI is a natural generalization of differential invariants for differential equations [Pla10, PC08, Pla12c] to differential games. Its quantifier pattern matches the information pattern of the differential game.

The only difficulty is its soundness proof. The premise of DGI shows that, at every point in space x, a control action y exists for Demon that will, for all control actions z that Angel could respond with, make the Lie-derivative $F'^{f(x,y,z)}_{x'}$ true. In conventional (non-game) wisdom, this makes the truth-value of F never decrease. However, it is not particularly obvious whether those various local control actions for each x at various points of the state space can be reassembled into a single coherent control signal that is measurable and passes muster on leading the whole differential game to a successful response for every strategy of Angel. Certainly, the original quantification over nonanticipative strategies and measurable control signals from the semantics is hard to capture in useful (first-order) proof rules.

It also took 30 years to justify Isaacs equations for differential games, however innocent they may look. Fortunately, unlike Isaacs [Isa67], differential game invariants have most required advances of mathematics already at their disposal. This section proves *soundness* of the differential game proof rules. That is: If their premise is valid, then so is their respective conclusion. Proving soundness, thus, assumes the *premise* (above rule bar) to be valid and considers a state $\xi \in \mathcal{S}$ in which the *antecedent* (left side of $\rightarrow$) of the *conclusion* (below rule bar) is true to show that its *succedent* (right side of $\rightarrow$) is true in ξ, too.

4.1 Differential Game Refinement

DGR can be proved sound using the notions introduced so far. The key is to exploit the Borel measurability and existence of semialgebraic Skolem functions to extract measurable and nonanticipative correspondence functions from its premise.

Remark 6 (Preimage). The preimage $f^{-1}(A) = \{x : f(x) \in A\}$ of set A under function f satisfies the usual properties:
1. $A \subseteq B$ implies $f^{-1}(A) \subseteq f^{-1}(B)$
2. $f^{-1}(A^{\complement}) = (f^{-1}(A))^{\complement}$ for the complement $A^{\complement}$ of A
3. $f^{-1}(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} f^{-1}(A_i)$ for any index family I
4. $(f \circ g)^{-1}(A) = g^{-1}(f^{-1}(A)))$

For the sequel note: by Case 4, the composition $f \circ g$ is $\mathcal{M}$-measurable if f is *Borel* measurable and g is $\mathcal{M}$-measurable.

Lemma 7. *Semialgebraic functions[10] are Borel measurable.*

Proof. Let f be a semialgebraic function. The proof of its Borel measurability is by induction along the Borel hierarchy.

1) The preimage $f^{-1}(A)$ of any semialgebraic set A is semialgebraic [BPR06, Proposition 2.83], thus, Borel measurable.

2) The preimage $f^{-1}(A^{\complement}) = (f^{-1}(A))^{\complement}$ of the complement $A^{\complement}$ of any set A whose preimage $f^{-1}(A)$ is Borel is a complement of a Borel set and, thus, Borel.

3) The preimage $f^{-1}(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} f^{-1}(A_i)$ of an intersection of any family of sets A_i whose preimage $f^{-1}(A_i)$ is Borel is an intersection of Borel sets and, thus, Borel. □

Theorem 8 (Differential game refinement)**.** *Differential game refinements (rule DGR) are sound.*

Proof. The formulas $u \in U, v \in V, y \in Y, z \in Z$ only have the indicated free variables, so write $[\![Z]\!]$ for the set of values for z that satisfy $z \in Z$, etc. The premise implies

$$\vDash \exists y \in Y \, \forall z \in Z \, \exists v \in V \, \forall x \, (f(x, y, z) = g(x, u, v))$$

Since this formula and its parts describe semialgebraic sets and real-closed fields have definable Skolem functions [Mar02, Corollary 3.3.26], this induces a *semialgebraic*, so by Lemma 7 Borel measurable, function $\bar{y} : [\![U]\!] \to [\![Y]\!]$ such that[11]

$$\vDash \forall z \in Z \, \exists v \in V \, \forall x \, (f(x, \bar{y}(u), z) = g(x, u, v)) \tag{4}$$

The validity (4) similarly induces a semialgebraic, thus, Borel measurable function $\bar{v} : [\![U]\!] \times [\![Z]\!] \to [\![V]\!]$ such that

$$\vDash \forall x \, (f(x, \bar{y}(u), z) = g(x, u, \bar{v}(u, z))) \tag{5}$$

To show validity of the conclusion, consider any state ξ in which its antecedent is true and show that its succedent is. That is, assume $\xi \in \delta_{x'=g(x,u,v) \, \& \, u \in U^d \& v \in V}([\![F]\!])$, which implies

$$\forall T{\geq}0 \, \forall \gamma \in \mathcal{S}_{U \to V} \, \exists u \in \mathcal{M}_U \, \forall 0{\leq}\zeta{\leq}T \, x_g(\zeta; \xi, u, \gamma(u)) \in [\![F]\!] \tag{6}$$

It remains to be shown that $\xi \in \delta_{x'=f(x,y,z) \, \& \, y \in Y^d \& z \in Z}([\![F]\!])$, which is

$$\forall T{\geq}0 \, \forall \beta \in \mathcal{S}_{Y \to Z} \, \exists y \in \mathcal{M}_Y \, \forall 0{\leq}\zeta{\leq}T \, x_f(\zeta; \xi, y, \beta(y)) \in [\![F]\!] \tag{7}$$

Consider any $T \geq 0$ and $\beta \in \mathcal{S}_{Y \to Z}$. From (6), obtain some $u \in \mathcal{M}_U$ corresponding to the strategy γ defined as

$$\gamma(u)(s) \stackrel{\text{def}}{=} \bar{v}\big(u(s), \beta(\bar{y} \circ u)(s)\big)$$

---

[10]i.e. a function between semialgebraic sets whose graph is semialgebraic.

[11]Substitution of a semialgebraic function $\bar{y}(u)$ for y into a formula $F(u, y)$ of real arithmetic is definable, e.g., by $\forall y \, (y = \bar{y}(u) \to F(u, y))$. The subsequent proof only needs $\bar{y}$ to be measurable, which the measurable selection theorem [RS98, §6 Theorem 6.13] guarantees. The only inconvenience then is that $\bar{y}$ cannot be syntactically inserted into the logical formulas but their mathematical equivalents need to be used.

which defines a function $\gamma : \mathcal{M}_U \to \mathcal{M}_V$, because the composition $\bar{y} \circ u$ of Borel measurable function $\bar{y}$ with measurable u is measurable, which makes $\beta(\bar{y} \circ u)$ measurable and so is its composition with the Borel measurable $\bar{v}$ while u was measurable to begin with. The function γ is also nonanticipative, hence, $\gamma \in \mathcal{S}_{U \to V}$, because for all $\eta \le s \le T$ and $u, \hat{u} \in \mathcal{M}_U$:

$$
\begin{aligned}
&\text{if } u(\tau) = \hat{u}(\tau) && \text{for a.e. } \eta \le \tau \le s \\
&\text{so } (\bar{y} \circ u)(\tau) = (\bar{y} \circ \hat{u})(\tau) && \text{for a.e. } \eta \le \tau \le s \\
&\text{then } \beta(\bar{y} \circ u)(\tau) = \beta(\bar{y} \circ \hat{u})(\tau) && \text{for a.e. } \eta \le \tau \le s \\
&\text{hence } \gamma(u)(\tau) = \gamma(\hat{u})(\tau) && \text{for a.e. } \eta \le \tau \le s
\end{aligned}
$$

because $\beta \in \mathcal{S}_{Y \to Z}$ and the compositions with Borel measurable functions $\bar{y}$ and $\bar{v}$ preserve equality a.e.[12] Define the control y for strategy β by $y(s) \stackrel{\text{def}}{=} (\bar{y} \circ u)(s) = \bar{y}(u(s))$. The corresponding responses x_f and x_g of the respective differential games satisfy

$$
\begin{aligned}
x'_f(s) &= f(x_f(s), y(s), \beta(y)(s)) = f(x_f(s), (\bar{y} \circ u)(s), \beta(\bar{y} \circ u)(s)) \\
x'_g(s) &= g(x_g(s), u(s), \gamma(u)(s)) = g(x_g(s), u(s), \bar{v}(u(s), \beta(\bar{y} \circ u)(s)))
\end{aligned}
$$

which (5) equates

$$
f(x_f(s), \bar{y}(u(s)), \beta(\bar{y} \circ u)(s)) = g(x_f(s), u(s), \bar{v}(u(s), \beta(\bar{y} \circ u)(s)))
$$

so that the response x_f solves the same differential equation that x_g does, which shows $x_f = x_g$ by uniqueness (Lemma 1). Consequently, the antecedent (6) implies (7), which shows the conclusion of DGR to be valid since ξ was arbitrary. $\qquad\square$

4.2 Values of Differential Games

Using the response $x(s) = x(s; \xi, y, \beta(y))$ of differential game (1) for initial condition $x(\eta) = \xi$, the *lower value* of differential game (1) with the player for Z minimizing payoff and the player for Y maximizing captures the optimal payoff with nonanticipative strategies for minimizer for Z, i.e. when minimizer moves last [EK74, ES84, BRP99]. It is defined as:

$$
V(\eta, \xi) = \inf_{\beta \in \mathcal{S}_{Y \to Z}} \sup_{y \in \mathcal{M}_Y} g(x(T; \xi, y, \beta(y))) \tag{8}
$$

$$
= \inf_{\beta \in \mathcal{S}_{Y \to Z}} \sup_{y \in \mathcal{M}_Y} V(\eta + \sigma, x(\eta + \sigma; \xi, y, \beta(y))) \tag{9}
$$

with (9) being the dynamic programming optimality condition [EK74, ES84, Thm 3.1] for any $0 \le \eta < \eta + \sigma \le T$ and $\xi \in \mathbb{R}^n$ for payoff g. With response $x(s) = x(s; \xi, \alpha(z), z)$, the *upper value* of differential game (1) captures the optimal payoff when maximizer for Y moves last and is defined as:

$$
U(\eta, \xi) = \sup_{\alpha \in \mathcal{S}_{Z \to Y}} \inf_{z \in \mathcal{M}_Z} g(x(T; \xi, \alpha(z), z)) \tag{10}
$$

$$
= \sup_{\alpha \in \mathcal{S}_{Z \to Y}} \inf_{z \in \mathcal{M}_Z} U(\eta + \sigma, x(\eta + \sigma; \xi, \alpha(z), z)) \tag{11}
$$

---

[12]If f is function and $g(\tau) = \hat{g}(\tau)$ for a.e. τ, then $f(g(\tau)) = f(\hat{g}(\tau))$ for a.e. τ, because $\{\tau : f(g(\tau)) \ne f(\hat{g}(\tau))\} \subseteq \{\tau : g(\tau) \ne \hat{g}(\tau)\}$ is contained in a set of measure 0.

for any $0 \le \eta < \eta + \sigma \le T$ and $\xi \in \mathbb{R}^n$, again with (11) being the dynamic programming optimality condition.

Theorem 9 (Continuous values [EK74, ES84, Thm 3.2]). *V and U are bounded and Lipschitz (in η, ξ).*

The following observation relates signs of values to the existence of strategies and controls for winning their corresponding differential game at time T.

Lemma 10 (Signs of value). *Let $T > 0$.*

1. $V(0, \xi) > 0$ *iff* $\exists b > 0\ \forall \beta \in \mathcal{S}_{Y \to Z}\ \exists y \in \mathcal{M}_Y\ g(x(T; \xi, y, \beta(y))) > b$.

2. $V(0, \xi) < 0$ *iff* $\exists b < 0\ \exists \beta \in \mathcal{S}_{Y \to Z}\ \forall y \in \mathcal{M}_Y\ g(x(T; \xi, y, \beta(y))) < b$.

3. $V(0, \xi) \ge 0$ *iff* $\forall b < 0\ \forall \beta \in \mathcal{S}_{Y \to Z}\ \exists y \in \mathcal{M}_Y\ g(x(T; \xi, y, \beta(y))) \ge b$.

4. $V(0, \xi) \le 0$ *iff* $\forall b > 0\ \exists \beta \in \mathcal{S}_{Y \to Z}\ \forall y \in \mathcal{M}_Y\ g(x(T; \xi, y, \beta(y))) \le b$.

5. $V(0, \xi) = 0$ *iff* $\forall b > 0$:
   $\forall \beta \in \mathcal{S}_{Y \to Z}\ \exists y \in \mathcal{M}_Y\ g(x(T; \xi, y, \beta(y))) \ge -b$ *and*
   $\exists \beta \in \mathcal{S}_{Y \to Z}\ \forall y \in \mathcal{M}_Y\ g(x(T; \xi, y, \beta(y))) \le b$.

   Similar relations hold for the upper value, e.g.:

6. $U(0, \xi) > 0$ *iff* $\exists b > 0\ \exists \alpha \in \mathcal{S}_{Z \to Y}\ \forall z \in \mathcal{M}_Z\ g(x(T; \xi, \alpha(z), z)) > b$.

7. $U(0, \xi) \ge 0$ *iff* $\exists b > 0\ \exists \alpha \in \mathcal{S}_{Z \to Y}\ \forall z \in \mathcal{M}_Z\ g(x(T; \xi, \alpha(z), z)) \ge b$.

Proof. Case 3 is the contrapositive of Case 2, which proves as follows. If $V(0, \xi) < 0$, then $V(0, \xi) < 2b$ for some $b < 0$. Thus, by (8), $\exists \beta \in \mathcal{S}_{Y \to Z}$ such that $\sup_{y \in \mathcal{M}_Y} g(x(T; \xi, y, \beta(y))) < b$. Hence, $\exists \beta \in \mathcal{S}_{Y \to Z}\ \forall y \in \mathcal{M}_Y\ g(x(T; \xi, y, \beta(y))) < b < 0$. The converse proves accordingly, where b, β are witnesses for $V(0, \xi) = \inf_{\beta \in \mathcal{S}_{Y \to Z}} \sup_{y \in \mathcal{M}_Y} g(x(T; \xi, y, \beta(y))) \le b < 0$.

Case 4 is the contrapositive of Case 1, which proves as follows. If $V(0, \xi) > 0$, then $V(0, \xi) > 2b$ for some $b > 0$. Thus, by (8), $\forall \beta \in \mathcal{S}_{Y \to Z} \sup_{y \in \mathcal{M}_Y} g(x(T; \xi, y, \beta(y))) > 2b$. Hence, $\forall \beta \in \mathcal{S}_{Y \to Z}\ \exists y \in \mathcal{M}_Y\ g(x(T; \xi, y, \beta(y))) > b$. $\square$

The following Lemma 11 is a stronger version of Case 3 and shows that the simplicity of Case 1 continues to hold for $\ge$ instead of $>$. The proof is more complex based on the results developed in the remainder of this section using Tychonoff's theorem, the Borel swap, and a continuous dependency result for Carathéodory solutions that justifies continuous responses of differential games. This stronger version, Lemma 11, makes it possible to lift differential game invariants to closed sets.

Lemma 11 (Closed values). *Let $T > 0$. Then $V(0, \xi) \ge 0$ iff*

$$\forall \beta \in \mathcal{S}_{Y \to Z}\ \exists y \in \mathcal{M}_Y\ g(x(T; \xi, y, \beta(y))) \ge 0$$

Proof. " $\Leftarrow$ ": This direction follows from Case 3 of Lemma 10 because $0 \geq b$ for all $b < 0$.

"$\Rightarrow$": Let $\bar{B} \stackrel{\text{def}}{=} \{\bar{b} : \mathcal{M}_Y \to \mathcal{M}_Z \text{ Borel-measurable}\}$. Note $\bar{B} \subseteq \mathcal{S}_{Y \to Z}$, because the mapping $\bar{b}(y)(s) \stackrel{\text{def}}{=} \bar{b}(y(s))$ is nonanticipative. The infimum over bigger sets is smaller, thus, by $V(0, \xi) \geq 0$:

$$
\begin{aligned}
0 &\leq \inf_{\beta \in \mathcal{S}_{Y \to Z}} \sup_{y \in \mathcal{M}_Y} g(x(T; \xi, y, \beta(y))) \\
&\leq \inf_{\bar{b} \in \bar{B}} \sup_{y \in \mathcal{M}_Y} g(x(T; \xi, y, \bar{b}(y)))) \\
&= \max_{y \in \mathcal{M}_Y} \min_{z \in \mathcal{M}_Z} g(x(T; \xi, y, z)) \qquad\qquad \text{by Lemma 12}
\end{aligned}
$$

Hence, $\exists y \in \mathcal{M}_Y \, \forall z \in \mathcal{M}_Z \, g(x(T; \xi, y, z)) \geq 0$ as $\min, \max$ extrema will be assumed. Since this applies for all possible values $\beta(y) \in \mathcal{M}_Z$ of any $\beta \in \mathcal{S}_{Y \to Z}$, this implies

$$
\forall \beta \in \mathcal{S}_{Y \to Z} \, \exists y \in \mathcal{M}_Y \, g(x(T; \xi, y, \beta(y))) \geq 0
$$

Interestingly, this last step is essentially Herbrandization.

It remains to show that Lemma 12 is applicable. The $[0, T]$-fold product $\{y : [0, T] \to Y\}$ of compact space Y is compact by Tychonoff's theorem with respect to the product topology, i.e. $y_n \to y$ for $n \to \infty$ iff $y_n(s) \to y(s)$ for $n \to \infty$ for all s. Since pointwise limits of measurable functions are measurable [Wal95, 9.9], $\mathcal{M}_Y$ is a closed subset, thus, remains compact [Wal95, 1.16]. Similarly, $\mathcal{M}_Z$ is compact. That $g(x(T; \xi, y, z))$ is continuous (in the product topology which is the one of pointwise convergence) as a functional of y and z follows from Lemma 14 and the continuity of g. $\qquad\square$

Lemma 11 would not hold for infinite time horizon $T = \infty$ or non-compact control sets Y, Z. Lemma 12 is insightful as a separate step, explaining the game's quantifier swaps [Qui11].

Lemma 12 (Borel swap [Qui11]). *If $G : A \times B \to \mathbb{R}$ is continuous on compact A, B and $\bar{B} \stackrel{\text{def}}{=} \{\bar{b} : A \to B \text{ Borel-measurable}\}$:*

$$
\min_{a \in A} \max_{b \in B} G(a, b) = \sup_{\bar{b} \in \bar{B}} \inf_{a \in A} G(a, \bar{b}(a))
$$

$$
\max_{a \in A} \min_{b \in B} G(a, b) = \inf_{\bar{b} \in \bar{B}} \sup_{a \in A} G(a, \bar{b}(a))
$$

Proof. The proof is for the first equation, the second proves by duality. "$\geq$": Fix $\bar{b} \in \bar{B}$. Then for any a: $G(a, \bar{b}(a)) \leq \sup_{b \in B} G(a, b)$. Thus, $\inf_{a \in A} G(a, \bar{b}(a)) \leq \inf_{a \in A} \sup_{b \in B} G(a, b)$. Since $\bar{b}$ was arbitrary $\sup_{\bar{b} \in \bar{B}} \inf_{a \in A} G(a, \bar{b}(a)) \leq \inf_{a \in A} \sup_{b \in B} G(a, b)$.

"$\leq$": Fix $\varepsilon > 0$. For any $a \in A$ choose a $b_a \in B$ such that $G(a, b_a) \geq \sup_{b \in B} G(a, b) - \frac{\varepsilon}{2}$. Since G is continuous and B compact, the function $a \mapsto \sup_{b \in B} G(a, b)$ is continuous. There, thus, is a finite open cover $O_i \subseteq B$ and $b_i \in A$ such that $G(a, b_i) \geq \sup_{b \in B} G(a, b) - \varepsilon$ for $a \in O_i$. Thus, $G(a, b(a)) \geq \sup_{b \in B} G(a, b) - \varepsilon$ for all $a \in A$ when defining the Borel measurable function

$$
b(a) \stackrel{\text{def}}{=} b_i \text{ if } a \in O_i \setminus \bigcup_{j < i} O_j
$$

16

Consequently, $\inf_{a \in A} G(a, b(a)) \geq \inf_{a \in A} \sup_{b \in B} G(a, b) - \varepsilon$. So, $\sup_{\bar{b} \in \bar{B}} \inf_{a \in A} G(a, \bar{b}(a)) \geq \inf_{a \in A} \sup_{b \in B} G(a, b) - \varepsilon \to \inf_{a \in A} \sup_{b \in B} G(a, b)$ for $\varepsilon \to 0$. $\qquad \square$

Carathéodory solutions depend continuously on the right-hand side if uniformly bounded and uniformly Lipschitz.

Lemma 13 (Continuous dependence). *Let $h_n:[\eta, T] \times \mathbb{R}^n \to \mathbb{R}^n$ a sequence of functions that are measurable in t, uniformly L-Lipschitz in x, and of common supremum bound. If $h_n \to h$ for $n \to \infty$ pointwise and x, x_n are Carathéodory solutions:*

$$x'(s) = h(s, x(s)) \quad x_n'(s) = h_n(s, x_n(s)) \quad x(\eta) = x_n(\eta)$$

Then $x_n \to x$ uniformly[13] on $[\eta, T]$ for $n \to \infty$.

Proof. The assumptions are $|h_n(t, x)| \leq B$, $|h_n(t, x) - h_n(t, y)| \leq L|x - y|$ for all n, t, x, y and $h_n(t, x) \to h(t, x)$ for $n \to \infty$ and all t, x. By [Wal00, §10.XIX], x and x_n are Carathéodory solutions of their respective differential equation iff they satisfy corresponding integral equations:

$$x(t) = x(\eta) + \int_\eta^t h(s, x(s))ds$$

$$x_n(t) = x_n(\eta) + \int_\eta^t h_n(s, x_n(s))ds$$

Thus

$$\begin{aligned} |x(t) - x_n(t)| &= |\int_\eta^t h(s, x(s)) - h_n(s, x_n(s))ds| \\ &= |\int_\eta^t h(s, x(s)) - h_n(s, x(s)) + h_n(s, x(s)) - h_n(s, x_n(s))ds| \\ &\leq \int_\eta^t |h(s, x(s)) - h_n(s, x(s))|ds + \int_\eta^t |h_n(s, x(s)) - h_n(s, x_n(s))|ds \\ &\leq \int_\eta^t |h(s, x(s)) - h_n(s, x(s))|ds + \int_\eta^t L|x(s) - x_n(s)|ds \end{aligned}$$

Due to its norm, the first term is nondecreasing, hence Grönwall's inequality implies:

$$|x(t) - x_n(t)| \leq \underbrace{e^{\int_\eta^t Lds}}_{e^{Lt}} \int_\eta^t |h(s, x(s)) - h_n(s, x(s))|ds \to 0$$

for $n \to \infty$ by dominated convergence [Wal95, 9.14], because $|h(s, x(s)) - h_n(s, x(s))| \to 0$ for all s and $|h(s, x(s)) - h_n(s, x(s))|$ is bounded by the Lebesgue-integrable function $2B$ since all h_n are bounded by the same B and so is h as their pointwise limit. $\qquad \square$

---

[13]i.e. convergent in supremum norm $\|x_n - x\|_\infty \to 0$ for $n \to \infty$, which is equivalent to: $\forall \varepsilon > 0 \, \exists n_0 \, \forall n \geq n_0 \, \forall s \, |x_n(s) - x(s)| < \varepsilon$

Lemma 14 (Continuous response). *Responses of a differential game depend continuously on the controls: if $y_n \to y$ and $z_n \to z$ for $n \to \infty$ pointwise, then $x(\cdot; \xi, y_n, z_n) \to x(\cdot; \xi, y, z)$ for $n \to \infty$ uniformly on $[\eta, T]$.*

Proof. Let $y_n \to y$ and $z_n \to z$ for $n \to \infty$ pointwise. Then the respective right-hand sides of (1) converge: $f(s, x, y_n(s), z_n(s)) \to f(s, x, y(s), z(s))$ for $n \to \infty$ pointwise by continuity of f. The response $x(\cdot; \xi, y, z)$ solves (1), which, with $h(s, x) \stackrel{\text{def}}{=} f(s, x, y(s), z(s))$, is

$$x'(s) = h(s, x(s)) \qquad x(\eta) = \xi \tag{12}$$

Likewise, response $x_n(s) \stackrel{\text{def}}{=} x(s; \xi, y_n, z_n)$ solves

$$x_n'(s) = h_n(s, x_n(s)) \qquad x_n(\eta) = \xi \tag{13}$$

with $h_n(s, x) \stackrel{\text{def}}{=} f(s, x, y_n(s), z_n(s)) \to h(s, x)$ pointwise for $n \to \infty$. By Def. 3, h_n and h satisfy the assumptions of Lemma 13 using the Lipschitz constant L of f in x and a bound on f, which implies $x_n \to x$ uniformly for $n \to \infty$. $\square$

Controls are usually not continuous over time nor continuous functions of the state [Háj75, §2.2]. Lemma 14 merely entails that the responses depend continuously on the controls in the product topology. Lemma 14 may not hold when replacing z_n by $\beta(y_n)$, because the nonanticipative strategy β does not have to depend continuously on y_n, so $\beta(y_n)$ may not converge to $\beta(y)$. This is despite the fact that $\mathcal{S}_{Y \to Z}$ is compact, which holds as follows. By Tychonoff's theorem, the product $\{\beta : \mathcal{M}_Y \to \mathcal{M}_Z\}$ is compact since $\mathcal{M}_Z$ is compact (proof of Lemma 11). Since pointwise limits of nonanticipative functions are nonanticipative[14], $\mathcal{S}_{Y \to Z}$ is a closed subset, thus, still compact.

4.3 Viscosity Solutions

The lower and upper values of a differential game satisfy the Isaacs partial differential equation when generalizing the concept of solution beyond classical differentiable cases.

This section recalls viscosity solutions, which are the appropriate notion of weak solutions for Hamilton-Jacobi type partial differential equations [CL83, Bar13], using an elegant characterization with sub- and superdifferentials, which capture all derivatives from below and from above a function [Bre, Bar13]. They are based on single-sided understandings of the gradient operator $D = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$. To emphasize the affected variables x, D is also written as D_x. Another common notation for a single variable t is to write x_t for $D_t x$.

---

[14]Let $\beta_n \to \beta$, i.e. $\beta_n(y) \to \beta(y)$ for all y, which, because of the nested product topology, is $\beta_n(y)(s) \to \beta(y)(s)$ for all s and all y. Let $y(\tau) = \tilde{y}(\tau)$ for a.e. $\tau \leq s$. Then, $\beta_n(y)(\tau) = \beta_n(\tilde{y})(\tau)$ for a.e. $\tau \leq s$, as $\beta_n \in \mathcal{S}_{Y \to Z}$ for all n. This equality (a.e.) is preserved for both limits $\beta_n(y)(\tau) \to \beta(y)(\tau)$ and $\beta_n(\tilde{y})(\tau) \to \beta(\tilde{y})(\tau)$ such that $\beta(y)(\tau) = \beta(\tilde{y})(\tau)$ for a.e. $\tau \leq s$.

Definition 7 (Subdifferentials, superdifferentials)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open. The *superdifferential* $D^+ u(x)$ of $u : \Omega \to \mathbb{R}$ at $x \in \Omega$ and the *subdifferential* $D^- u(x)$ of u at x are defined as

$$D^+ u(x) \stackrel{\text{def}}{=} \{p \in \mathbb{R}^n : \limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \leq 0\}$$

$$D^- u(x) \stackrel{\text{def}}{=} \{p \in \mathbb{R}^n : \liminf_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \geq 0\}$$

Both $D^+ u(x)$, $D^- u(x)$ are closed, convex [AF90, §6.4.3].

Lemma 15 (Characterizations [Bre, Lem. 2.2, 2.5] [Bar13, Thm. 3.3])**.** *Let $u \in C(\Omega)$ on an open set $\Omega \subseteq \mathbb{R}^n$. Then*

1. *$p \in D^+ u(x)$ iff the hyperplane $y \mapsto u(x) + p \cdot (y - x)$ is tangent from above to the graph of u at x. That is:*

$$u(x) + p \cdot (y - x) \geq u(y) \text{ for all } y \text{ sufficiently close to } x$$

   Similarly, $p \in D^- u(x)$ iff it is tangent from below:

$$u(x) + p \cdot (y - x) \leq u(y) \text{ for all } y \text{ sufficiently close to } x$$

2. *$p \in D^+ u(x)$ iff there is a $v \in C^1(\Omega)$ such that $Dv(x) = p$ and $u - v$ has a local maximum[15] at x.*

3. *$p \in D^- u(x)$ iff there is a $v \in C^1(\Omega)$ such that $Dv(x) = p$ and $u - v$ has a local minimum at x.*

4. *If $D^+ u(x) \neq \emptyset$ and $D^- u(x) \neq \emptyset$, then u is differentiable at x.*

5. *If u is differentiable at x, then $D^+ u(x) = D^- u(x) = \{Du(x)\}$ coincides with the gradient $Du(x)$.*

6. *$\{x \in \Omega : D^+ u(x) \neq \emptyset\}$ and $\{x \in \Omega : D^- u(x) \neq \emptyset\}$ are dense in Ω.*

Lemma 16. *The superdifferential $D^+ u(x)$ of the pointwise minimum $u(x) \stackrel{\text{def}}{=} \min_i u_i(x)$ of the functions $u_1, \ldots, u_k : \Omega \to \mathbb{R}$ at $x \in \Omega$ is the convex hull of their support (similar for $D^- \max_i u_i(x)$):*

$$D^+ u(x) = \text{Conv} \bigcup_{u_i(x) = u(x)} D^+ u_i(x)$$

---

[15]$v(x) = u(x)$ can be assumed without loss of generality in both cases. Furthermore, $u - v$ can be assumed a strict local maximum/minimum at x. The property is also equivalent when using smooth $v \in C^\infty(\Omega)$ instead [CEL84].

Proof. "⊆": Let $p \in D^+ u_i(x)$ for some i with $u_i(x) = u(x)$, then $p \in D^+ u(x)$ because:

$$\limsup_{y \to x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|}$$

$$\leq \limsup_{y \to x} \frac{u_i(y) - u_i(x) - p \cdot (y - x)}{|y - x|} \leq 0$$

because $u_i(x) = u(x)$ and $u(y) \leq u_i(y)$ for all y and i. Since $D^+ u(x)$ is convex [AF90, §6.4.3], $D^+ u(x)$ thus contains the convex hull of all such p, which gives the right-hand side.

"⊇": Consider any x and let j such that $u(x) = u_j(x)$. Let $p \in D^+ u(x)$, i.e. for all y close to x:

$$p \cdot (y - x) \leq u(y) - u(x) = \min_i u_i(y) - u_j(x) \leq u_j(y) - u_j(x) \qquad \square$$

Definition 8 (Viscosity solution). Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ continuous with an open $\Omega \subseteq \mathbb{R}^n$. A continuous function $u \in C(\Omega)$ is a *viscosity solution* of the first-order *partial differential equation* (PDE)

$$F(x, u(x), Du(x)) = 0 \tag{14}$$

for *terminal* boundary problems iff it satisfies both:

$$\text{subsolution: } F(x, u(x), p) \geq 0 \text{ for all } p \in D^+ u(x), x \in \Omega$$
$$\text{supersolution: } F(x, u(x), p) \leq 0 \text{ for all } p \in D^- u(x), x \in \Omega$$

By Lemma 15, viscosity solutions are classical solutions, i.e. (14) holds with gradient $Du(x)$, at points x where they are differentiable. PDEs are not extensional, though: (14) and $-F(x, u, Du) = 0$ can have different viscosity solutions [Bre].

The partial differential equation of relevance here is the terminal[16] evolutionary *Hamilton-Jacobi equation*

$$\begin{cases} u_t + H(t, x, Du) = 0 & \text{in } (0, T) \times \mathbb{R}^n & (15a) \\ u(T, x) = g(x) & \text{in } \mathbb{R}^n & (15b) \end{cases}$$

with a continuous Hamiltonian $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and a bounded and uniformly continuous $g : \mathbb{R}^n \to \mathbb{R}$. Bounded, uniformly continuous solutions will suffice for this paper. The Hamiltonian H describes the time derivative of u but simultaneously depends on the space derivatives of u.

The major workhorse for PDEs are comparison theorems [Bre, Thm. 5.3] [Bar13, Thm. 5.2] [BRP99, §2, Thm. 3.3].[16]

Theorem 17 (Comparison). *Let u, v bounded, uniformly continuous sub- and supersolutions of (15a) and $u \leq v$ on $\{T\} \times \mathbb{R}^n$, then $u \leq v$ on $[0, T] \times \mathbb{R}^n$ provided one of the following conditions is true:*

---

[16]Signs in *terminal* value problems reverse compared to *initial* value problems [ES84, Eva10, Chapter 10.3]. A terminal subsolution u of (15a) induces an initial subsolution $w(t, x) = u(T-t, x)$ of $w_t + J(t, x, Dw) = 0, w(0, x) = g(x)$ where $J(t, x, p) = -H(T - t, x, p)$ and likewise for supersolutions.

1. *H is Lipschitz, i.e. there is a C such that*

$$|H(t, x, p) - H(t, x, q)| \leq C|p - q|$$
$$|H(t, x, p) - H(s, y, p)| \leq C(|t - s| + |x - y|)(1 + |p|)$$

2. *u is Lipschitz in x uniformly in t*
3. *v is Lipschitz in x uniformly in t, i.e.*
    $|v(t, x) - v(t, y)| \leq L|x - y|$ *for all* x, y, t

Generalizations to bounded open Ω or modules of continuity exist. If u is growing faster and ends $u \leq v$, it must have been smaller all along, which remains true for viscosity solutions:

Corollary 18 (Monotone comparison). *Assume one of the conditions of Theorem 17 holds or that Hamiltonian J is Lipschitz. Let u be a viscosity subsolution of* (15a) *and v a viscosity supersolution of*

$$v_t + J(t, x, Dv) = 0 \quad in \ (0, T) \times \mathbb{R}^n$$

If $u \leq v$ *on* $\{T\} \times \mathbb{R}^n$ *and* $H \leq J$, *then* $u \leq v$ *on* $[0, T] \times \mathbb{R}^n$.

Proof. v is a supersolution of $v_t + J(t, x, Dv) = 0$ if:

$$\tau + J(t, x, p) \leq 0 \quad \forall (\tau, p) \in D^- v(x) \tag{16}$$

Thus, v is also a supersolution of $v_t + H(t, x, Dv) = 0$, i.e.

$$\tau + H(t, x, p) \leq 0 \quad \forall (\tau, p) \in D^- v(x)$$

which follows from (16) using $H \leq J$. In the case where the conditions of Theorem 17 are satisfied, this implies $u \leq v$ by Theorem 17. Otherwise J is Lipschitz, and the proof proceeds as follows. u is a subsolution of $u_t + H(t, x, Du) = 0$ if:

$$\tau + H(t, x, p) \geq 0 \quad \forall (\tau, p) \in D^+ u(x) \tag{17}$$

Thus, u is also a subsolution of $u_t + J(t, x, Du) = 0$, i.e.

$$\tau + J(t, x, p) \geq 0 \quad \forall (\tau, p) \in D^+ u(x)$$

which follows from (17) using $J \geq H$. Since J is Lipschitz, this implies $u \leq v$ by Theorem 17. □

4.4 Isaacs Equations

Seminal results [Sou85,BEJ84,ES84] characterize the upper and lower values of differential games by the Isaacs equations [Isa67]. For reference, a proof of Theorem 19 is in Appendix C.

Theorem 19 (Isaacs PDE [ES84, Thm 4.1]). *The lower value V from* (8) *of* (1) *is the unique bounded, uniformly continuous viscosity solution of lower Isaacs equation*

$$\begin{cases} V_t + H^-(t, x, D_x V) = 0 & (0 \leq t \leq T, x \in \mathbb{R}^n) \\ V(T, x) = g(x) & (x \in \mathbb{R}^n) \end{cases} \tag{18}$$
$$H^-(t, x, p) = \max_{y \in Y} \min_{z \in Z} f(t, x, y, z) \cdot p$$

and U (10) *the unique such solution of upper Isaacs equation*

$$\begin{cases} U_t + H^+(t, x, D_x U) = 0 & (0 \leq t \leq T, x \in \mathbb{R}^n) \\ U(T, x) = g(x) & (x \in \mathbb{R}^n) \end{cases} \tag{19}$$
$$H^+(t, x, p) = \min_{z \in Z} \max_{y \in Y} f(t, x, y, z) \cdot p$$

Corollary 20 (Minimax [ES84, Corollary 4.2]). *$V \leq U$ holds. If $H^+(t, x, p) = H^-(t, x, p)$ for all $0 \leq t \leq T, x, p \in \mathbb{R}^n$, then $V = U$, i.e. the game has value.*

Proof. $H^- \leq H^+$ holds, so Corollary 18 implies $V \leq U$. If $H^- = H^+$ holds, too, then Corollary 18 implies $U \leq V$. $\qquad\square$

The fact $V \leq U$ follows from the observation that the player who chooses last is at advantage for optimizing the resulting value. The assumption $H^+(t, x, p) = H^-(t, x, p)$ corresponds to order independence, which implies $V = U$.

Corollary 21. *The Isaacs equations for U, V hold classically a.e. in $(0, T) \times \mathbb{R}^n$, i.e. except on a subset of a measure 0 set.*

Proof. By Theorem 9, U and V are Lipschitz and, hence, by Rademacher's theorem, differentiable a.e., which implies by Lemma 15 that the Isaacs equations that U and V satisfy by Theorem 19 hold classically in those points. $\qquad\square$

4.5 Frozen Games

The results from Sect. 4.2 and 4.4 characterize winning regions of differential games by signs of the solutions of their PDEs if Angel chooses $\zeta = T$. Lifting these characterizations to the case where Angel decides to stop early by choosing $\zeta < T$ is possible by repeating the same analysis for minimum payoff games [Ser02]. This leads to less convenient PDEs, though.

An easier way is to add an extra freeze input [MBT05] for Angel, which she can control to slow down or lock the system in place. The freeze factor $c \in [0, 1]$ multiplies the differential game and is under Angel's control, which will keep the system unmodified ($c = 1$), in stasis ($c = 0$), or in slow motion ($0 < c < 1$). Angel controls time ζ and freeze factor c. So the frozen system does not need early stopping, because she can freeze $c = 0$ instead. The quantifier for ζ in Def. 5 is, thus, irrelevant.

Lemma 22 (Frozen values). *For any atomically open F: $\xi \in \delta_{x'=cf(x,y,z)\,\&\,y\in Y^d\&z\in Z\wedge c\in[0,1]}(\llbracket F \rrbracket)$ iff its lower value satisfies $V(0,\xi) > 0$ for all $T \geq 0$ when using the realization $g \stackrel{\text{def}}{=} F^{\Re}$ as payoff. Accordingly for atomically closed F.*

Proof. "$\Rightarrow$": by Case 1 of Lemma 10 using Lemma 3.

"$\Leftarrow$": By Lemma 3 and Case 1 of Lemma 10, it only remains to be shown that ζ can always be instantiated to T in Def. 5 for this game. Instead of stopping prematurely at $\zeta < T$, Angel can set her extra freeze input c to 0 at ζ, because $c = 0$ will keep x constant.

The proof for closed F uses Lemma 11 instead. $\qquad\square$

When replacing all differential games with their frozen version, Lemma 22 implies that the results from Sect. 4.2 and 4.4 characterize their winning regions by signs of values. Yet, it is more efficient to exploit the structure of the frozen game to get rid of c with a minimal change in the Hamiltonian.

Lemma 23 (Frozen Isaacs). *According to Theorem 19, let H^- and H^+ the Hamiltonians for the lower and upper values of*

$$x' = f(x, y, z) \,\&\, y \in Y^d \& z \in Z \tag{20}$$

Then the lower and upper values of the frozen differential game

$$x' = cf(x, y, z) \,\&\, y \in Y^d \& z \in Z \wedge c \in [0, 1] \tag{21}$$

respect the lower (18) and upper (19) Isaacs equations with the following Hamiltonians instead:

$$J^-(t, x, p) = \min(0, H^-(t, x, p)) \tag{22}$$
$$J^+(t, x, p) = \min(0, H^+(t, x, p))$$

Proof. By Theorem 19, the lower value and upper value of (21) satisfy the lower and upper Isaacs equations with the following Hamiltonians:

$$
\begin{aligned}
J^-(t, x, p) &= \max_{y \in Y} \min_{z \in Z} \min_{c \in [0,1]} cf(t, x, y, z) \cdot p \\
&= \max_{y \in Y} \min_{z \in Z} \min(0, f(t, x, y, z) \cdot p) \\
&= \min(0, \max_{y \in Y} \min_{z \in Z} f(t, x, y, z) \cdot p) \\
J^+(t, x, p) &= \min_{c \in [0,1]} \min_{z \in Z} \max_{y \in Y} cf(t, x, y, z) \cdot p \\
&= \min(0, \min_{z \in Z} \max_{y \in Y} f(t, x, y, z) \cdot p)
\end{aligned}
$$

since $\min$ and $\max$ are mutually distributive. By Corollary 18, those transformations do not change the solution. $\qquad\square$

When starting both differential games in the same initial state with the same payoff, the lower and upper value of (20), thus, dominate the lower and upper value, respectively, of (21), by Corollary 18, because $J^-(t, x, p) \leq H^-(t, x, p)$ and $J^+(t, x, p) \leq H^+(t, x, p)$. The freeze input c can be removed from the Hamiltonian by Lemma 23. Indeed, c does not ever need to be introduced into differential games explicitly either, because both winning regions are identical, based on [MBT05]:

Lemma 24 (Superfluous freezing). *Let $X \subseteq \mathcal{S}$. Then*

$$\delta_{x'=f(x,y,z)\,\&\,y\in Y^d\&z\in Z}(X) = \delta_{x'=cf(x,y,z)\,\&\,y\in Y^d\&z\in Z\wedge c\in[0,1]}(X)$$

$$\varsigma_{x'=f(x,y,z)\,\&\,y\in Y^d\&z\in Z}(X) = \varsigma_{x'=cf(x,y,z)\,\&\,y\in Y^d\&z\in Z\wedge c\in[0,1]}(X)$$

Proof. Angel can set her extra freeze input c to 0 to freeze the evolution of the game any time, which, by smart freezing, will make the final outcome of the frozen game at time T equal to its value at freeze time. Hence, frozen systems do not need to be stopped earlier (and the quantifier for ζ in Def. 5 is irrelevant). The addition of c does not otherwise affect the system behavior or capabilities, because its only effect is a time dilation. Note that time is not observable unless the differential game includes a clock $t' = 1$, in which case that clock will be frozen when changing c. The idea is to speed up the system such that c is 1 first and then drops and stays at zero. And then to show that such an evolution of c can be emulated by the other system by stopping when c drops to zero. By Theorem 2, the equations imply each other, so the proof only considers $\delta_{...}(X)$.
"$\supseteq$": This inclusion follows from the soundness of the DGR proof step:

$$\frac{\forall u\in Y\,\exists y\in Y\,\forall z\in Z\,\exists v\in Z,\,c\in[0,1]\,\forall x\,(f(x,y,z)=cf(x,u,v))}{[x'{=}cf(x,u,v)\,\&\,u\in Y^d\&v\in Z\wedge c\in[0,1]]F \to [x'{=}f(x,y,z)\,\&\,y\in Y^d\&z\in Z]F}$$

whose premise proves using $y \stackrel{\text{def}}{=} u$, $v \stackrel{\text{def}}{=} z$, $c \stackrel{\text{def}}{=} 1$.
"$\subseteq$": This direction has been shown elsewhere [MBT05, Corollary 5]. $\qquad\qquad\square$

In a similar way, differential games restricted to evolution domains are definable by the *dual freezing game* that gives another freeze factor b to Demon with which he can suspend the system should Angel ever try to leave the domain. A differential game with evolution domain C has to remain in C and stop before leaving it. But only Angel is in control of time. She might try to leave C temporarily and sneak back before Demon notices. Adding the dual freeze factor b to the game gives Demon the option of slowing it down and challenging Angel to demonstrate to still be in C. Ensuring that Demon does not slow the game down just to prevent Angel's progress to victory is possible by exploiting hybrid games around it:

$$t := x_0;\ (x' = bf(x,y,z), t' = 1\,\&\,y \in Y \wedge b \in [0,1]^d\&z \in Z);$$
$$?C;\,?(x_0 = t)^d$$

This reduction assumes that the (vectorial) differential game $x' = f(x,y,z)$ contains a deterministic clock $x_0' = 1$ and adds a separate unfrozen absolute clock $t' = 1$ starting from the same value after the assignment $t := x_0$. To slow the system down, Demon needs to choose $b < 1$ on a set of non-zero measure (otherwise $b = 1$ a.e., which has no effect). That will slow down the frozen $x_0' = b$ compared to the unfrozen $t' = 1$, so that Demon fails his time synchronicity test $?(x_0 = t)^d$ and loses. Unless he correctly points out that the system left the domain C, in which case Angel will lose because she fails her test $?C$ first. Even though Demon has no influence on Angel's choice of time ζ, he can choose $b = 0$ to force the game into stasis any time. He just needs to use that

power wisely or else he will lose the game for false allegations. This is the differential game analogon of the "there and back again game" for differential equations with evolution domains [Pla14]. The hybrid game enables simpler differential games compared to incorporating state constraints into differential games [Rap98].

4.6   Soundness of Differential Game Invariants

This completes the background required for proving soundness of DGI. The soundness proof proves the arithmetized postcondition to be a viscosity subsolution (Sect. 4.3) of the lower Isaacs PDE that characterizes (Sect. 4.4) the lower value whose sign characterizes (Sect. 4.2) winning regions (Sect. 2) independently of premature stopping (Sect. 4.5).

Theorem 25 (Soundness of differential game invariants). *Differential game invariants (rule DGI) are sound.*

Proof. To prove soundness, assume the premise to be valid and the antecedent of the conclusion true in a state ξ:

$$\vDash \exists y \in Y \, \forall z \in Z \, F'^{f(x,y,z)}_{x'} \tag{23}$$

$$\xi \vDash F \tag{24}$$

To make the proof easier to follow, the proof first considers the case where F is a simpler formula even if that follows from subsequent cases.

1) Consider the case where F is of the form $F \equiv (g > 0)$ for a (smooth) term g. Then the (valid) premise (23) of DGI specializes to $\exists y \in Y \, \forall z \in Z \, (g > 0)'^{f(x,y,z)}_{x'}$, which is

$$\vDash \exists y \in Y \, \forall z \in Z \, (\nabla(g)^{f(x,y,z)}_{x'} \geq 0) \tag{25}$$

When $\xi \in \mathcal{S}$ is a state, adopt the usual mathematical liberties of writing $g(\xi)$ for the value $[\![g]\!]_\xi$ of term g in state $\xi \in \mathcal{S}$ to simplify notation substantially and keep it closer to mathematical practice. Similarly for $f(x, y, z)$, since it will be clear from the context whether the term $f(x, y, z)$ or its value is being referred to. If all the x, y, z are variables, $f(x, y, z)$ is a term. If, instead, ξ, η, ζ are all (vectors of) reals, $f(\xi, \eta, \zeta)$ refers to the corresponding value $[\![f(x, y, z)]\!]_{\sigma^{\xi\eta\zeta}_{x_y z}}$ instead. Mixed cases where some x, y, z are variables and others are reals are not defined to avoid confusion.

Consider any time horizon $T \geq 0$ of Angel's choosing. The proof first shows that the time-invariant extension $\bar{g}(t, x) \stackrel{\text{def}}{=} g(x)$ is a subsolution of the lower Isaacs equation (18) with unique solution V (Theorem 19), which, by Theorem 17, implies $\bar{g} \leq V$, because both functions coincide at time T. Since $\bar{g}$ is smooth, it, by Lemma 15, is a subsolution iff it satisfies the subsolution inequality classically at every (η, ξ):

$$\underbrace{\bar{g}_t(\eta, \xi)}_{0} + \underbrace{\max_{y \in Y} \min_{z \in Z} f(\xi, y, z) \cdot D_x \bar{g}(\eta, \xi)}_{\geq 0} \geq 0 \tag{26}$$

25

which holds since $\bar{g}$ is time-invariant so its time-derivative $\bar{g}_t$ vanishes and by premise (25), recalling that $f(\xi, y, z) \cdot D_x\bar{g}(\eta, \xi) = [\![\nabla(g)_{x'}^{f(x,y,z)}]\!]_\xi$ for all ζ, y, z by Lemma 5, so that (25) implies:

$$\exists y \in Y \, \forall z \in Z \, f(\xi, y, z) \cdot D_x\bar{g}(\eta, \xi) = [\![\nabla(g)_{x'}^{f(x,y,z)}]\!]_\xi \geq 0$$

By (26), $\bar{g}$ is a subsolution of (18), so $g(\xi) = \bar{g}(\eta, \xi) \leq V(\eta, \xi)$ for all η, ξ by Theorem 17, which is applicable because V is bounded and uniformly continuous by Theorem 19, and Lipschitz in x, t by Theorem 9, thus, Lipschitz in x *uniformly in* t since t is bounded by T so the maximum Lipschitz bound among $t \in [0, T]$ is finite. For the applicability of Theorem 17, note that g and $\bar{g}$ are bounded and Lipschitz (on the domain from Lemma 1) by Def. 3 and, thus, uniformly continuous by Footnote 4.

So $V(\eta, \xi) \geq g(\xi) > 0$ for all η and any initial state ξ that satisfies the antecedent $F \equiv (g > 0)$ of the conclusion of DGI, i.e. (24) which is $g(\xi) > 0$. Hence, Case 1 of Lemma 10 implies

$$\forall \beta \in \mathcal{S}_{Y \to Z} \, \exists y \in \mathcal{M}_Y \, g(x(T; \xi, y, \beta(y))) > 0$$

This shows that Demon can achieve $g > 0$ from any initial state ξ where $g > 0$ holds if Angel decides to evolve the full duration T. Since $g(\xi) \leq V(t, \xi)$ is a time-independent lower bound for all times t and all time horizons T, Angel cannot achieve a lower value of g by stopping earlier:

Part 1. If the payoff g is a time-independent subsolution of (18) with $g(\xi) > 0$, then

$$\xi \in \delta_{x'=f(x,y,z) \, \& \, y \in Y^d \& z \in Z}([\![g > 0]\!]) \tag{27}$$

The case $g(\xi) \geq 0$ is accordingly with $[\![g \geq 0]\!]$ instead.

Subproof: The function g is a subsolution of (18) iff:

$$\underbrace{\tau}_{0} + H^-(t, x, p) \geq 0 \text{ for all } (\tau, p) \in D^+ g(t, x) \text{ and all } t, x$$

Thus, g is a subsolution of the frozen lower Isaacs equation with Hamiltonian (22) from Lemma 23:

$$\underbrace{\tau}_{0} + \min(0, H^-(t, x, p)) \geq 0 \text{ for } (\tau, p) \in D^+ g(t, x) \text{ and } t, x$$

Thus, the lower value of the frozen game (21) has lower bound g. By Lemma 22, it does not need premature stopping, so that Lemma 10 proves

$$\xi \in \delta_{x'=cf(x,y,z) \, \& \, y \in Y^d \& z \in Z \wedge c \in [0,1]}([\![g > 0]\!])$$

since $T \geq 0$ was arbitrary. The "$\supseteq$" inclusion of Lemma 24, which follows from Theorem 8, then implies (27), concluding the subproof.

Alternatively, without any freezing, g is a subsolution of the Isaacs equation for infimum cost [Ser02]

$$\min(v_t(t, x) + h^-(x, v(t, x), D_x v(t, x)), \, g(x) - v(t, x)) = 0$$

$$h^-(x, r, p) = \begin{cases} \max_{y \in Y} \min_{z \in Z} f(x, y, z) \cdot p & \text{if } g(x) \leq r \\ \infty & \text{if } g(x) > r \end{cases}$$

26

that the infimum cost value over time solves

$$v(\eta, \xi) = \inf_{\beta \in \mathcal{S}_{Y \to Z}} \sup_{y \in \mathcal{M}_Y} \min_{t \leq T} g(x(t; \xi, y, \beta(y)))$$

because the choice of $g(x)$ for $v(t, x)$ satisfies

$$\min(\tau + h^-(x, \bar{g}(t, x), p), \, g(x) - \bar{g}(t, x)) \geq 0 \,\, \mathbb{V}(\tau, p) \in D^+ \, \bar{g}(x)$$

Lemma 10 carries over to v with an extra $\exists t \leq T$ for time, so that $0 < g(\xi) \leq v(0, \xi)$ directly shows

$$\xi \in \delta_{x'=f(x,y,z) \, \& \, y \in Y^d \& z \in Z}(\llbracket g > 0 \rrbracket)$$

The downside of this alternative proof, which also works for time-dependent g, though, is that the PDE assumes a convex image of f under Y and under Z to facilitate discontinuous games [Ser02], which are not needed here. $\qquad\square$

2) Consider the case where F is of the form $F \equiv (g \geq 0)$ for a (smooth) term g. Then the proof proceeds as in Case 1, since the premise of DGI is still (25), because $\nabla(g \geq 0)$ is equivalent to $\nabla(g > 0)$ by Def. 6. In that case, the antecedent (24) only implies $\xi \models g \geq 0$ in the initial state ξ, thus, $V(\eta, \xi) \geq g(\xi) \geq 0$ for all η. Yet, then Lemma 11 still implies

$$\mathbb{V}\beta \in \mathcal{S}_{Y \to Z} \, \exists y \in \mathcal{M}_Y \, g(x(T; \xi, y, \beta(y))) \geq 0$$

which shows the conclusion of DGI by Part 1.

3) Consider the case where F is atomically open. By congruence, it is enough to consider the case where F is normalized by $(a < b) \equiv (b - a > 0)$ so that it is built with $\wedge, \vee$ from formulas of the form $g_i > 0$. Let $I \stackrel{\text{def}}{=} \{i : g_i(\xi) > 0\} \neq \emptyset$ the set of all indices whose atomic formula is true in the initial state ξ. Part 2 shows that the time-invariant minimum $\bar{g}(t, x) \stackrel{\text{def}}{=} \min_{i \in I} g_i(x)$ of the involved continuously differentiable g_i is a subsolution of the lower Isaacs equation.

Part 2. $\bar{g}$ is a subsolution of lower Isaacs equation (18).

Validity of the conclusion of DGI follows from Part 2 like for Case 1 with Part 1 and the observation that the combination of subformulas of F that were true initially will stay true using Lemma 3, because $0 < \bar{g}(\eta, \xi) \leq V(\eta, \xi)$ for all η and the initial state ξ, which satisfies the antecedent (24).

Subproof of Part 2: Now, the proof from Case 1 no longer works, because $\bar{g}$ has no differentials at points where the minimum switches from one g_i to another g_j unless their differentials happen to align. A similar idea applies, however.

The premise (23) in this case yields

$$\models \forall x \, \exists y \in Y \, \forall z \in Z \, \bigwedge_i (g_i \geq 0)'^{f(x,y,z)}_{x'} \tag{28}$$

which, in mathematical metalanguage is

$$\mathbb{V}x \, \exists y \in Y \, \mathbb{V}z \in Z \, f(x, y, z) \cdot Dg_i(x) \geq 0 \text{ for all } i \tag{29}$$

because $(g_i \geq 0)'^{f(x,y,z)}_{x'}$ is $\nabla(g_i)^{f(x,y,z)}_{x'} \geq 0$, which is $f(x,y,z) \cdot Dg_i(x) \geq 0$ by Lemma 5. Proving that $\bar{g}$ is a subsolution of lower Isaacs (18) requires proving

$$\underbrace{\tau}_{0} + \max_{y \in Y} \min_{z \in Z} f(x,y,z) \cdot p \geq 0 \qquad (30)$$

for all $(\tau, p) \in D^+\bar{g}(t,x)$ and all $x \in \mathcal{S}$. Since $\bar{g}$ is time-invariant, it is differentiable by t with derivative 0 everywhere, hence the time component of its superdifferential coincides with the gradient $\tau = 0$ by Lemma 15. Dropping time from the notation simplifies (30) to:

$$\max_{y \in Y} \min_{z \in Z} f(x,y,z) \cdot p \geq 0 \text{ for all } p \in D^+\bar{g}(x) \text{ and all } x \qquad (31)$$

That is, it remains to show:

$$\forall x \, \forall p \in D^+\bar{g}(x) \, \exists y \in Y \, \forall z \in Z \, f(x,y,z) \cdot p \geq 0$$

For any x, using the corresponding $y \in Y$ from (29), it is the case that for all $z \in Z$ and all i:

$$f(x,y,z) \cdot D^+g_i(x) \geq 0$$

because $D^+g_i(x) = \{Dg_i\}$ by Lemma 15. According to Lemma 16, all convex generators of $D^+\bar{g}$, thus, satisfy this property, which continues to hold for convex combinations, since for any $p, q \in D^+\bar{g}(x)$ and $\lambda \in [0,1]$:

$$f(x,y,z) \cdot (\lambda p + (1-\lambda)q) = \lambda f(x,y,z) \cdot p + (1-\lambda)f(x,y,z) \cdot q \geq 0$$

This proves (31), so that $\bar{g}$ is a subsolution of (18). $\qquad \square \qquad\qquad \square$

4) The case where F is atomically closed proceeds as in Case 3. The premise of DGI is equivalent to the premise in Case 3, because $\nabla(a \geq b)$ and $\nabla(a > b)$ are equivalent by Def. 6. The additional thought is as for Case 2. Since $\bar{g}$ is a subsolution, the same combination of subformulas of F that were true initially will stay true.

5) The case where F is any first-order formula (quantifier-free by quantifier elimination [Tar51]) reduces to Case 4. By congruence, it is enough to consider the case where F is normalized by $(a < b) \equiv (b - a > 0)$ and $(a = b) \equiv (a - b \geq 0 \wedge b - a \geq 0)$ etc. so that it is built with $\wedge, \vee$ from formulas of the form $g_i \geq 0$ or $h_j > 0$. Replace every strict inequality $h_j > 0$ in F that is true in the initial state ξ by a new weak inequality $g_j \geq 0$ with the term $g_j \stackrel{\text{def}}{=} h_j - a_j$, which is still true in the initial state when choosing the constant $a_j \stackrel{\text{def}}{=} h_j(\xi) > 0$. Replace every strict inequality $h_j > 0$ that is not true in the initial state ξ by $-1 \geq 0$. The resulting formula G is closed, true in the initial state, and, if Demon has a strategy to achieve G, then, by monotonicity of winning regions, he also has a strategy to achieve the original F, because $\vDash G \rightarrow F$. Case 4 implies that Demon can achieve G, because the premise of DGI that Case 4 assumes for G is implied by the premise for F since $\nabla(h_j > 0)$ is equivalent to $\nabla(h_j \geq 0)$ which is equivalent to $\nabla(h_j - a_j \geq 0)$ by Def. 6 as $\nabla(a_j) = 0$ for constant a_j. Likewise $\nabla(-1 \geq 0) \equiv (0 \geq 0)$ is trivially implied.

This concludes the proof of Theorem 25.

4.7 Soundness of Differential Game Variants

Since DGV settles for a conservative quantifier pattern, the soundness proof for DGI can be adapted more easily for DGV.

Theorem 26 (Soundness of differential game variants). *Differential game variants (rule DGV) are sound.*

Proof. Let $\xi \models g < 0$, i.e. $g(\xi) < 0$, otherwise Angel wins by choosing $T = 0$. The proof follows the same principle as the proof of Theorem 25 by using the duality Theorem 2, since the same game is played in $[x' = f(x, y, z) \& y \in Y^d \& z \in Z]$ and $\langle x' = f(x, y, z) \& y \in Y^d \& z \in Z \rangle$ with the same partition of control advantage and information just from another player's perspective. To facilitate proof reuse, rule DGV uses a conservative information pattern, so that the duality allows to swap player controls and consider $[x' = f(x, y, z) \& z \in Z^d \& y \in Y](g \geq 0)$. This formula cannot be expected to be true, since the current state does not need to satisfy $g \geq 0$ for Angel would stop right away then. Yet, the study of its value will still prove to be informative and, in particular, reuse the proof of Theorem 25. The only, but critical, change is that DGV does not assume the postcondition to hold in the beginning and, instead, requires a proof that it will finally be reached. This leads to the following variation on the choice of the subsolution for the comparison theorem. Let ε the value whose existence the premise shows. For postcondition $g \geq 0$, consider $\bar{g}(t, x) \overset{\text{def}}{=} g(x) + \frac{\varepsilon}{2}(T - t)$. This $\bar{g}$ is smooth, so, by Lemma 15, a subsolution of the lower Isaacs equation (18) iff:

$$\underbrace{\bar{g}_t(t, x)}_{-\frac{\varepsilon}{2}} + \underbrace{\max_{y \in Y} \min_{z \in Z} f(x, y, z) \cdot D_x \bar{g}(t, x)}_{\geq \varepsilon} \geq 0 \tag{32}$$

which again holds by premise using Lemma 5 if its assumption $g(x) \leq 0$ holds. The left-hand side of (32) is $\geq \frac{\varepsilon}{2}$ on the closed set $[\![g \leq 0]\!]$, and is a continuous functions, so it continues to be > 0 on sufficiently small neighborhoods of $[\![g \leq 0]\!]$. Thus, the argument from Theorem 25 continues to work when restricting the domain to a sufficiently small open neighborhood $\mathcal{U}$ of $[\![g \leq 0]\!]$. Since $\bar{g}(\eta, \xi) \leq V(\eta, \xi)$ follows from Theorem 17 as in Theorem 25 and Lemma 11 proves the conclusion of DGV from $0 \leq V(0, \xi)$, this will happen for large enough T according to the definition of $\bar{g}$. In particular, $0 < \bar{g}(\eta, \xi) \leq V(\eta, \xi)$ when T is sufficiently large, e.g. $T > -\frac{2}{\varepsilon}g(\xi) > 0$, which is under Angel's control. The existence of a (unique) solution of such a duration T follows from Perron's existence theorem for Hamilton Jacobi PDEs [Bar13, Thm. 7.1].

For this T, by Lemma 10, Demon of the flipped game, who plays for Angel's controls of the original differential game, will ultimately be in a state where $g \geq 0$, if he just happens to be lucky that such a long time is played and the game does not stop prematurely, so $\zeta = T$ such that (18) characterizes the lower value (otherwise the frozen Isaacs Hamiltonian (22) would apply so that (32) stops holding). For the original differential game, in which Angel is in charge of controlling the time, this means that she can win $g \geq 0$ by playing long enough, which is under her control, and by limiting herself to $\zeta = T$, which is her choice. Since $0 < \bar{g}(\eta, \xi) \leq V(\eta, \xi)$ for all η and $g(x(s; \xi, y, \beta(y)))$ is continuous in s (Lemma 1), Angel will win into $[\![g \geq 0]\!]$ before leaving the open neighborhood $\mathcal{U}$ of $[\![g \leq 0]\!]$.

It is of apparent significance for DGV that the lower bound ε holds for all x, not just that there is an ε for every x. Otherwise, the progress might converge (long) before $g \geq 0$ is reached. Note that it is also possible to prove soundness of DGV based on the soundness proof of DGR. That works by replacing the Hamiltonian in (32) by a uniformly continuous continuation J (which exists by Tietze [Wal95, 2.19]) to the full space, which agrees with the Hamiltonian from (32) on the open neighborhood $\mathcal{U}$ of $[\![g \leq 0]\!]$ and shares the same lower bound ε, but globally. The proof then uses soundness of the $\langle \cdot \rangle$ dual of DGR to show that the original game has a winning strategy since the game corresponding to J has a winning strategy for $g \geq 0$. The only additional thought is that it is enough to restrict the premise of DGR to the set of x that can occur during the game starting from ξ, which is where the values of the original game and the one for the Hamiltonian J coincide by Tietze. $\qquad \square$

5 Differential Game Embeddings

The previous sections have immersed differential games within hybrid games to form differential hybrid games and studied how their properties can be proved. This is a useful approach in practice. The alternative is to understand how differential games relate to (non-differential) hybrid games from a theoretical perspective. Tracing in $\mathsf{dG}\mathcal{L}$ the characterizations developed in this paper for open or closed postconditions gives:

Theorem 27 (Differential game characterization)**.** *Differential games are hybrid games, i.e. differential game logic of differential hybrid games ($\mathsf{dG}\mathcal{L}_{DHG}$) and differential game logic of hybrid games ($\mathsf{dG}\mathcal{L}_{HG}$) are equally expressive:*[17] $\mathsf{dG}\mathcal{L}_{HG} \equiv \mathsf{dG}\mathcal{L}_{DHG}$*.*

Proof. The proof uses the results in Appendix A. The nontrivial direction $\mathsf{dG}\mathcal{L}_{DHG} \equiv \mathsf{dG}\mathcal{L}_{HG}$ can be shown by a careful analysis of the constructions involved in characterizing differential games. The original definition of differential games and their behavior in terms of nonanticipative strategies and measurable functions of control input does not lend itself naturally to a characterization without facing substantial challenges of having to characterize higher-order quantification in large function classes. The indirect characterization of a differential game in terms of its Isaacs equations proves to be more useful. Using expressiveness results for the base logic [Pla12a, Pla14], it is enough to consider the new elementary cases

$$[x' = f(x, y, z) \, \& \, y \in Y^d \& z \in Z]F \tag{33}$$

and $\langle x' = f(x, y, z) \, \& \, y \in Y^d \& z \in Z \rangle F$. By Theorem 2 it is enough to consider (33).
1) Consider the case where F is atomically open. By Lemma 24, (33) is equivalent to its frozen analogon[18] $[x' = cf(x, y, z) \, \& \, y \in Y^d \& z \in Z \wedge c \in [0, 1]]F$. By Lemma 22, the latter needs no premature stopping and is true in a state ξ iff $V(0, \xi) > 0$ for all $T \geq 0$, using the realization

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[17]Logic $\mathcal{B}$ is at least as expressive as $\mathcal{A}$, written $\mathcal{A} \leq \mathcal{B}$ if every formula of $\mathcal{A}$ can be expressed by an equivalent formula of $\mathcal{B}$. Further, $\mathcal{A} \equiv \mathcal{B}$ if $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \equiv \mathcal{A}$. And $\mathcal{A} < \mathcal{B}$ if $\mathcal{A} \leq \mathcal{B}$ but not $\mathcal{B} < \mathcal{A}$.

[18]As in Part 1 of Theorem 25, Theorem 27 can alternatively be proved using the Isaacs equations for infimum cost [Ser02] instead of the frozen differential game from Lemma 24.

$g \overset{\text{def}}{=} F^{\Re}$ as payoff. By Lemma 23, V satisfies the lower Isaacs equation (18) with the Hamiltonian (22). Thus, (33) is true in ξ iff $V(0, \xi) > 0$ for all $T \geq 0$. The quantification over T is definable in first-order real arithmetic. So is the condition whether the state characterized by a variable vector x satisfies $V(0, x) > 0$ if only V and its evaluation can be characterized, which is what Corollary 32 in Appendix A shows. Postponing the evaluation of the (by Theorem 9 continuous) V and its existential quantification until Corollary 32, the proof proceeds with the characterization of V in logic. By Theorem 19, V is the unique bounded, uniformly continuous viscosity solution of the lower Isaacs equation (18) with the Hamiltonian (22) from Lemma 23. Boundedness and uniform continuity are characterizable in first-order real arithmetic, just as long as evaluation of V is. The terminal condition, $V(T, x) = g(x)$ for all x, is characterizable by quantification and evaluation. The fact that V solves the (by Lemma 23 frozen) Isaacs equation

$$V_t + \max_{y \in Y} \min_{z \in Z} \min_{c \in [0,1]} cf(x, y, z) \cdot DV = 0$$

can be characterized by the definable condition

$$\tau + \max_{y \in Y} \min_{z \in Z} \min_{c \in [0,1]} cf(x, y, z) \cdot p \geq 0 \ \ \mathbb{V}(\tau, p) \in D^+ V(t, x)$$

provided quantification over all superdifferentials (τ, p) is definable. Once that succeeds, the argument is then the same to characterize that V is a viscosity supersolution.

Dropping the time coordinates t, τ for notational simplicity, Def. 7 implies that $p \in D^+ V(x)$ iff

$$\limsup_{y \to x} \frac{V(y) - V(x) - p \cdot (y - x)}{|y - x|} \leq 0$$

which is characterizable as follows. Abbreviating the definable

$$(V(y) - V(x) - p \cdot (y - x)) / (|y - x|)$$

by $h(y)$,

$$\limsup_{y \to x} h(y) = \inf_{\varepsilon > 0} \sup \{ h(y) : 0 < |y - x| < \varepsilon \}$$

Whether, for a $\varepsilon > 0$, the inner $\sup$ has value s is definable:

$$\forall y \left(0 < |y - x| < \varepsilon \to s \geq h(y) \right) \wedge \forall b \left(\forall y \left(0 < |y - x| < \varepsilon \to s \geq h(y) \right) \to s \leq b \right)$$

A similar first-order formula around this one characterizes the value of the outer $\inf$ in terms of s.

Consequently, the set of states where $\mathsf{dG\mathcal{L}}$ formula (33) is true is characterizable in $\mathsf{dG\mathcal{L}}$ without using differential games.

2) The case of closed formulas F is accordingly, using the criterion Case 3 from Lemma 10 or Lemma 11 instead.

$\square$

Theorem 28 (Expressive power). *Differential games are strictly less expressive than hybrid games, i.e. differential game logic of differential games ($\mathsf{dG\mathcal{L}}_{DG}$) is less expressive than differential game logic of hybrid games: $\mathsf{dG\mathcal{L}}_{DG} < \mathsf{dG\mathcal{L}}_{HG}$.*

Proof. The proof of Theorem 27 does not rely on special features of hybrid games but continues to work when characterizing differential games in $d\mathcal{L}$, the corresponding logic of hybrid systems [Pla12a]. Thus, [Pla14, Thm. 19] shows that hybrid systems are strictly less expressive than hybrid games. □

This is surprising, because the contrary holds for hybrid systems. Hybrid systems are equivalently reducible to differential equations [Pla12a]. Theorem 28 shows that this situation is quite different for differential games versus hybrid games.

6 Related Work

A general overview of the long history of differential games since their conception [Isa67, Fri71] and breakthroughs of their viscosity understanding [Sou85, BEJ84, ES84] is in the literature [BRP99]. This discussion of related work focuses on differential games as they relate to hybrid games. Hybrid games themselves [NRY96, HHM99, TLS00, DR06, BBC10, VPVD11] are discussed elsewhere [Pla14]. See [MBT05] for a helpful broader overview of hybrid systems verification and how Lagrangian verification relates to Eulerian verification. The relationship of differential games to (robust) control theory [BM08], which is interesting for piecewise continuous controls or under linearity assumptions but does not give a provably sound approach for general differential games, is elaborated in the literature [BRP99, MBT05, CQSP07].

Previous techniques for handling and understanding differential games revolve around solving the PDEs that they induce [BRP99, Isa67, MBT05], corresponding viability theory [CQSP07], or classically by considering lower and upper time-discrete approximations with strategies changing at finitely many points and then passing to the limit [Fri71]. The latter is hard to implement and its theoretical understanding has been revolutionized by the invention of viscosity solutions [CL83, ES84, Bar13]. The former are interesting but also difficult, because PDEs are highly nontrivial to solve, and make it hard to obtain formal proofs that give 100% correct answers. Mitchell et al. [MBT05], for example, report a number of subtle soundness issues with prior work depending on the shape of the sets. Their own numerical scheme cannot provide correctness guarantees. Unlike in $dG\mathcal{L}$, the PDEs also only give answers for a fixed time horizon T.

Viability theory provides geometric notions for the PDEs of differential games [KNRY95, CQSP07, BCSP07]. It is easier to give an internally consistent answer with viability theory than with PDEs, but the errors off grid can be unbounded leading to soundness issues [MBT05], and inherent discontinuities in the value function complicate the matter [MBT05]. Yet, viability theory enables guaranteed approximation (on the grid, not off grid) and handles some cases of discontinuous dynamics [BRP99, Chapter 4]. Viability alternatives for hybrid systems are also pursued by Gao et al. [GLQ07] for affine dynamics with convexity assumptions and if no input ever influences a discrete state. Focusing on cases such as continuous controls or strategies [SP04] or convex control images with affine dynamics [Car96] as well as relaxing to limits of extra separators are common to make the problem more feasible; see [CQSP07, BRP99] for detailed comparisons.

Special purpose cases for differential games where players play hybrid input have been considered [DR06]. There is an argument to be made in favor of more modular designs such as $dG\mathcal{L}$,

where discrete and continuous games are integrated side-by-side in a modular way, as opposed to all intermingled. The same complicated applications will still be described that way, but each of their elements will have a conceptually easier description. This observation is paramount also in hybrid systems, whose key success relates to the fact that they can describe even complicated systems, but with elegantly separated smaller pieces of more narrow effect [Pla12b].

Differential game logic for hybrid games *without differential games* has been introduced along with an axiomatization and theoretical analysis in prior work [Pla14]. The present paper extends this approach with an integration of differential games into hybrid games. The focus in this paper is on the characterization, study, and proof principles of differential games.

7   Future Work

Differential game invariants, variants, and refinements are simple inductive proof techniques for differential games. Induction can be defined in different ways for differential equations. Similar flexibility is expected for differential games, for which differential game invariants are the first induction principle. In passing, Theorem 25 showed soundness of superdifferentials for differential invariants, which will be investigated in future work. Recent advances in generating differential invariants should also generalize to differential game invariants.

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A   Encoding Proofs for Embedding

The hybrid systems logic $\mathsf{d}\mathcal{L}$ [Pla12a] is the sublogic of $\mathsf{dG}\mathcal{L}$ that has differential equations but neither ^{d} nor differential games. By A^B denote the set of functions $B \to A$. The proof of Theorem 27 is based on the results in Appendix A.

Lemma 29 ($\mathbb{R}$-Gödel encoding [Pla08, Lem. 4]). *The formula $\mathrm{at}(Z, n, j, z)$, which holds iff Z is a real number that represents a Gödel encoding of a sequence of n real numbers with real value z at position j (for $1 \le j \le m$), is definable in $\mathsf{d}\mathcal{L}$. For a formula $\phi(z)$ abbreviate $\exists z\,(\mathrm{at}(Z, n, j, z) \wedge \phi(z))$ by $\phi(Z_j^{(n)})$.*

Corollary 30 (Infinite $\mathbb{R}$-Gödel encoding). *The bijection $\mathbb{R}\tilde{\to}\mathbb{R}^{\mathbb{N}}$ is characterizable in $\mathsf{d}\mathcal{L}$ by a formula $\mathrm{at}(Z, \infty, j, z)$, which holds iff Z is a real number that represents a Gödel encoding of a ω-infinite sequence of real numbers with real value z at position j. For a formula $\phi(z)$, abbreviate $\exists z\,(\mathrm{at}(Z, \infty, j, z) \wedge \phi(z))$ by $\phi(Z_j^{(\infty)})$.*

Proof. $\mathrm{at}(Z, \infty, j, z)$ is definable by repeated unpairing

$$\left(\langle ((j := j - 1; Z := Z_2^{(2)})^* \rangle (j = 0 \wedge z = Z_1^{(2)})\right)$$

Note that the use of an abbreviation formula like $Z_2^{(2)}$ inside a hybrid game is definable (e.g. in rich-test $\mathsf{d}\mathcal{L}$). □

Corollary 31. *The bijections $\mathbb{N}\tilde{\to}\mathbb{Q}$ and $\mathbb{R}\tilde{\to}\mathbb{R}^{\mathbb{Q}}$ are characterizable in $\mathsf{d}\mathcal{L}$.*

Proof. $\mathsf{d}\mathcal{L}$ can define the formula $\mathrm{rat}(n, p, q)$, which holds iff $\frac{p}{q}$ is the n-th rational number (in some fixed order):

$$\mathrm{rat}(n, p, q) \leftrightarrow p = n_1^{(2)} \wedge q = n_2^{(2)} \wedge q > 0 \qquad\qquad □$$

Corollary 32. *The bijection $\mathbb{R}\tilde{\to}C(\mathbb{R}, \mathbb{R})$ from reals to the continuous functions on the reals is characterizable in $\mathsf{d}\mathcal{L}$.*

Proof. Since continuous functions are uniquely defined by their values on the rationals, Corollary 31 shows that $\mathsf{d}\mathcal{L}$ can characterize the bijection by

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall \frac{p}{q} : \mathbb{Q} \, \forall n : \mathbb{N}$$

$$\left(\mathrm{rat}(n, p, q) \wedge |x - \frac{p}{q}| < \delta \rightarrow |z - F_n^{(\infty)}| < \varepsilon \right)$$

Observe that the enumeration of $\frac{p}{q}$ from Corollary 31 enumerates identical fractions with different denominators repeatedly, which would allow for the definition of inconsistent F that give different values at $\frac{p}{q}$ and $\frac{2p}{2q}$. This is easily overcome, e.g., by skipping fractions that cancel, which can be checked by divisibility or Euclid's gcd algorithm, which are both definable with programs in $\mathsf{d}\mathcal{L}$. $\qquad \square$

B Non-differential Hybrid Game Axiomatization

For reference, Figure 2 shows a sound and complete axiomatization for the case of differential game logic for hybrid games with differential equations but without differential games from prior work [Pla14]. The axiomatization is designed on top of the first-order Hilbert calculus (modus ponens, uniform substitution, and Bernays' $\forall$-generalization) with all instances of valid formulas of first-order logic as axioms, including first-order real arithmetic. The only change of Figure 2 compared to prior work [Pla14] is the use of dualization to convert $\langle \cdot \rangle$ axioms into $[\cdot]$ axioms. This is a cosmetic change to make it easier for the reader to appreciate how differential game invariants (proof rule DGI) integrate seamlessly into the proof calculus for the other operators of differential hybrid games.

C Proof of Isaacs Equations

This section shows a proof of Theorem 19 that is simplified compared to its original version [ES84]. The proof of Theorem 19 uses two lemmas.

Lemma 33. *Let $v \in C^1((0, T) \times \mathbb{R}^n)$. The upper value U of (1) satisfies for any $0 \leq \eta \leq \eta + \sigma \leq T$:*

$$U(\eta, \xi) - v(\eta, \xi) = \sup_{\alpha \in \mathcal{S}_{Z \rightarrow Y}} \inf_{z \in \mathcal{M}_Z}$$

$$\left(\int_{\eta}^{\eta + \sigma} \nabla_f v(s) ds + U(\eta + \sigma, x(\eta + \sigma)) - v(\eta + \sigma, x(\eta + \sigma)) \right)$$

where $x(\zeta) = x(\zeta; \xi, \alpha(z), z)$ is the response of (1) for $\alpha(z)(\cdot)$ and $z(\cdot)$ and

$$\nabla_f v(s) \stackrel{\mathrm{def}}{=} v_t(s, x(s)) + f(s, x(s), \alpha(z)(s), z(s)) \cdot D_x v(s, x(s))$$

$\langle \cdot \rangle$ $\langle \alpha \rangle \phi \leftrightarrow \neg [\alpha] \neg \phi$

$[:=]$ $[x := \theta] \phi(x) \leftrightarrow \phi(\theta)$

$[']$ $[x' = f(x)] \phi \leftrightarrow \forall t \geq 0 \, [x := y(t)] \phi$ $\qquad (y'(t) = f(y))$

$[?]$ $[?C] \phi \leftrightarrow (C \rightarrow \phi)$

$[\cup]$ $[\alpha \cup \beta] \phi \leftrightarrow [\alpha] \phi \wedge [\beta] \phi$

$[;]$ $[\alpha; \beta] \phi \leftrightarrow [\alpha][\beta] \phi$

$[^*]$ $\phi \wedge [\alpha][\alpha^*] \phi \leftarrow [\alpha^*] \phi$

$[^d]$ $[\alpha^d] \phi \leftrightarrow \neg [\alpha] \neg \phi$

M $\dfrac{\phi \rightarrow \psi}{[\alpha] \phi \rightarrow [\alpha] \psi}$

ind $\dfrac{\psi \rightarrow [\alpha] \psi}{\psi \rightarrow [\alpha^*] \psi}$

Figure 2: Differential game logic axiomatization for hybrid games without differential games

Proof. The result follows from the dynamic programming optimality condition Sect. 4.2 with step size σ. Recall $U(\eta, \xi)$

$$\sup_{\alpha \in \mathcal{S}_{Z \rightarrow Y}} \inf_{z \in \mathcal{M}_Z} U(\eta + \sigma, x(\eta + \sigma))$$

using the fundamental theorem of calculus [Wal95, Thm. 9.23] (since v is differentiable on the open interval $(\eta, \eta + \sigma)$ and continuous on the closed interval $[\eta, \eta + \sigma]$):

$$v(\eta + \sigma, x(\eta + \sigma)) - v(\eta, \xi) = \int_{\eta}^{\eta + \sigma} \frac{\mathsf{d}v(t, x(t))}{\mathsf{d}t}(s) ds = \int_{\eta}^{\eta + \sigma} \nabla_f v(s) ds$$

$\square$

Lemma 34 ([ES84, Lem. 4.3]). *Let* $v \in C^1((0, T) \times \mathbb{R}^n)$. *If*

$$v_t(\eta, \xi) + H^+(\eta, \xi, Dv(\eta, \xi)) \leq -\theta < 0 \tag{34}$$

then $\forall \sigma \, \exists z \in \mathcal{M}_Z \, \forall \alpha \in \mathcal{S}_{Z \rightarrow Y}$

$$\int_{\eta}^{\eta + \sigma} \nabla_f v(s) ds \leq -\frac{\sigma \theta}{2}$$

If

$$v_t(\eta, \xi) + H^+(\eta, \xi, Dv(\eta, \xi)) \geq \theta > 0 \tag{35}$$

then $\forall \sigma \, \exists \alpha \in \mathcal{S}_{Z \rightarrow Y} \, \forall z \in \mathcal{M}_Z$

$$\int_{\eta}^{\eta + \sigma} \nabla_f v(s) ds \geq \frac{\sigma \theta}{2}$$

Proof. To simplify the left-hand side, abbreviate

$$\Lambda(t, x, y, z) \stackrel{\text{def}}{=} v_t(t, x) + f(t, x, y, z) \cdot D_x v(t, x)$$

First prove the first inequality. By the definition of H^+, (34) is

$$\min_{z \in Z} \max_{y \in Y} \Lambda(\eta, \xi, y, z) \leq -\theta < 0$$

which implies for some $z^* \in Z$ that

$$\max_{y \in Y} \Lambda(\eta, \xi, y, z^*) \leq -\theta < 0$$

Since $\Lambda(t, x, y, z)$ is (uniformly) continuous

$$\max_{y \in Y} \Lambda(s, x(s), y, z^*) \leq -\frac{\theta}{2}$$

for $s \in [\eta, \eta + \sigma]$ with a sufficiently small σ when $x(\cdot)$ is the response of (1) for any $y(\cdot), z(\cdot)$ with initial condition $x(\eta) = \xi$. Consequently, for the constant control $z(\cdot) \stackrel{\text{def}}{=} z^*$, any $\alpha \in \mathcal{S}_{Z \to Y}$ gives

$$\Lambda(s, x(s), \alpha(z)(s), z(s)) \leq -\frac{\theta}{2}$$

Now, prove the second inequality (35), which is

$$\min_{z \in Z} \max_{y \in Y} \Lambda(\eta, \xi, y, z) \geq \theta > 0$$

which implies that, for each $z \in Z$, there is a $y \in Y$ such that

$$\Lambda(\eta, \xi, y, z) \geq \theta$$

Since $\Lambda(t, x, y, z)$ is (uniformly) continuous

$$\Lambda(\eta, \xi, y, \zeta) \geq \frac{3\theta}{4} \tag{36}$$

for all $\zeta \in Z$ in an open ball around z. Since this holds for all $z \in Z$ and Z is compact, there is a finite open covering of Z with open balls B_i within which (36) holds for all $\zeta \in B \cap Z$. Pick a function $c : Z \to Y$ such that $c(z)$ is the center of the closest ball B_i to z (breaking ties arbitrarily). Then, for all $z \in Z$:

$$\Lambda(\eta, \xi, c(z), z) \geq \frac{3\theta}{4}$$

Since $\Lambda(t, x, y, z)$ is (uniformly) continuous,

$$\Lambda(\eta, \xi, c(z), z) \geq \frac{\theta}{2} \tag{37}$$

40

for $s \in [\eta, \eta + \sigma]$ with a sufficiently small σ when $x(\cdot)$ is the response of (1) for any $y(\cdot), z(\cdot)$ with initial condition $x(\eta) = \xi$. Construct $\alpha \in \mathcal{S}_{Z \to Y}$ for $z \in \mathcal{M}_Z$ as $\alpha(z)(s) \stackrel{\text{def}}{=} c(z(s))$ for all s. Then (37) implies

$$\Lambda(s, x(s), \alpha(z)(s), z(s)) \geq \frac{\theta}{2}$$

for all $s \in [\eta, \eta + \sigma]$, which implies the desired inequality by integration from η to $\eta + \sigma$. $\qquad \square$

Proof of Theorem 19. U can be shown to be the viscosity solution of the upper Isaacs equation. First, $U(T, \xi) = g(x(T)) = g(\xi)$ for all $\xi \in \mathbb{R}^n$. Second, consider any $v \in C^1((0, T) \times \mathbb{R}^n)$. If $U - v$ attains a local maximum at $(\eta, \xi) \in (0, T) \times \mathbb{R}^n$, i.e.

$$U(\eta, \xi) - v(\eta, \xi) \geq U(\eta + \sigma, x(\eta + \sigma)) - v(\eta + \sigma, x(\eta + \sigma)) \tag{38}$$

for sufficiently small σ and $x(\cdot)$ solving (1) with initial condition $x(\eta) = \xi$, then we need to show

$$v_t(\eta, \xi) + H^+(\eta, \xi, Dv(\eta, \xi)) \geq 0 \tag{39}$$

Otherwise, there were a θ such that

$$v_t(\eta, \xi) + H^+(\eta, \xi, Dv(\eta, \xi)) \leq -\theta < 0 \tag{34*}$$

By Lemma 33, (38) implies for any $0 \leq \eta \leq \eta + \sigma \leq T$

$$\sup_{\alpha \in \mathcal{S}_{Z \to Y}} \inf_{z \in \mathcal{M}_Z} \int_\eta^{\eta + \sigma} \nabla_f v(s) ds \geq 0 \tag{40}$$

By Lemma 34, (34) implies
$\forall \sigma \, \exists z \in \mathcal{M}_Z \, \forall \alpha \in \mathcal{S}_{Z \to Y}$

$$\int_\eta^{\eta + \sigma} \nabla_f v(s) ds \leq -\frac{\sigma\theta}{2}$$

This choice of z (that is even common for all α) implies in particular

$$\sup_{\alpha \in \mathcal{S}_{Z \to Y}} \inf_{z \in \mathcal{M}_Z} \int_\eta^{\eta + \sigma} \nabla_f v(s) ds \leq -\frac{\sigma\theta}{2} \tag{41}$$

Equation (40) contradicts (41) and, thus, refutes (34) and proves (39).

Third, if $U - v$ attains a local minimum at $(\eta, \xi) \in (0, T) \times \mathbb{R}^n$, i.e.

$$U(\eta, \xi) - v(\eta, \xi) \leq U(\eta + \sigma, x(\eta + \sigma)) - v(\eta + \sigma, x(\eta + \sigma)) \tag{42}$$

for sufficiently small σ and $x(\cdot)$ solving (1) with initial condition $x(\eta) = \xi$, then we need to show

$$v_t(\eta, \xi) + H^+(\eta, \xi, Dv(\eta, \xi)) \leq 0 \tag{43}$$

Otherwise, there were a θ such that

$$v_t(\eta, \xi) + H^+(\eta, \xi, Dv(\eta, \xi)) \geq \theta > 0 \tag{35*}$$

By Lemma 33, (42) implies for any $0 \leq \eta \leq \eta + \sigma \leq T$

$$\sup_{\alpha \in \mathcal{S}_{Z \to Y}} \inf_{z \in \mathcal{M}_Z} \int_{\eta}^{\eta+\sigma} \nabla_f v(s) ds \leq 0 \tag{44}$$

By Lemma 34, (35) implies
$\forall \sigma \, \exists \alpha \in \mathcal{S}_{Z \to Y} \forall z \in \mathcal{M}_Z$

$$\int_{\eta}^{\eta+\sigma} \nabla_f v(s) ds \geq \frac{\sigma \theta}{2}$$

This choice of α demonstrates the lower bound

$$\sup_{\alpha \in \mathcal{S}_{Z \to Y}} \inf_{z \in \mathcal{M}_Z} \int_{\eta}^{\eta+\sigma} \nabla_f v(s) ds \geq \frac{\sigma \theta}{2} \tag{45}$$

Equation (44) contradicts (45) and, thus, refutes (35) and proves (43). $\qquad \square$