Weak Focusing for Ordered Linear Logic

Robert J. Simmons **Frank Pfenning**

April 7, 2011
CMU-CS-10-147

School of Computer Science
Carnegie Mellon University
Pittsburgh, PA 15213

Support for this research was provided by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the Carnegie Mellon Portugal Program under Grant NGN-44 and by an X10 Innovation Award from IBM.

Keywords: Focusing, Linear Logic, Ordered Linear Logic, Cut Admissibility, Identity, Hypothetical focus, Equality

Abstract

Ever since Andreoli's pioneering work in linear logic, it has been understood that the technique of *focusing* can be used to obtain the operational semantics of a logic programming language from a logic. In previous work, Polakow presented ordered linear logic and gave a backward-chaining operational semantics semantics for the uniform fragment of the logic, and the authors have previously presented a fragment of ordered linear logic suitable to a forward-chaining operational semantics. In this report we give a so-called *weakly focused* sequent calculus for a polarized first-order ordered linear logic with equality assumptions, generalizing and extending both of these previous proposals. The polarized sequent calculus and the cut admissibility theorem for this logic are standard, but the proof of the identity theorem requires a new technique. We show how the cut and identity theorem can be used in a straightforward manner to establish the completeness of the weakly focused sequent calculus with respect to an non-focused sequent calculus for ordered linear logic.

1 Ordered linear logic

The sequent calculus for first-order ordered logic with equality is pretty straightforward, though the definition of *resource contexts* is non-standard. We will leave out the units of disjunction (**0**) and additive conjunction ($\top$), but both could be included without difficulty.

$$
\begin{array}{llcl}
\textit{Propositions:} & A, B, C & ::= & Q \mid\ !A \mid\ ¡A \mid \mathbf{1} \mid A \bullet B \mid A \twoheadrightarrow B \mid A \rightarrowtail B \mid \\
 & & & A \oplus B \mid A \,\&\, B \mid \forall x.A \mid \exists x.A \mid t \doteq s \\
\textit{Variable contexts:} & \Sigma & ::= & \cdot \mid \Sigma, x \\
\textit{Persistent contexts:} & \Gamma & ::= & \cdot \mid \Gamma, A \\
\textit{Resource contexts:} & \Delta & ::= & \cdot \mid A \mid \underline{A} \mid \Delta, \Delta'
\end{array}
$$

1.1 Context equivalence

When we write A in a resource context, it should be seen as shorthand for the judgment "A *ordered*" representing an ordered resource, and when we write $\underline{A}$ in a resource context, it should be seen as shorthand for the judgment "A *mobile*" representing a mobile (linear) resource. The intuition is that the relative position of ordered resources is significant, but the relative order of linear resources relative to each other and relative to the ordered resources is unimportant.

This intuition is validated by the context equivalence $\Delta \equiv \Delta'$, which is the least equivalence relation closed under the following:

- $\underline{A}, \underline{B} \equiv \underline{B}, \underline{A}$
- $\underline{A}, B \equiv B, \underline{A}$
- $\cdot, \Delta \equiv \Delta \equiv \Delta, \cdot$
- $(\Delta_1, \Delta_2), \Delta_3 \equiv \Delta_1, (\Delta_2, \Delta_3)$
- $\Delta \equiv \Delta'$ implies $\Delta_L, \Delta, \Delta_R \equiv \Delta_L, \Delta', \Delta_R$

This makes context equivalence a special case of an algebraic structure known as a *trace monoid*.

1.2 Unfocused sequent calculus

The only sequent is $\Gamma; \Delta \vdash_\Sigma C$, where all the variables in Σ are presumed to be unique and any free variables in Γ, Δ, or C are required to be present in Σ.

Many rules (for instance $\bullet R$, $!R$, and every left rule) make some presumption about the shape of the context; we adopt the view that these left rules are "really" shorthand for a more precise rule that references the structural equivalence explicitly. For instance, the precise version of $\mathbf{1}L$ looks like this:

$$\frac{\Delta \equiv \Delta_L, \mathbf{1}, \Delta_R \quad \Gamma; \Delta_L, \Delta_R \vdash_\Sigma C}{\Gamma; \Delta \vdash_\Sigma C}\ \mathbf{1}L$$

The notation $\underline{\Delta}$ in $¡L$ indicates that this rules application is restricted to the case where the resource context Δ contains only mobile resources $\underline{A}$.

The following are the sequent calculus rules:

1.2.1 Structural rules

$$\frac{A \in \Gamma \quad \Gamma; \Delta_L, A, \Delta_R \vdash_\Sigma C}{\Gamma; \Delta_L, \Delta_R \vdash_\Sigma C}\ \mathit{copy} \qquad \frac{\Gamma; \Delta_L, A, \Delta_R \vdash_\Sigma C}{\Gamma; \Delta_L, \underline{A}, \Delta_R \vdash_\Sigma C}\ \mathit{place} \qquad \frac{}{\Gamma; Q \vdash_\Sigma Q}\ \mathit{init}$$

1.2.2 Exponentials

$$\frac{\Gamma; \cdot \vdash_\Sigma A}{\Gamma; \cdot \vdash_\Sigma\, !A}\ !R \qquad \frac{\Gamma, A; \Delta_L, \Delta_R \vdash_\Sigma C}{\Gamma; \Delta_L, !A, \Delta_R \vdash_\Sigma C}\ !L \qquad \frac{\Gamma; \underline{\Delta} \vdash_\Sigma A}{\Gamma; \underline{\Delta} \vdash_\Sigma\, ¡A}\ ¡R \qquad \frac{\Gamma; \Delta_L, \underline{A}, \Delta_R \vdash_\Sigma C}{\Gamma; \Delta_L, ¡A, \Delta_R \vdash_\Sigma C}\ ¡L$$

1.2.3 Multiplicative connectives

$$\frac{}{\Gamma; \cdot \vdash_\Sigma \mathbf{1}}\ \mathbf{1}R \qquad \frac{\Gamma; \Delta_L, \Delta_R \vdash_\Sigma C}{\Gamma; \Delta_L, \mathbf{1}, \Delta_R \vdash_\Sigma C}\ \mathbf{1}L$$

$$\frac{\Gamma; \Delta_L \vdash_\Sigma A \quad \Gamma; \Delta_R \vdash_\Sigma B}{\Gamma; \Delta_L, \Delta_R \vdash_\Sigma A \bullet B}\ \bullet R \qquad \frac{\Gamma; \Delta_L, A, B, \Delta_R \vdash_\Sigma C}{\Gamma; \Delta_L, A \bullet B, \Delta_R \vdash_\Sigma C}\ \bullet L$$

$$\frac{\Gamma; \Delta, A \vdash_\Sigma B}{\Gamma; \Delta \vdash_\Sigma A \twoheadrightarrow B}\ {\twoheadrightarrow}R \qquad \frac{\Gamma; \Delta_A \vdash A \quad \Gamma; \Delta_L, B, \Delta_R \vdash_\Sigma C}{\Gamma; \Delta_L, A \twoheadrightarrow B, \Delta_A, \Delta_R \vdash_\Sigma C}\ {\twoheadrightarrow}L$$

$$\frac{\Gamma; A, \Delta \vdash_\Sigma B}{\Gamma; \Delta \vdash_\Sigma A \rightarrowtail B}\ {\rightarrowtail}R \qquad \frac{\Gamma; \Delta_A \vdash A \quad \Gamma; \Delta_L, B, \Delta_R \vdash_\Sigma C}{\Gamma; \Delta_L, \Delta_A, A \rightarrowtail B, \Delta_R \vdash_\Sigma C}\ {\rightarrowtail}L$$

1.2.4 Additive connectives

$$\frac{\Gamma; \Delta \vdash_\Sigma A_i}{\Gamma; \Delta \vdash_\Sigma A_1 \oplus A_2}\ {\oplus}R_i \qquad \frac{\Gamma; \Delta_L, A, \Delta_R \vdash_\Sigma C \quad \Gamma; \Delta_L, B, \Delta_R \vdash_\Sigma C}{\Gamma; \Delta_L, A \oplus B, \Delta_R \vdash_\Sigma C}\ {\oplus}L$$

$$\frac{\Gamma; \Delta \vdash_\Sigma A \quad \Gamma; \Delta \vdash_\Sigma B}{\Gamma; \Delta \vdash_\Sigma A \,\&\, B}\ \&R \qquad \frac{\Gamma; \Delta_L, A_i, \Delta_R \vdash_\Sigma C}{\Gamma; \Delta_L, A_1 \,\&\, A_2, \Delta_R \vdash_\Sigma C}\ \&L_i$$

1.2.5 First-order connectives

The left rule for equality uses a higher-order judgment: it says that, if we assume that t and s are equal, then we have to prove the sequent under *all* unifying substitutions of t and s, of which there may be a countably infinite number. In reading the $\doteq L$ rule, it is important to distinguish $t \doteq s$, which is a proposition in ordered logic, from $t = s$, which is a meta-level judgment.

$$\frac{\Gamma; \Delta \vdash_{\Sigma,x} A}{\Gamma; \Delta \vdash_\Sigma \forall x.A}\ \forall R^x \qquad \frac{\Sigma \vdash t \quad \Gamma; \Delta_L, A[t/x], \Delta_R \vdash_\Sigma C}{\Gamma; \Delta_L, \forall x.A, \Delta_R \vdash_\Sigma C}\ \forall L$$

$$\frac{\Sigma \vdash t \quad \Gamma; \Delta \vdash_\Sigma A[t/x]}{\Gamma; \Delta \vdash_\Sigma \exists x.A}\ \exists R \qquad \frac{\Gamma; \Delta_L, A, \Delta_R \vdash_{\Sigma,x} C}{\Gamma; \Delta_L, \exists x.A, \Delta_R \vdash_\Sigma C}\ \exists L^x$$

$$\frac{}{\Gamma; \cdot \vdash_\Sigma t \doteq t}\ {\doteq}R \qquad \frac{\forall(\Sigma' \vdash \theta : \Sigma)\text{:} \quad \theta t = \theta s \quad \longrightarrow \quad \theta\Gamma; \theta\Delta_L, \theta\Delta_R \vdash_{\Sigma'} \theta C}{\Gamma; \Delta_L, t \doteq s, \Delta_R \vdash_\Sigma C}\ {\doteq}L$$

2 Polarized ordered linear logic

While many presentations of focusing do not change the language of propositions, this presentation is a "shifty" presentation of focusing that does change the language of propositions. In addition to splitting propositions into positive and negative propositions, we also make new syntactic categories of persistent and linear propositions. We could add other propositions to those categories, and even introduce both positive and negative persistent and mobile propositions, but at this point we are only interested having atoms (which act as positive atoms) in both the persistent and mobile syntactic categories.

Persistent propositions	Ap	$::=$	$A^- \mid Qp$
Mobile (linear) propositions	Al	$::=$	$A^- \mid Ql$
Positive propositions	A^+, B^+	$::=$	$Q^+ \mid {\downarrow}A^+ \mid {!}Ap \mid {¡}Al \mid \mathbf{1} \mid A^+ \bullet B^+ \mid A^+ \oplus B^+ \mid$ $\exists x.A^+ \mid t \doteq s$
Negative propositions	A^-, B^-	$::=$	$Q^- \mid {\uparrow}A^- \mid A^+ \twoheadrightarrow B^- \mid A^+ \rightarrowtail B^- \mid A^- \,\&\, B^- \mid \forall x.A^-$
Focal contexts:	$\aleph$	$::=$	$\cdot \mid \alpha{::}[A^+] \mid \gamma{::}[A^-] \mid \aleph, \aleph'$
Variable contexts:	Σ	$::=$	$\cdot \mid \Sigma, x$
Persistent contexts:	Γ	$::=$	$\cdot \mid \Gamma, A^- \mid \Gamma, Qp$
Resource contexts:	Δ	$::=$	$\cdot \mid \alpha \mid A^+ \mid \underline{Al} \mid \Delta, \Delta'$
Conclusions:	C	$::=$	$\gamma \mid A^-$

The equivalence on resource contexts is the same as before. Our formulation of polarized ordered logic contains an innovation, the use of a *focal context*, written as $\aleph$, that obeys a linear resource discipline (so conjunction of focal contexts is treated as being associative and commutative with unit $\cdot$). Focal contexts are critical to the completeness lemmas for ordered linear logic, though the linearity of the focal context is not critical – we could also allow focal contexts to obey a persistent resource discipline with contraction, exchange, and weakening.

2.1 Weakly focused sequent calculus

A neutral sequent has the form $\aleph; \Gamma; \Delta \Rightarrow_\Sigma C$, a right-focused sequent has the form $\aleph; \Gamma; \Delta \Rightarrow_\Sigma [A^+]$, and a left-focused sequent has the form $\aleph; \Gamma; \Delta_L[A^-]\Delta_R \Rightarrow_\Sigma C$. Once again, all the variables in Σ are presumed to be unique and any free variables in $\aleph$, Γ, Δ, or C are required to be present in Σ.

2.1.1 Focal hypotheses

$$\frac{}{\alpha{::}[A^+]; \Gamma; \alpha \Rightarrow_\Sigma [A^+]}\ \mathit{init}\alpha \qquad \frac{}{\gamma{::}[A^-]; \Gamma; [A^-] \Rightarrow_\Sigma \gamma}\ \mathit{init}\gamma$$

2.1.2 Entering focus, shifts

$$\frac{A^- \in \Gamma \quad \aleph; \Gamma; \Delta_L[A^-]\Delta_R \Rightarrow_\Sigma C}{\aleph; \Gamma; \Delta_L, \Delta_R \Rightarrow_\Sigma C}\ \mathit{copy} \qquad \frac{\aleph; \Gamma; \Delta_L[A^-]\Delta_R \Rightarrow_\Sigma C}{\aleph; \Gamma; \Delta_L, \underline{A^-}, \Delta_R \Rightarrow_\Sigma C}\ \mathit{place}$$

$$\frac{}{\cdot; \Gamma; [Q^-] \Rightarrow_\Sigma Q^-}\ \mathit{init}^- \qquad \frac{\aleph; \Gamma; \Delta \Rightarrow_\Sigma [A^+]}{\aleph; \Gamma; \Delta \Rightarrow_\Sigma {\uparrow}A^+}\ {\uparrow}R \qquad \frac{\aleph; \Gamma; \Delta_L, A^+, \Delta_R \Rightarrow_\Sigma C}{\aleph; \Gamma; \Delta_L[{\uparrow}A^+]\Delta_R \Rightarrow_\Sigma C}\ {\uparrow}L$$

$$\frac{}{\cdot;\Gamma;Q^+ \Rightarrow_\Sigma [Q^+]}\ init^+ \qquad \frac{\aleph;\Gamma;\Delta \Rightarrow_\Sigma A^-}{\aleph;\Gamma;\Delta \Rightarrow_\Sigma [{\downarrow}A^-]}\ {\downarrow}R \qquad \frac{\aleph;\Gamma;\Delta_L[A^-]\Delta_R \Rightarrow_\Sigma C}{\aleph;\Gamma;\Delta_L,{\downarrow}A^-,\Delta_R \Rightarrow_\Sigma C}\ {\downarrow}L$$

2.1.3 Exponentials

$$\frac{\aleph;\Gamma,Ap;\Delta_L,\Delta_R \Rightarrow_\Sigma C}{\aleph;\Gamma;\Delta_L,!Ap,\Delta_R \Rightarrow_\Sigma C}\ !L \qquad \frac{\cdot;\Gamma;\cdot \Rightarrow_\Sigma A^-}{\cdot;\Gamma;\cdot \Rightarrow_\Sigma [!A^-]}\ !R \qquad \frac{Qp \in \Gamma}{\cdot;\Gamma;\cdot \Rightarrow_\Sigma [!Qp]}\ !Rq$$

$$\frac{\aleph;\Gamma;\Delta_L,\underline{Al},\Delta_R \Rightarrow_\Sigma C}{\aleph;\Gamma;\Delta_L,¡Al,\Delta_R \Rightarrow_\Sigma C}\ ¡L \qquad \frac{\cdot;\Gamma;\underline{\Delta} \Rightarrow_\Sigma A^-}{\cdot;\Gamma;\underline{\Delta} \Rightarrow_\Sigma [¡A^-]}\ ¡R \qquad \frac{}{\cdot;\Gamma;\underline{Ql} \Rightarrow_\Sigma [¡Ql]}\ ¡Rq$$

2.1.4 Multiplicative connectives

$$\frac{}{\cdot;\Gamma;\cdot \Rightarrow_\Sigma [\mathbf{1}]}\ \mathbf{1}R \qquad \frac{\aleph;\Gamma;\Delta_L,\Delta_R \Rightarrow_\Sigma C}{\aleph;\Gamma;\Delta_L,\mathbf{1},\Delta_R \Rightarrow_\Sigma C}\ \mathbf{1}L$$

$$\frac{\aleph_1;\Gamma;\Delta_L \Rightarrow_\Sigma [A^+] \quad \aleph_2;\Gamma;\Delta_R \Rightarrow_\Sigma [B^+]}{\aleph_1,\aleph_2;\Gamma;\Delta_L,\Delta_R \Rightarrow_\Sigma [A^+ \bullet B^+]}\ \bullet R \qquad \frac{\aleph;\Gamma;\Delta_L,A^+,B^+,\Delta_R \Rightarrow_\Sigma C}{\aleph;\Gamma;\Delta_L,A^+ \bullet B^+,\Delta_R \Rightarrow_\Sigma C}\ \bullet L$$

$$\frac{\aleph;\Delta,A^+ \Rightarrow_\Sigma B^-}{\aleph;\Gamma;\Delta \Rightarrow_\Sigma A^+ \twoheadrightarrow B^-}\ {\twoheadrightarrow}R \qquad \frac{\aleph_1;\Gamma;\Delta_A \Rightarrow_\Sigma [A^+] \quad \aleph_2;\ \Delta_L[B^-]\Delta_R \Rightarrow_\Sigma C}{\aleph_1,\aleph_2;\Gamma;\Delta_L[A^+ \twoheadrightarrow B^-]\Delta_A,\Delta_R \Rightarrow_\Sigma C}\ {\twoheadrightarrow}L$$

$$\frac{\aleph;A^+,\Delta \Rightarrow_\Sigma B^-}{\aleph;\Gamma;\Delta \Rightarrow_\Sigma A^+ \rightarrowtail B^-}\ {\rightarrowtail}R \qquad \frac{\aleph_1;\Gamma;\Delta_A \Rightarrow_\Sigma [A^+] \quad \aleph_2;\ \Delta_L[B^-]\Delta_R \Rightarrow_\Sigma C}{\aleph_1,\aleph_2;\Gamma;\Delta_L,\Delta_A[A^+ \rightarrowtail B^-]\Delta_R \Rightarrow_\Sigma C}\ {\rightarrowtail}L$$

2.1.5 Additive connectives

$$\frac{\aleph;\Gamma;\Delta \Rightarrow_\Sigma [A_i^+]}{\aleph;\Gamma;\Delta \Rightarrow_\Sigma [A_1^+ \oplus A_2^+]}\ \oplus R_i \qquad \frac{\aleph;\Gamma;\Delta_L,A^+,\Delta_R \Rightarrow_\Sigma C \quad \aleph;\Gamma;\Delta_L,B^+,\Delta_R \Rightarrow_\Sigma C}{\aleph;\Gamma;\Delta_L,A^+ \oplus B^+,\Delta_R \Rightarrow_\Sigma C}\ \oplus L$$

$$\frac{\aleph;\Gamma;\Delta \Rightarrow_\Sigma A \quad \aleph;\Gamma;\Delta \Rightarrow_\Sigma B}{\aleph;\Gamma;\Delta \Rightarrow_\Sigma A \,\&\, B}\ \&R \qquad \frac{\aleph;\Gamma;\Delta_L[A_i^-]\Delta_R \Rightarrow_\Sigma C}{\aleph;\Gamma;\Delta_L[A_1^- \,\&\, A_2^-]\Delta_R \Rightarrow_\Sigma C}\ \&L_i$$

2.1.6 First-order connectives

$$\frac{\aleph;\Gamma;\Delta \Rightarrow_{\Sigma,x} A^-}{\aleph;\Gamma;\Delta \Rightarrow_\Sigma \forall x.A^-}\ \forall R^x \qquad \frac{\Sigma \vdash t \quad \aleph;\Gamma;\Delta_L[A^-[t/x]]\Delta_R \Rightarrow_\Sigma C}{\aleph;\Gamma;\Delta_L[\forall x.A^-]\Delta_R \Rightarrow_\Sigma C}\ \forall L$$

$$\frac{\Sigma \vdash t \quad \aleph;\Gamma;\Delta \Rightarrow_\Sigma [A^+[t/x]]}{\aleph;\Gamma;\Delta \Rightarrow_\Sigma [\exists x.A^+]}\ \exists R \qquad \frac{\aleph;\Gamma;\Delta_L,A^+\Delta_R \Rightarrow_{\Sigma,x} C}{\aleph;\Gamma;\Delta_L,\exists x.A^+,\Delta_R \Rightarrow_\Sigma C}\ \exists L^x$$

$$\frac{}{\aleph;\Gamma;\cdot \Rightarrow_\Sigma [t \doteq t]}\ \doteq R \qquad \frac{\forall(\Sigma' \vdash \theta : \Sigma):\quad \theta t = \theta s \quad \longrightarrow \quad \theta\aleph;\theta\Gamma;\theta\Delta_L,\theta\Delta_R \Rightarrow_{\Sigma'} \theta C}{\aleph;\Gamma;\Delta_L,t \doteq s,\Delta_R \Rightarrow_\Sigma C}\ \doteq L$$

2.2 Notation

In may cases (for example, the proof of soundness) we expect the focal context $\aleph$ to be empty; in these cases we will occasionally omit $\aleph$ altogether and write $(\Gamma; \Delta \Rightarrow_\Sigma C)$ instead of $(\cdot; \Gamma; \Delta \Rightarrow_\Sigma C)$.

3 Metatheory

In this section, we will give an overview of the metatheory of the weakly focused sequent calculus. The standard results are those of generalized weakening, both for persistent contexts (Theorem 1) and variable contexts (Theorem 2); cut admissibility (Theorem 3), which implies that the logic is internally sound; and identity expansion (Theorem 6), which implies that the logic is internally complete.

In addition, we have three new properties of the logic that we need to consider. The first, substitution of focal hypotheses (Theorem 5), is a natural consequence of the generalization of the weakly focused sequent calculus to include the focal context $\aleph$. The second, unfocused admissibility of the focused rules (Theorem 8), establishes the logical completeness of the weakly focused sequent calculus relative to the unfocused calculus presented in Section 1.

3.1 Weakening

Critical to the proof of the cut admissibility theorem is the weakening property, which the judgmental methodology treats as one of the language's defining principles. In the case of our sequent calculus, we will separate out these principles into three parts: we show that we can add, exchange, or contract the persistent premises in Section 3.1.1 and we show that we can apply substitutions to change the variable context in Section 3.1.2. In Section 3.1.3, we define a notion of the size of contexts that allows us to apply both of these kinds of weakening to derivations and then call an induction hypothesis on the result; this third part is a prerequisite for the cut admissibility theorem in Section 3.2.

3.1.1 Generalized weakening (context renaming)

The generalized weakening theorem – which is also known as context renaming – subsumes the usual properties of contraction, exchange, and weakening.

Theorem 1 (Generalized weakening)**.** *If* $\Gamma \subseteq \Gamma'$*, then*

- *If* $(\aleph; \Gamma; \Delta \Rightarrow_\Sigma C)$*, then* $(\aleph; \Gamma'; \Delta \Rightarrow_\Sigma C)$*,*
- *If* $(\aleph; \Gamma; \Delta \Rightarrow_\Sigma [A^+])$*, then* $(\aleph; \Gamma'; \Delta \Rightarrow_\Sigma [A^+])$*, and*
- *If* $(\aleph; \Gamma; \Delta_L[A^-]\Delta_R \Rightarrow_\Sigma C)$*, then* $(\aleph; \Gamma'; \Delta_L[A^-]\Delta_R \Rightarrow_\Sigma C)$*.*

Proof. Straightforward mutual induction over derivations. □

3.1.2 Variable weakening

In order to deal with equality, we also must consider a variable weakening lemma, where we "weaken" the variable context by applying a substitution θ to all parts of the derivation.

Theorem 2 (Variable weakening)**.** *If* $\Sigma' \vdash \theta : \Sigma$*, then*

- *If* $(\aleph; \Gamma; \Delta \Rightarrow_\Sigma C)$*, then* $(\theta\aleph; \theta\Gamma; \theta\Delta \Rightarrow_{\Sigma'} \theta C)$*,*
- *If* $(\aleph; \Gamma; \Delta \Rightarrow_\Sigma [A^+])$*, then* $(\theta\aleph; \theta\Gamma; \theta\Delta \Rightarrow_{\Sigma'} [\theta A^+])$*, and*
- *If* $(\aleph; \Gamma; \Delta_L[A^-]\Delta_R \Rightarrow_\Sigma C)$*, then* $(\theta\aleph; \theta\Gamma'; \theta\Delta_L[\theta A^-]\theta\Delta_R \Rightarrow_{\Sigma'} \theta C)$.

Proof. Let $\mathcal{D}$ be the derivation $\Sigma' \vdash \theta : \Sigma$; we will give two of the critical cases.

$$\textbf{Case: } \mathcal{E} = \dfrac{\forall(\Sigma'' \vdash \sigma : \Sigma) : \quad \sigma t \doteq \sigma s \quad \overset{\mathcal{E}_1}{\longrightarrow} \quad \sigma\aleph; \sigma\Gamma; \sigma\Delta_L, \sigma\Delta_R \Rightarrow_{\Sigma''} \sigma C}{\aleph; \Gamma; \Delta_L, t \doteq s, \Delta_R \Rightarrow_\Sigma C} \doteq\! L$$

To show: $\theta\aleph; \theta\Gamma; \theta\Delta_L, \theta t \doteq \theta s, \theta\Delta_R \Rightarrow_{\Sigma'} \theta C$

(1) $\forall(\Sigma'' \vdash \tau : \Sigma')$: $\tau(\theta t) \doteq \tau(\theta s) \longrightarrow \tau(\theta\aleph); \tau(\theta\Gamma); \tau(\theta\Delta_L), \tau(\theta\Delta_R) \Rightarrow_{\Sigma''} \tau(\theta C)$
by the following hypothetical reasoning:

Assume an arbitrary Σ'' and τ such that (2) $\Sigma'' \vdash \tau : \Sigma'$ and (3) $\tau(\theta t) \doteq \tau(\theta s)$

(4) $\Sigma'' \vdash \tau \circ \theta : \Sigma$ (composition of substitutions on $\mathcal{D}$ and (2))

(5) $(\tau \circ \theta)t \doteq (\tau \circ \theta)s$ (associativity of substitutions on (3))

(6) $(\tau \circ \theta)\aleph; (\tau \circ \theta)\Gamma; (\tau \circ \theta)\Delta_L, (\tau \circ \theta)\Delta_R \Rightarrow_{\Sigma''} (\tau \circ \theta)C$ ($\mathcal{E}_1$ on (4) and (5))

$\tau(\theta\aleph); \tau(\theta\Gamma); \tau(\theta\Delta_L), \tau(\theta\Delta_R) \Rightarrow_{\Sigma''} \tau(\theta C)$ (associativity of substitutions on (6))

$\theta\aleph; \theta\Gamma; \theta\Delta_L, \theta t \doteq \theta s, \theta\Delta_R \Rightarrow_{\Sigma'} \theta C$ ($\doteq L$ on (1))

$$\textbf{Case: } \mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\aleph; \Gamma; \Delta \Rightarrow_{\Sigma,x} A^-}}{\aleph; \Gamma; \Delta \Rightarrow_\Sigma \forall x.A^-} \forall R^x$$

To show: $\theta\aleph; \theta\Gamma; \theta\Delta \Rightarrow_{\Sigma'} \forall x.(\theta, x/x)A^-$

(1) $\Sigma, x \vdash \theta, x/x : \Sigma', x$ (extending the substitution $\mathcal{D}$)

(2) $(\theta, x/x)\aleph; (\theta, x/x)\Gamma; (\theta, x/x)\Delta \Rightarrow_{\Sigma',x} (\theta, x/x)A^-$ (IH on $\mathcal{E}_1$)

(3) $\theta\aleph; \theta\Gamma; \theta\Delta \Rightarrow_{\Sigma',x} (\theta, x/x)A^-$ (x not free in $\aleph, \Gamma, \Delta$ in (2))

$\theta\aleph; \theta\Gamma; \theta\Delta \Rightarrow_{\Sigma'} \forall x.(\theta, x/x)A^-$ ($\forall L$ on (3))

The other cases are just straightforward mutual induction over derivations. □

3.1.3 The size of derivations

In the cut admissibility theorem in the next section, we will frequently need to apply the two preceding weakening theorems, and often we will apply the induction hypothesis to the weakened derivation. Shapes are indexed by variable contexts, and a Ψ-shape captures the shape of a derivation of $\tau(\aleph; \Gamma; \Delta \Rightarrow_\Psi C)$, which is equal to $\tau\aleph; \tau\Gamma; \tau\Delta \Rightarrow_\Sigma \tau C$, for any substitution $\Sigma \vdash \tau : \Psi$.

- The Ψ-shape of the $\mathbf{1}R$ axiom is a unit.
- Given the Ψ-shape s_1 of the derivation $\mathcal{D} :: \tau(\aleph; \Gamma; \Delta_L, \Delta_R \Rightarrow_\Psi C)$, the Ψ-shape of the $\mathbf{1}L$ rule applied to $\mathcal{D}$ is a one-element tuple containing s_1.

- Given the Ψ-shape s_1 of the derivation $\mathcal{D}_1 :: \tau(\aleph_1; \Gamma; \Delta_L \Rightarrow_\Psi [A^+])$ and the Ψ-shape s_2 of the derivation $\mathcal{D}_2 :: \tau(\aleph_2; \Gamma; \Delta_R \Rightarrow_\Psi [B^+])$, the Ψ-shape of the $\bullet R$ rule applied to $\mathcal{D}_1$ and $\mathcal{D}_2$ is a two-element tuple containing s_1 and s_2.

- ... the other derivations follow exactly the same pattern save for the left rule for equality ...

- Given a function s_f from substitutions $\Sigma' \vdash \theta : \Psi$ where $\theta t \doteq \theta s$ to Σ'-shapes and a function $\mathcal{D}$ from substitutions $\Sigma' \vdash \sigma : \Sigma$ where $\sigma(\tau t) \doteq \sigma(\tau s)$ to derivations of $\mathcal{D}(\sigma) :: (\sigma \circ \tau)\aleph; (\sigma \circ \tau)\Gamma; (\sigma \circ \tau)\Delta_L, (\sigma \circ \tau)\Delta_R \Rightarrow_{\Sigma'} (\sigma \circ \tau)C$ with the property that for all σ, $s_f(\sigma \circ \tau)$ is the shape of $D(\sigma)$, the Ψ-shape of the $\doteq L$ rule applied to $\mathcal{D}$ is a one-element tuple containing s_f.

This critical last case is where all of the interesting work happens. We rely on two lemmas, which are again straightforward to prove: first, that we can derive a Σ-shape for any derivation $\Gamma; \Delta \Rightarrow_\Sigma C$, and second, that we can weaken a derivation without changing its shape. This second lemma will implicitly justify our calls to the induction hypothesis in the next section.

Both our formulation of first-order equality and our treatment of the size of derivations differs from the approach in the literature, which is detailed by Schroeder-Heister [2]. Our particular treatment is influenced by the first author's investigations of formalizing self-reflective logics in dependent type theory [5].

3.2 Cut admissibility

We prove cut admissibility with an empty focal context, so we just omit the focal context from mention for this section.

Theorem 3 (Cut admissibility)**.**

- *For all A^-:*

  1. *If* $(\Gamma; \Delta \Rightarrow_\Sigma A^-)$ *and* $(\Gamma; \Delta_L[A^-]\Delta_R \Rightarrow_\Sigma C)$*, then* $(\Gamma; \Delta_L, \Delta, \Delta_R \Rightarrow_\Sigma C)$.
  2. *If* $(\Gamma; \Delta'_L[B^-]\Delta'_R \Rightarrow_\Sigma A^-)$ *and* $(\Gamma; \Delta_L[A^-]\Delta_R \Rightarrow_\Sigma C)$*, then* $(\Delta_L, \Delta'_L[B^-]\Delta'_R, \Delta_R \Rightarrow_\Sigma C)$.

- *For all A^+:*

  3. *If* $(\Gamma; \Delta \Rightarrow_\Sigma [A^+])$ *and* $(\Gamma; \Delta_L, A^+, \Delta_R \Rightarrow_\Sigma C)$*, then* $(\Gamma; \Delta_L, \Delta, \Delta_R \Rightarrow_\Sigma C)$.
  4. *If* $(\Gamma; \Delta \Rightarrow_\Sigma [A^+])$ *and* $(\Gamma; \Delta_L, A^+, \Delta_R \Rightarrow_\Sigma [B^+])$*, then* $(\Gamma; \Delta_L, \Delta, \Delta_R \Rightarrow_\Sigma [B^+])$.
  5. *If* $(\Gamma; \Delta \Rightarrow_\Sigma [A^+])$ *and* $(\Gamma; \Delta_L, A^+, \Delta'_L[B^-]\Delta_R \Rightarrow_\Sigma C)$*,*
     then $(\Gamma; \Delta_L, \Delta, \Delta'_L[B^-]\Delta_R \Rightarrow_\Sigma C)$.
  6. *If* $(\Gamma; \Delta \Rightarrow_\Sigma [A^+])$ *and* $(\Gamma; \Delta_L[B^-]\Delta_R, A^+, \Delta'_R \Rightarrow_\Sigma C)$*,*
     then $(\Gamma; \Delta_L[B^-]\Delta_R, \Delta, \Delta'_R \Rightarrow_\Sigma C)$.

Proof. By lexicographic induction, first on the size of the formula A^+ or A^- (where we get the size of a formula by simply erasing all the terms from atomic propositions), and second on the Ψ-sizes of the two input derivations $\mathcal{D}$ and $\mathcal{E}$ (as discussed in Section 3.1.3 – note that we do not require Ψ to be equal to Σ). The proof is basically standard and has been presented elsewhere; we give the cases for equality $t \doteq s$ since those cases are not standard.

Case: (Part 3, principal cut)

$$\mathcal{D} = \frac{}{\Gamma; \cdot \Rightarrow_\Sigma [t \doteq t]} \doteq R \qquad \mathcal{E} = \frac{\forall(\Sigma' \vdash \theta : \Sigma): \quad \theta t = \theta t \quad \stackrel{\mathcal{E}_1}{\longrightarrow} \quad \theta\Gamma; \theta\Delta_L, \theta\Delta_R \Rightarrow'_\Sigma \theta C}{\Gamma; \Delta_L, t \doteq t, \Delta_R \Rightarrow_\Sigma C} \doteq L$$

To show: $\Gamma; \Delta_L, \Delta_R \Rightarrow_\Sigma C$

(1) $t = t$ (reflexivity of equality)

(2) $\Gamma; \Delta_L, \Delta_R \Rightarrow_\Sigma C$ ($\mathcal{E}_1$ on the identity substitution and (1)

Case: (Part 3, right commutative cut)

$$\stackrel{\mathcal{D}}{\Gamma; \Delta \Rightarrow_\Sigma [A^+]} \qquad \mathcal{E} = \frac{\forall(\Sigma' \vdash \theta : \Sigma): \quad \theta t \doteq \theta s \quad \stackrel{\mathcal{E}_1}{\longrightarrow} \quad \theta\Gamma; \theta\Delta_{L1}, \theta\Delta_{L2}, \theta A^+, \theta\Delta_R \Rightarrow_{\Sigma'} \theta C}{\Gamma; \Delta_{L1}, t \doteq s, \Delta_{L2}, A^+, \Delta_R \Rightarrow_\Sigma C} \doteq L$$

To show: $\Gamma; \Delta_{L1}, t \doteq s, \Delta_{L2}, \Delta_R \Rightarrow_\Sigma C$

(1) $\forall(\Sigma' \vdash \theta : \Sigma)$: $\theta t \doteq \theta s \longrightarrow \theta\Gamma; \theta\Delta_{L1}, \theta\Delta_{L2}, \theta\Delta, \theta\Delta_R \Rightarrow_{\Sigma'} \theta C$

by the following hypothetical reasoning:

Assume that for arbitrary Σ' and θ we have (2) $\Sigma' \vdash \theta : \Sigma$ and (3) $\theta t \doteq \theta s$

(4) $\theta\Gamma; \theta\Delta \Rightarrow_{\Sigma'} [\theta A^+]$ (variable weakening on $\mathcal{D}$)

(5) $\theta\Gamma; \theta\Delta_{L1}, \theta\Delta_{L2}, \theta A^+, \theta\Delta_R \Rightarrow_{\Sigma'} \theta C$ ($\mathcal{E}_1$ on (2) and (3))

$\theta\Gamma; \theta\Delta_{L1}, \theta\Delta_{L2}, \theta\Delta, \theta\Delta_R \Rightarrow_{\Sigma'} \theta C$ (IH(θA^+) on (4) and (5))

$\Gamma; \Delta_{L1}, t \doteq s, \Delta, \Delta_R \Rightarrow_\Sigma C$ ($\doteq L$ on (1))

Note that in this case we were relying on the fact that the size of θA^+ is the same as the size as A^+ and on the fact that the Ψ-size of $\mathcal{D}$ under the substitution θ is the same as the Ψ-size of $\mathcal{D}$.

Case: (Part 3, right commutative cut)

$$\stackrel{\mathcal{D}}{\Gamma; \Delta \Rightarrow_\Sigma [A^+]} \qquad \mathcal{E} = \frac{\forall(\Sigma' \vdash \theta : \Sigma): \quad \theta t \doteq \theta s \quad \stackrel{\mathcal{E}_1}{\longrightarrow} \quad \theta\Gamma; \theta\Delta_L, \theta A^+, \theta\Delta_{R1}, \theta\Delta_{R2} \Rightarrow_{\Sigma'} \theta C}{\Gamma; \Delta_L, A^+, \Delta_{R1}, t \doteq s, \Delta_{R2} \Rightarrow_\Sigma C} \doteq L$$

(Essentially the same as the previous case)

Case: (Part 1, left commutative cut)

$$\mathcal{D} = \frac{\forall(\Sigma' \vdash \theta : \Sigma): \quad \theta t \doteq \theta s \quad \stackrel{\mathcal{D}_1}{\longrightarrow} \quad \theta\Gamma; \theta\Delta, \theta\Delta' \Rightarrow_{\Sigma'} \theta A^-}{\Gamma; \Delta, t \doteq s, \Delta' \Rightarrow_\Sigma A^-} \doteq L \qquad \stackrel{\mathcal{E}}{\Gamma; \Delta_L, A^-, \Delta_R \Rightarrow_\Sigma C}$$

To show: $\Gamma; \Delta_L, \Delta, t \doteq s, \Delta', \Delta_R \Rightarrow_\Sigma C$

(1) $\forall(\Sigma' \vdash \theta : \Sigma)$: $\theta t \doteq \theta s \longrightarrow \theta\Gamma; \theta\Delta_L, \theta\Delta, \theta\Delta', \theta\Delta_R \Rightarrow_{\Sigma'} \theta C$

by the following hypothetical reasoning:

Assume that for arbitrary Σ' and θ we have (2) $\Sigma' \vdash \theta : \Sigma$ and (3) $\theta t \doteq \theta s$

(4) $\theta\Gamma; \theta\Delta, \theta\Delta' \Rightarrow_{\Sigma'} \theta A^-$ ($\mathcal{D}_1$ on (2) and (3))

(5) $\theta\Gamma; \theta\Delta_L, \theta A^-, \theta\Delta_R \Rightarrow_{\Sigma'} \theta C$ (variable weakening on $\mathcal{E}$)

$\theta\Gamma; \theta\Delta_L, \theta\Delta, \theta\Delta', \theta\Delta_R \Rightarrow_{\Sigma'} \theta C$ (IH(θA^-) on (4) and (5)

$\Gamma; \Delta_L, \Delta, t \doteq s, \Delta', \Delta_R \Rightarrow_\Sigma C$ ($\doteq L$ on (1))

The cases for other connectives are straightforward extensions of standard cut-admissibility arguments. □

3.2.1 Does cut admissibility make sense in a non-empty focal context?

It greatly simplifies matters to prove cut admissibility over proofs where the focal context is empty. Even if the statement of cut in the obvious extension with the focal context is true, the theorem introduces a large number of new cases, such as the following one, which I am unclear how to finish:

Case: (Principal(?) cut)

$$\mathcal{D} = \dfrac{}{\alpha{::}[A^+ \bullet B^+]; \Gamma; \alpha \Rightarrow_\Sigma [A^+ \bullet B^+]}\ init\alpha \qquad \mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1 \\ \aleph; \Gamma; \Delta_L, A^+, B^+, \Delta_R \Rightarrow_\Sigma C\end{array}}{\aleph'; \Gamma; \Delta_L, A^+ \bullet B^+, \Delta_R \Rightarrow_\Sigma C}\ \bullet L$$

To show: $\aleph', \alpha{::}[A^+ \bullet B^+]; \Gamma; \Delta_L, \alpha, \Delta_R \Rightarrow_\Sigma C$

(1)	$\alpha_1{::}[A^+]; \Gamma; \alpha_1 \Rightarrow_\Sigma [A^+]$	(Rule $init\alpha$)
(2)	$\alpha_2{::}[B^+]; \Gamma; \alpha_2 \Rightarrow_\Sigma [B^+]$	(Rule $init\alpha$)
(3)	$\alpha_1{::}[A^+], \alpha_2{::}[B^+]; \Gamma; \alpha_1, \alpha_2 \Rightarrow_\Sigma [A^+ \bullet B^+]$	(Rule $\bullet R$ on (1) and (2))
(4)	$\aleph, \alpha_1{::}[A^+]; \Gamma; \Delta_L, \alpha_1, B^+, \Delta_R \Rightarrow_\Sigma C$	(Cut on A^+, (1), and $\mathcal{E}_1$)
(5)	$\aleph, \alpha_1{::}[A^+], \alpha_2{::}[B^+]; \Gamma; \Delta_L, \alpha_1, \alpha_2, \Delta_R \Rightarrow_\Sigma C$	(Cut on B^+, (2), and (4))
(6)	?	

3.2.2 Unfocused cut

This simple corollary of the cut admissibility theorem is useful:

Theorem 4 (Unfocused cut)**.**

- *If* $(\Gamma; \Delta \Rightarrow_\Sigma {\uparrow}A^+)$ *and* $(\Gamma; \Delta_L, A^+, \Delta_R \Rightarrow_\Sigma C)$*, then* $(\Gamma; \Delta_L, \Delta, \Delta_R \Rightarrow_\Sigma C)$
- *If* $(\Gamma; \Delta \Rightarrow_\Sigma A^-)$ *and* $(\Gamma; \Delta_L, {\uparrow}A^-, \Delta_R \Rightarrow_\Sigma C)$*, then* $(\Gamma; \Delta_L, \Delta, \Delta_R \Rightarrow_\Sigma C)$

Proof. In the first case, given $(\Gamma; \Delta_L, A^+, \Delta_R \Rightarrow_\Sigma C)$, we can get $(\Gamma; \Delta_L[{\uparrow}A^+]\Delta_R \Rightarrow_\Sigma C)$ by rule ${\uparrow}L$, and then we can apply cut with the cut formula ${\uparrow}A^+$. In the second case, given $(\Gamma; \Delta \Rightarrow_\Sigma A^-)$, we can get $(\Gamma; \Delta \Rightarrow_\Sigma [{\downarrow}A^-])$ by rule ${\downarrow}R$, and then we can apply cut with the cut formula ${\downarrow}A^-$. □

3.3 Substitution

The meaning of the focal contexts is defined by substitution.

Theorem 5 (Substitution)**.**

- *If* $(\aleph; \Gamma; \Delta \Rightarrow_\Sigma [A^+])$ *and* $(\aleph', \alpha{::}[A^+]; \Gamma; \Delta_L, \alpha, \Delta_R \Rightarrow_\Sigma C)$*, then* $(\aleph, \aleph'; \Gamma; \Delta_L, \Delta, \Delta_R \Rightarrow_\Sigma C)$.
- *If* $(\aleph; \Gamma; \Delta_L[A^-]\Delta_R \Rightarrow_\Sigma C)$ *and* $(\aleph', \gamma{::}[A^-]; \Gamma; \Delta \Rightarrow_\Sigma \gamma)$*, then* $(\aleph, \aleph'; \Gamma; \Delta_L, \Delta, \Delta_R \Rightarrow_\Sigma C)$.

Proof. The two substitutions can be proven separately by generalizing the induction hypothesis.

Positive substitution – established by proving the following by mutual induction on the second premise.

If $(\aleph; \Gamma; \Delta \Rightarrow_\Sigma [A^+])$ and $(\aleph', \alpha{::}[A^+]; \Gamma; \Delta_L, \alpha, \Delta_R \Rightarrow_\Sigma C)$, then $(\aleph', \aleph; \Gamma; \Delta_L, \Delta, \Delta_R \Rightarrow_\Sigma C)$.

If $(\aleph; \Gamma; \Delta \Rightarrow_\Sigma [A^+])$ and $(\aleph', \alpha{::}[A^+]; \Gamma; \Delta_L, \alpha, \Delta_R \Rightarrow_\Sigma [B^+])$,
then $(\aleph', \aleph; \Gamma; \Delta_L, \Delta, \Delta_R \Rightarrow_\Sigma [B^+])$.

If $(\aleph; \Gamma; \Delta \Rightarrow_\Sigma [A^+])$ and $(\aleph', \alpha{::}[A^+]; \Gamma; \Delta_L, \alpha, \Delta'_L[B^-]\Delta_R \Rightarrow_\Sigma C)$,
then $(\aleph', \aleph; \Gamma; \Delta_L, \Delta, \Delta'_L[B^-]\Delta_R \Rightarrow_\Sigma C)$.

If $(\aleph; \Gamma; \Delta \Rightarrow_\Sigma [A^+])$ and $(\aleph', \alpha{::}[A^+]; \Gamma; \Delta_L[B^-]\Delta_R, \alpha, \Delta'_R \Rightarrow_\Sigma C)$,
then $(\aleph', \aleph; \Gamma; \Delta_L[B^-]\Delta_R, \Delta, \Delta'_R \Rightarrow_\Sigma C)$.

Negative substitution – established by proving the following by mutual induction on the second premise.

If $(\aleph; \Gamma; \Delta_L[A^-]\Delta_R \Rightarrow_\Sigma C)$ and $(\aleph', \gamma{::}[A^-]; \Gamma; \Delta \Rightarrow_\Sigma \gamma)$, then $(\aleph', \aleph; \Gamma; \Delta_L, \Delta, \Delta_R \Rightarrow_\Sigma C)$.

If $(\aleph; \Gamma; \Delta_L[A^-]\Delta_R \Rightarrow_\Sigma C)$ and $(\aleph', \gamma{::}[A^-]; \Gamma; \Delta'_L[B^-]\Delta'_R \Rightarrow_\Sigma \gamma)$,
then $(\aleph', \aleph; \Gamma; \Delta_L, \Delta'_L[B^-]\Delta'_R, \Delta_R \Rightarrow_\Sigma C)$.

The proof proceeds by straightforward induction on the Σ-size of the second derivation. □

3.4 Identity expansion

Whereas cut admissibility (soundness) avoided mentioning the focal context, the identity expansion lemma (completeness) uses it in an absolutely critical manner. This is not accidental; focal contexts were introduced in order to deal with the complexity of quantifiers in the completeness or identity expansion in a uniform way.

The $\aleph$ are, in fact, an attempt to capture the "regular worlds" definition used in a Twelf proof of weak focusing for a polarized propositional persistent logic [4]. Previous attempts to capture this reasoning on paper require a somewhat complex reasoning about which parts of a proof were parametric and which were not: see Appendix A.1 of [1], for instance, and note that this proof does not obviously generalize to a logic with positive conjunction. The $\aleph$-context, while unique to our presentation, simply embeds into the logic the uniform and parametric treatment of focal hypotheses that comes naturally in the Twelf encoding.

Theorem 6 (Identity expansion)**.**

- *For all A^+, for all Σ, $\aleph$, Γ, Δ_L, Δ_R, and C,*
  if $(\aleph, \alpha{::}[A^+]; \Gamma; \Delta_L, \alpha, \Delta_R \Rightarrow_\Sigma C)$*, then* $(\aleph; \Gamma; \Delta_L, A^+, \Delta_R \Rightarrow_\Sigma C)$.
- *For all A^-, for all $\aleph$, Γ, and Δ,*
  if $(\aleph, \gamma{::}[A^-]; \Gamma; \Delta \Rightarrow_\Sigma \gamma)$*, then* $(\aleph; \Gamma; \Delta \Rightarrow_\Sigma A^-)$.

Proof. By mutual induction on the size of the proposition A^+ or A^-.

Case: $A^+ = Q^+$

(1) $\aleph, \alpha{::}[Q^+]; \Gamma; \Delta_L, \alpha, \Delta_R \Rightarrow_\Sigma C$ (Given)
To show: $\aleph; \Gamma; \Delta_L, Q^+, \Delta_R \Rightarrow_\Sigma C$
(2) $\cdot; \Gamma; Q^+ \Rightarrow_\Sigma [Q^+]$ (Rule $init^+$)
$\aleph; \Gamma; \Delta_L, Q^+, \Delta_R \Rightarrow_\Sigma C$ (Substitution on (2) and (1))

Case: $A^+ = {\downarrow}A^-$

(1)	$\aleph, \alpha{::}[{\downarrow}A^-]; \Gamma; \Delta_L, \alpha, \Delta_R \Rightarrow_\Sigma C$	(Given)
To show:	$\aleph; \Gamma; \Delta_L, {\downarrow}A^-, \Delta_R \Rightarrow_\Sigma C$	
(2)	$\gamma{::}[A^-]; \Gamma; [A^-] \Rightarrow_\Sigma \gamma$	(Rule $init\gamma$)
(3)	$\gamma{::}[A^-]; \Gamma; {\downarrow}A^- \Rightarrow_\Sigma \gamma$	(Rule ${\downarrow}L$ on (2))
(4)	$\cdot; \Gamma; {\downarrow}A^- \Rightarrow_\Sigma A^-$	(IH(A^-) where $\aleph = \cdot$ and $\Delta = {\downarrow}A^-$ on (3))
(5)	$\cdot; \Gamma; {\downarrow}A^- \Rightarrow_\Sigma [{\downarrow}A^-]$	(Rule ${\downarrow}R$ on (4))
	$\aleph; \Gamma; \Delta_L, {\downarrow}A^-, \Delta_R \Rightarrow_\Sigma C$	(Substitution on (5) and (1))

Case: $A^+ = !A^-$

(1)	$\aleph, \alpha{::}[!A^-]; \Gamma; \Delta_L, \alpha, \Delta_R \Rightarrow_\Sigma C$	(Given)
To show:	$\aleph; \Gamma; \Delta_L, !A^-, \Delta_R \Rightarrow_\Sigma C$	
(2)	$\gamma{::}[A^-]; \Gamma, A^-; [A^-] \Rightarrow_\Sigma \gamma$	(Rule $init\gamma$)
(3)	$\gamma{::}[A^-]; \Gamma, A^-; \cdot \Rightarrow_\Sigma \gamma$	(Rule $copy$ on (2))
(4)	$\cdot; \Gamma, A^-; \cdot \Rightarrow_\Sigma A^-$	(IH(A^-) where $\aleph = \cdot$ and $\Gamma = (\Gamma, A^-)$ and $\Delta = \cdot$ on (2)
(5)	$\cdot; \Gamma, A^-; \cdot \Rightarrow_\Sigma [!A^-]$	(Rule $!R$ on (4))
(6)	$\aleph, \alpha{::}[!A^-]; \Gamma, A^-; \Delta_L, \alpha, \Delta_R \Rightarrow_\Sigma C$	(Weakening on (1))
(7)	$\aleph; \Gamma, A^-; \Delta_L, \Delta_R \Rightarrow_\Sigma C$	(Substitution on (5) and (6))
	$\aleph; \Gamma; \Delta_L, !A^-, \Delta_R \Rightarrow_\Sigma C$	(Rule $!L$ on (7))

Case: $A^+ = !Qp$

(1)	$\aleph, \alpha{::}[!Qp]; \Gamma; \Delta_L, \alpha, \Delta_R \Rightarrow_\Sigma C$	(Given)
To show:	$\aleph; \Gamma; \Delta_L, !Qp, \Delta_R \Rightarrow_\Sigma C$	
(2)	$\cdot; \Gamma, Qp; \cdot \Rightarrow_\Sigma [!Qp]$	(Rule $!Rp$)
(3)	$\aleph, \alpha{::}[!Qp]; \Gamma, Qp; \Delta_L, \alpha, \Delta_R \Rightarrow_\Sigma C$	(Weakening on (1))
(4)	$\aleph; \Gamma, Qp; \Delta_L, \Delta_R \Rightarrow_\Sigma C$	(Substitution on (2) and (3))
	$\aleph; \Gamma; \Delta_L, !Qp, \Delta_R \Rightarrow_\Sigma C$	(Rule $!L$ on (4))

Case: $A^+ = ¡A^-$ (similar to the case for $!A^-$)

Case: $A^+ = ¡Ql$ (similar to the case for $!Ql$)

Case: $A^+ = \mathbf{1}$

(1)	$\aleph, \alpha{::}[\mathbf{1}]; \Gamma; \Delta_L, \alpha, \Delta_R \Rightarrow_\Sigma C$	(Given)
To show:	$\aleph; \Gamma; \Delta_L, \mathbf{1}, \Delta_R \Rightarrow_\Sigma C$	
(2)	$\cdot; \Gamma; \cdot \Rightarrow_\Sigma [\mathbf{1}]$	(Rule $\mathbf{1}R$)
	$\aleph; \Gamma; \Delta_L, \Delta_R \Rightarrow_\Sigma C$	(Substitution on (2) and (1))

Case: $A^+ = A^+ \bullet B^+$

(1) $\aleph, \alpha{::}[A^+ \bullet B^+]; \Gamma; \Delta_L, \alpha, \Delta_R \Rightarrow_\Sigma C$ (Given)

To show: $\aleph; \Gamma; \Delta_L, A^+ \bullet B^+, \Delta_R \Rightarrow_\Sigma C$

(2) $\alpha_1{::}[A^+]; \Gamma; \alpha_1 \Rightarrow_\Sigma [A^+]$ (Rule $init\alpha$)

(3) $\alpha_2{::}[B^+]; \Gamma; \alpha_2 \Rightarrow_\Sigma [B^+]$ (Rule $init\alpha$)

(4) $\alpha_1{::}[A^+], \alpha_2{::}[B^+]; \Gamma; \alpha_1, \alpha_2 \Rightarrow_\Sigma [A^+ \bullet B^+]$ (Rule $\bullet R$ on (2) and (3))

(5) $\aleph, \alpha_1{::}[A^+], \alpha_2{::}[B^+]; \Gamma; \Delta_L, \alpha_1, \alpha_2, \Delta_R \Rightarrow_\Sigma C$ (Substitution on (4) and (1))

(6) $\aleph, \alpha_1{::}[A^+]; \Gamma; \Delta_L, \alpha_1, B^+, \Delta_R \Rightarrow_\Sigma C$

(IH(B^+) where $\aleph = \aleph, \alpha_1{::}[A^+]$ and $\Delta_L = \Delta_L, \alpha_1$ on (5))

(7) $\aleph; \Gamma; \Delta_L, A^+, B^+, \Delta_R \Rightarrow_\Sigma C$ (IH(A^+) where $\Delta_R = B^+, \Delta_R$ on (6))

$\aleph; \Gamma; \Delta_L, A^+ \bullet B^+, \Delta_R \Rightarrow_\Sigma C$ (Rule $\bullet L$ on (7))

Case: $A^+ = A^+ \oplus B^+$

(1) $\aleph, \alpha{::}[A^+ \oplus B^+]; \Gamma; \Delta_L, \alpha, \Delta_R \Rightarrow_\Sigma C$ (Given)

To show: $\aleph; \Gamma; \Delta_L, A^+ \oplus B^+, \Delta_R \Rightarrow_\Sigma C$

(2) $\alpha_1{::}[A^+]; \Gamma; \alpha_1 \Rightarrow_\Sigma [A^+]$ (Rule $init\alpha$)

(3) $\alpha_1{::}[A^+]; \Gamma; \alpha_1 \Rightarrow_\Sigma [A^+ \oplus B^+]$ (Rule $\oplus R_1$ on (2))

(4) $\aleph, \alpha_1{::}[A^+]; \Gamma; \Delta_L, \alpha_1, \Delta_R \Rightarrow_\Sigma C$ (Substitution on (3) and (1))

(5) $\alpha_2{::}[B^+]; \Gamma; \alpha_2 \Rightarrow_\Sigma [B^+]$ (Rule $init\alpha$)

(6) $\alpha_2{::}[B^+]; \Gamma; \alpha_2 \Rightarrow_\Sigma [A^+ \oplus B^+]$ (Rule $\oplus R_2$ on (5))

(7) $\aleph, \alpha_2{::}[B^+]; \Gamma; \Delta_L, \alpha_2, \Delta_R \Rightarrow_\Sigma C$ (Substitution on (6) and (1))

(8) $\aleph; \Gamma; \Delta_L, A^+, \Delta_R \Rightarrow_\Sigma C$ (IH(A^+) on (4))

(9) $\aleph; \Gamma; \Delta_L, B^+, \Delta_R \Rightarrow_\Sigma C$ (IH(B^+) on (7))

$\aleph; \Gamma; \Delta_L, A^+ \oplus B^+, \Delta_R \Rightarrow_\Sigma C$ (Rule $\oplus L$ on (8) and (9))

Case: $A^+ = \exists x.A^+$

(1) $\aleph, \alpha{::}[\exists x.A^+]; \Gamma; \Delta_L, \alpha, \Delta_R \Rightarrow_\Sigma C$

To show: $\aleph; \Gamma; \Delta_L, \exists x.A^+, \Delta_R \Rightarrow_\Sigma C$

(2) $\alpha{::}[A^+]; \Gamma; \alpha \Rightarrow_{\Sigma,x} [A^+]$ (Rule $init\alpha$)

(3) $\alpha{::}[A^+]; \Gamma; \alpha \Rightarrow_{\Sigma,x} [\exists x.A^+]$ (Rule $\exists R$ on (2))

(4) $\aleph, \alpha{::}[A^+]; \Gamma; \Delta_L, \alpha, \Delta_R \Rightarrow_{\Sigma,x} C$ (Substitution on (3) and (1))

(5) $\aleph; \Gamma; \Delta_L, A^+, \Delta_R \Rightarrow_{\Sigma,x} C$ (IH(A^+) on (4))

$\aleph; \Gamma; \Delta_L, \exists x.A^+, \Delta_R \Rightarrow_\Sigma C$ (Rule $\exists L$ on (6))

Case: $A^+ = t \doteq s$

(1) $\aleph, \alpha{::}[t \doteq s]; \Gamma; \Delta_L, \alpha, \Delta_R \Rightarrow_\Sigma C$

To show: $\aleph; \Gamma; \Delta_L, t \doteq s, \Delta_R \Rightarrow_\Sigma C$

(2) $\forall(\Sigma' \vdash \theta : \Sigma)$: $\theta t = \theta s \longrightarrow \theta\aleph; \theta\Gamma; \theta\Delta_L, \theta\Delta_R \Rightarrow_{\Sigma'} \theta C$

Assume an arbitrary $\Sigma' \vdash \theta : \Sigma$ such that (3) $\theta t = \theta s$

(4) $\cdot; \theta\Gamma; \cdot \Rightarrow_{\Sigma'} [\theta t \doteq \theta t]$ (Rule $\doteq R$)

(5) $\cdot; \theta\Gamma; \cdot \Rightarrow_{\Sigma'} [\theta t \doteq \theta s]$ (Equals-for-equals on (3) and (4))

(6) $\theta\aleph, \alpha{::}[\theta t \doteq \theta s]; \theta\Gamma; \theta\Delta_L, \alpha, \theta\Delta_R \Rightarrow_{\Sigma'} \theta C$ (Variable weakening on (1))

$\theta\aleph; \theta\Gamma; \theta\Delta_L, \theta\Delta_R \Rightarrow_{\Sigma'} \theta C$ (Substitution on (5) and (6)

$\aleph; \Gamma; \Delta_L, \Delta_R \Rightarrow_\Sigma C$ (Rule $\doteq L$ on (2))

Case: $A^- = Q^-$

(1) $\aleph, \gamma{::}[Q^-]; \Gamma; \Delta \Rightarrow_\Sigma \gamma$ (Given)

To show: $\aleph; \Gamma; \Delta \Rightarrow_\Sigma Q^-$

(2) $\cdot; \Gamma; [Q^-] \Rightarrow_\Sigma Q^-$ (Rule $init^-$)

$\aleph; \Gamma; \Delta \Rightarrow_\Sigma Q^-$ (Substitution on (2) and (1))

Case: $A^- = {\uparrow}A^+$

(1) $\aleph, \gamma{::}[{\uparrow}A^+]; \Gamma; \Delta \Rightarrow_\Sigma \gamma$ (Given)

To show: $\aleph; \Gamma; \Delta \Rightarrow_\Sigma {\uparrow}A^+$

(2) $\alpha{::}[A^+]; \Gamma; \alpha \Rightarrow_\Sigma [A^+]$ (Rule $init\alpha$)

(3) $\alpha{::}[A^+]; \Gamma; \alpha \Rightarrow_\Sigma {\uparrow}A^+$ (Rule ${\uparrow}R$ on (2))

(4) $\cdot; \Gamma; A^+ \Rightarrow_\Sigma {\uparrow}A^+$ (IH(A^+) where $\aleph = \cdot$ and $\Delta_L = \cdot$ and $\Delta_R = \cdot$ on (3))

(5) $\cdot; \Gamma; [{\uparrow}A^+] \Rightarrow_\Sigma {\uparrow}A^+$ (Rule ${\uparrow}L$ on (4))

$\aleph; \Gamma; \Delta \Rightarrow_\Sigma {\uparrow}A^+$ (Substitution on (5) and (1))

Case: $A^- = A^+ \twoheadrightarrow B^-$

(1) $\aleph, \gamma{::}[A^+ \twoheadrightarrow B^-]; \Gamma; \Delta \Rightarrow_\Sigma \gamma$ (Given)

To show: $\aleph; \Gamma; \Delta \Rightarrow_\Sigma A^+ \twoheadrightarrow B^-$

(2) $\alpha_1{::}[A^+]; \Gamma; \alpha_1 \Rightarrow_\Sigma [A^+]$ (Rule $init\alpha$)

(3) $\gamma_2{::}[B^-]; \Gamma; [B^-] \Rightarrow_\Sigma \gamma_2$ (Rule $init\gamma$)

(4) $\alpha_1{::}[A^+], \gamma_2{::}[B^-]; \Gamma; [A^+ \twoheadrightarrow B^-]\alpha_1 \Rightarrow_\Sigma \gamma_2$ (Rule $\twoheadrightarrow L$ on (2) and (3)

(5) $\aleph; \alpha_1{::}[A^+], \gamma_2{::}[B^-]; \Gamma; \Delta, \alpha_1 \Rightarrow_\Sigma \gamma_2$ (Substitution on (4) and (1))

(6) $\aleph, \alpha_1{::}[A^+]; \Gamma; \Delta, \alpha_1 \Rightarrow_\Sigma B^-$ (IH(B^-) where $\aleph = \aleph, \alpha_1{::}A^+$ and $\Delta = \Delta, \alpha_1$ on (5))

(7) $\aleph;\ \Delta, A^+ \Rightarrow_\Sigma B^-$ (IH(A^+) where $\Delta_L = \Delta$ and $\Delta_R = \cdot$ on (6))

$\aleph;\ \Delta \Rightarrow_\Sigma A^+ \twoheadrightarrow B^-$ (Rule $\twoheadrightarrow R$ on (7))

Case: $A^- = A^+ \rightarrowtail B^-$ (similar to the case for $A^+ \twoheadrightarrow B^-$)

Case: $A^- = A^- \,\&\, B^-$

(1)	$\aleph, \gamma{::}[A^- \,\&\, B^-]; \Gamma; \Delta \Rightarrow_\Sigma \gamma$	(Given)
To show:	$\aleph; \Gamma; \Delta \Rightarrow_\Sigma A^- \,\&\, B^-$	
(2)	$\gamma_1{::}[A^-]; \Gamma; [A^-] \Rightarrow_\Sigma \gamma_1$	(Rule $init\gamma$)
(3)	$\gamma_1{::}[A^-]; \Gamma; [A^- \,\&\, B^-] \Rightarrow_\Sigma \gamma_1$	(Rule $\&R_1$ on (2))
(4)	$\aleph, \gamma_1{::}[A^-]; \Gamma; \Delta \Rightarrow_\Sigma \gamma_1$	(Substitution on (3) and (1))
(5)	$\gamma_2{::}[B^-]; \Gamma; [B^-] \Rightarrow_\Sigma \gamma_2$	(Rule $init\gamma$)
(6)	$\gamma_2{::}[B^-]; \Gamma; [A^- \,\&\, B^-] \Rightarrow_\Sigma \gamma_2$	(Rule $\&R_2$ on (5))
(7)	$\aleph, \gamma_2{::}[B^-]; \Gamma; \Delta \Rightarrow_\Sigma \gamma_2$	(Substitution on (6) and (1))
(8)	$\aleph; \Gamma; \Delta \Rightarrow_\Sigma A^-$	(IH(A^-) on (4))
(9)	$\aleph; \Gamma; \Delta \Rightarrow_\Sigma B^-$	(IH(B^-) on (7)
	$\aleph; \Gamma; \Delta \Rightarrow_\Sigma A^- \,\&\, B^-$	(Rule $\&R$ on (8) and (9))

Case: $A^- = \forall x.A^-$

(1)	$\aleph, \gamma{::}[\forall x.A^-]; \Gamma; \Delta \Rightarrow_\Sigma \gamma$	(Given)
To show:	$\aleph; \Gamma; \Delta \Rightarrow_\Sigma \forall x.A^-$	
(2)	$\gamma'{::}[A^-]; \Gamma; [A^-] \Rightarrow_{\Sigma,x} \gamma'$	(Rule $init\gamma$)
(3)	$\gamma'{::}[A^-]; \Gamma; [\forall x.A^-] \Rightarrow_{\Sigma,x} \gamma'$	(Rule $\forall L$ on (2))
(4)	$\aleph, \gamma'{::}[A^-]; \Gamma; \Delta \Rightarrow_{\Sigma,x} \gamma'$	(Substitution on (3) and (1))
(5)	$\aleph; \Gamma; \Delta \Rightarrow_{\Sigma,x} A^-$	(IH(A^-) on (4))
(6)	$\aleph; \Gamma; \Delta \Rightarrow_\Sigma \forall x.A^-$	(Rule $\forall R$ on (5))

This concludes the proof. □

3.4.1 Unfocused identity

Now, our desired completeness theorem for the original logic is a straightforward corollary of the identity expansion lemma:

Theorem 7 (Unfocused identity)**.** *For all A^+, $\Gamma; A^+ \Rightarrow_\Sigma \uparrow A^+$, and for all A^-, $\Gamma; \downarrow A^- \Rightarrow_\Sigma A^-$.*

Proof. From rule *init*α we know that $\alpha{::}[A^+]; \Gamma; \alpha \Rightarrow_\Sigma [A^+]$,
and so by $\uparrow R$ we know $\alpha{::}[A^+]; \Gamma; \alpha \Rightarrow_\Sigma \uparrow A^+$.
Therefore, by the identity expansion theorem, we know $\cdot; \Gamma; A^+ \Rightarrow_\Sigma \uparrow A^+$.

From rule *init*γ we know that $\gamma{::}[A^-]; \Gamma; [A^-] \Rightarrow_\Sigma \gamma$,
and so by rule $\downarrow L$ we know $\gamma{::}[A^-]; \Gamma; \downarrow A^- \Rightarrow_\Sigma \gamma$.
Therefore, by the identity expansion theorem, we know $\cdot; \Gamma; \downarrow A^- \Rightarrow_\Sigma A^-$. □

3.5 Unfocused admissibility

Unfocused admissibility is what justifies the completeness of the weakly focused sequent calculus relative to the unfocused sequent calculus. If we then translate sequents into polarized logic by adding shifts everywhere, we can directly translate the a derivation in the unfocused, unpolarized logic into focused, polarized logic by using admissible rules. We give only the cases relevant to the logic programming fragment of ordered logic; the remaining cases are similar.

Theorem 8 (Unfocused admissibility)**.** *Shifted versions of in-focus rules are admissible when* $\aleph = \cdot$*:*

- $(copy')$ — *If* $(A^- \in \Gamma)$ *and* $(\Gamma; \Delta_L, {\downarrow}A^-, \Delta_R \Rightarrow_\Sigma C)$*, then* $(\Gamma; \Delta_L, \Delta_R \Rightarrow_\Sigma C)$*.*
- $(place')$ — *If* $(\Gamma; \Delta_L, {\downarrow}A^-, \Delta_R \Rightarrow_\Sigma C)$*, then* $(\Gamma; \Delta_L, \underline{A^-}, \Delta_R \Rightarrow_\Sigma C)$*.*
- $(\mathbf{1}R')$ — $(\Gamma; \cdot \Rightarrow_\Sigma {\uparrow}\mathbf{1})$*.*
- $(\bullet R')$ — *If* $(\Gamma; \Delta_1 \Rightarrow_\Sigma {\uparrow}A^+)$ *and* $(\Gamma; \Delta_2 \Rightarrow_\Sigma {\uparrow}B^+)$*, then* $(\Gamma; \Delta_1, \Delta_2 \Rightarrow_\Sigma {\uparrow}(A^+ \bullet B^+))$*.*
- $(\exists R')$ — *If* $(\Gamma; \Delta \Rightarrow_\Sigma {\uparrow}A^+[t/x])$*, then* $(\Gamma; \Delta \Rightarrow_\Sigma {\uparrow}\exists x.A^+)$*.*
- $(\twoheadrightarrow L')$ — *If* $(\Gamma; \Delta_A \Rightarrow_\Sigma {\uparrow}A^+)$ *and* $(\Gamma; \Delta_L, {\downarrow}B^-, \Delta_R \Rightarrow_\Sigma C)$*,*
  then $(\Gamma; \Delta_L, {\downarrow}(A^+ \twoheadrightarrow B^-), \Delta_A, \Delta_R \Rightarrow_\Sigma C)$*.*
- $(\forall L')$ — *If* $(\Gamma; \Delta_L, {\downarrow}A[t/x], \Delta_R \Rightarrow_\Sigma C)$*, then* $(\Gamma; \Delta_L, {\downarrow}\forall x.A^-, \Delta_R \Rightarrow_\Sigma C)$*.*

Proof. Each case involves an appeal to identity expansion and unfocused cut admissibility.

Admissible rule $copy'$

(1) $A^- \in \Gamma$ (Given)
(2) $\cdot; \Gamma; \Delta_L, {\downarrow}A^-, \Delta_R \Rightarrow_\Sigma C$ (Given)
To show: $\cdot; \Gamma; \Delta_L, \Delta_R \Rightarrow_\Sigma C$
(3) $\gamma{::}[A^-]; \Gamma; [A^-] \Rightarrow_\Sigma \gamma$ (Rule $init\gamma$)
(4) $\gamma{::}[A^-]; \Gamma; \cdot \Rightarrow_\Sigma \gamma$ (Rule *copy* on (1) and (3))
(5) $\cdot; \Gamma; \cdot \Rightarrow_\Sigma A^-$ (Identity expansion on (4))
$\cdot; \Gamma; \Delta_L, \Delta_R \Rightarrow_\Sigma C$ (Unfocused cut on (5) and (2))

Admissible rule $place'$

(1) $\cdot; \Gamma; \Delta_L, {\downarrow}A^-, \Delta_R \Rightarrow_\Sigma C$ (Given)
To show: $\cdot; \Gamma; \Delta_L, \underline{A^-}, \Delta_R \Rightarrow_\Sigma C$
(2) $\gamma{::}[A^-]; \Gamma; [A^-] \Rightarrow_\Sigma \gamma$ (Rule $init\gamma$)
(3) $\gamma{::}[A^-]; \Gamma; \underline{A^-} \Rightarrow_\Sigma \gamma$ (Rule *place* on (2))
(5) $\cdot; \Gamma; \underline{A^-} \Rightarrow_\Sigma A^-$ (Identity expansion on (4))
$\cdot; \Gamma; \Delta_L, \underline{A^-}, \Delta_R \Rightarrow_\Sigma C$ (Unfocused cut on (6) and (2))

Admissible rule $\mathbf{1}R'$

To show: $\cdot; \Gamma; \cdot \Rightarrow_\Sigma {\uparrow}\mathbf{1}$
(1) $\cdot; \Gamma; \cdot \Rightarrow_\Sigma [\mathbf{1}]$ (Rule $\mathbf{1}R$)
(2) $\cdot; \Gamma; \cdot \Rightarrow_\Sigma {\uparrow}\mathbf{1}$ (Rule ${\uparrow}R$)

Admissible rule $\bullet R'$

(1)	$\cdot; \Gamma; \Delta_L \Rightarrow_\Sigma \uparrow A^+$	(Given)
(2)	$\cdot; \Gamma; \Delta_R \Rightarrow_\Sigma \uparrow B^+$	(Given)
To show:	$\cdot; \Gamma; \Delta_L, \Delta_R \Rightarrow_\Sigma \uparrow(A^+ \bullet B^+)$	
(3)	$\alpha_1{::}[A^+]; \Gamma; \alpha_1 \Rightarrow_\Sigma [A^+]$	(Rule $init\alpha$)
(4)	$\alpha_2{::}[B^+]; \Gamma; \alpha_2 \Rightarrow_\Sigma [B^+]$	(Rule $init\alpha$)
(5)	$\alpha_1{::}[A^+], \alpha_2{::}[B^+]; \Gamma; \alpha_1, \alpha_2 \Rightarrow_\Sigma [A^+ \bullet B^+]$	(Rule $\bullet R$ on (3) and (4))
(6)	$\alpha_1{::}[A^+], \alpha_2{::}[B^+]; \Gamma; \alpha_1, \alpha_2 \Rightarrow_\Sigma \uparrow(A^+ \bullet B^+)$	(Rule $\uparrow R$ on (5))
(7)	$\alpha_1{::}[A^+]; \Gamma; \alpha_1, B^+ \Rightarrow_\Sigma \uparrow(A^+ \bullet B^+)$	(Identity expansion on (6))
(8)	$\cdot; \Gamma; A^+, B^+ \Rightarrow_\Sigma \uparrow(A^+ \bullet B^+)$	(Identity expansion on (7))
(9)	$\cdot; \Gamma; A^+, \Delta_R \Rightarrow_\Sigma \uparrow(A^+ \bullet B^+)$	(Unfocused cut on (2) and (8))
	$\cdot; \Gamma; \Delta_L, \Delta_R \Rightarrow_\Sigma \uparrow(A^+ \bullet B^+)$	(Unfocused cut on (1) and (9))

Admissible rule $\exists R'$

(1)	$\cdot; \Gamma; \Delta \Rightarrow_\Sigma \uparrow A^+[t/x]$	(Given)
To show:	$\cdot; \Gamma; \Delta \Rightarrow_\Sigma \uparrow \exists x.A^+$	
(2)	$\alpha{::}[A^+[t/x]]; \Gamma; \alpha \Rightarrow_\Sigma [A^+[t/x]]$	(Rule $init\alpha$)
(3)	$\alpha{::}[A^+[t/x]]; \Gamma; \alpha \Rightarrow_\Sigma [\exists x.A^+]$	(Rule $\exists R$ on (2))
(4)	$\alpha{::}[A^+[t/x]]; \Gamma; \alpha \Rightarrow_\Sigma \uparrow \exists x.A^+$	(Rule $\uparrow R$ on (3))
(5)	$\cdot; \Gamma; A^+[t/x] \Rightarrow_\Sigma \uparrow \exists x.A^+$	(Identity expansion on (4))
	$\cdot; \Gamma; \Delta \Rightarrow_\Sigma \uparrow \exists x.A^+$	(Unfocused cut on (1) and (5))

Admissible rule $\twoheadrightarrow L'$

(1)	$\cdot; \Gamma; \Delta_A \Rightarrow_\Sigma \uparrow A^+$	(Given)
(2)	$\cdot; \Gamma; \Delta_L, \uparrow B^-, \Delta_R \Rightarrow_\Sigma C$	(Given)
To show:	$\cdot; \Gamma; \Delta_L, \uparrow(A^+ \twoheadrightarrow B^-), \Delta_A, \Delta_R \Rightarrow_\Sigma C$	
(3)	$\alpha{::}[A^+]; \Gamma; \alpha \Rightarrow_\Sigma [A^+]$	(Rule $init\alpha$)
(4)	$\gamma{::}[B^-]; \Gamma; [B^-] \Rightarrow_\Sigma \gamma$	(Rule $init\gamma$)
(5)	$\alpha{::}[A^+], \gamma{::}[B^-]; \Gamma; [A^+ \twoheadrightarrow B^-], \alpha \Rightarrow_\Sigma \gamma$	(Rule $\twoheadrightarrow L$ on (3) and (4))
(6)	$\alpha{::}[A^+], \gamma{::}[B^-]; \Gamma; \uparrow(A^+ \twoheadrightarrow B^-), \alpha \Rightarrow_\Sigma \gamma$	(Rule $\uparrow L$ on (5))
(7)	$\alpha{::}[A^+]; \Gamma; \uparrow(A^+ \twoheadrightarrow B^-), \alpha \Rightarrow_\Sigma B^-$	(Identity expansion on (6))
(8)	$\cdot; \Gamma; \uparrow(A^+ \twoheadrightarrow B^-), A^+ \Rightarrow_\Sigma B^-$	(Identity expansion on (7))
(9)	$\cdot; \Gamma; \Delta_L, \uparrow(A^+ \twoheadrightarrow B^-), A^+, \Delta_R \Rightarrow_\Sigma C$	(Unfocused cut on (8) and (2))
	$\cdot; \Gamma; \Delta_L, \uparrow(A^+ \twoheadrightarrow B^-), \Delta_A, \Delta_R \Rightarrow_\Sigma C$	(Unfocused cut on (1) and (9))

Admissible rule $\forall R'$

(1)	$\cdot; \Gamma; \Delta_L, \uparrow A^-[t/x], \Delta_R \Rightarrow_\Sigma C$	(Given)
To show:	$\cdot; \Gamma; \Delta_L, \uparrow(\forall x.A^-), \Delta_R \Rightarrow_\Sigma C$	
(2)	$\gamma{::}[A^-[t/x]]; \Gamma; [A^-[t/x]] \Rightarrow_\Sigma \gamma$	(Rule $init\gamma$)
(3)	$\gamma{::}[A^-[t/x]]; \Gamma; [\forall x.A^-] \Rightarrow_\Sigma \gamma$	(Rule $\forall L$ on (2))
(4)	$\gamma{::}[A^-[t/x]]; \Gamma; \uparrow(\forall x.A^-) \Rightarrow_\Sigma \gamma$	(Rule $\uparrow L$ on (3))
(5)	$\cdot; \Gamma; \uparrow(\forall x.A^-) \Rightarrow_\Sigma A^-[t/x]$	(Identity expansion on (4))
	$\cdot; \Gamma; \Delta_L, \uparrow(\forall x.A^-), \Delta_R \Rightarrow_\Sigma C$	(Unfocused cut on (1) and (5))

This concludes the proof. □

4 Soundness and completeness

In this section, we outline the use of unfocused admissibility (Theorem 8) to show soundness and completeness between the unfocused, unpolarized logic given in Section 1 and the focused, polarized logic given in Section 2. Note that we can use these soundness and completeness results to establish the admissibility of cut and identity for the unfocused, unpolarized logic by translation into the focused, polarized logic.

The obvious erasure of polarized formulas to unpolarized formulas, given in Figure 1 below. $(A^+)^\oplus$ is the erasure of a positive proposition, $(A^-)^\ominus$ is the erasure of a negative proposition, and they are mutually inductive on the structure of propositions. One important point is that Q^+, Q^-, Qp, and Ql are all erased to "plain old atomic propositions," but they always remain distinct – an unpolarized atomic proposition remains marked with a polarity assignment even though the polarized framework cannot distinguish them.

$$
\begin{aligned}
(Q^+)^\oplus &= Q^+ \\
({\downarrow}A^-)^\oplus &= (A^+)^\ominus \\
(!A^-)^\oplus &= !(A^-)^\ominus \\
(!Qp)^\oplus &= !Qp \\
(¡A^-)^\oplus &= ¡(A^-)^\ominus \\
(¡Ql)^\oplus &= ¡Ql \\
(\mathbf{1})^\oplus &= \mathbf{1} \\
(A^+ \bullet B^+)^\oplus &= (A^+)^\oplus \bullet (B^+)^\oplus \\
(A^+ \oplus B^+)^\oplus &= (A^+)^\oplus \oplus (A^+)^\oplus \\
(\exists x.A^+)^\oplus &= \exists x.(A^+)^\oplus \\
(t \doteq s)^\oplus &= t \doteq s
\end{aligned}
\qquad
\begin{aligned}
(Q^-)^\ominus &= Q^- \\
({\uparrow}A^+)^\ominus &= (A^+)^\oplus \\
(A^+ \twoheadrightarrow B^-)^\ominus &= (A^+)^\oplus \twoheadrightarrow (A^-)^\ominus \\
(A^+ \rightarrowtail B^-)^\ominus &= (A^+)^\oplus \rightarrowtail (A^-)^\ominus \\
(A^- \,\&\, B^-)^\ominus &= (A^-)^\ominus \,\&(B^-)^\ominus \\
(\forall x.A^-)^\ominus &= \forall x.(A^-)^\ominus \\
(\cdot)^\ominus &= \cdot \\
(\Gamma, A^-)^\ominus &= (\Gamma)^\ominus, (A^-)^\ominus \\
(\cdot)^\oplus &= \cdot \\
(\Delta, \Delta')^\oplus &= (\Delta)^\oplus, (\Delta')^\oplus \\
(\underline{A^-})^\oplus &= \underline{(A^-)^\ominus} \\
(\underline{Ql})^\oplus &= \underline{Ql}
\end{aligned}
$$

Figure 1: Erasing the polarization of propositions and contexts.

One strategy that we could use would be to define a *particular* strategy for going in the other direction and polarizing unpolarized propositions. We will do something a bit more general and consider *all* strategies for polarizing formulas, which are necessarily contained in the (non-deterministic) inverse of the erasure function. It is important that we know there is *some* total strategy for polarizing unpolarized formulas, but this is straightforward.

A version of this proof for persistent logic (formalized in Twelf) was previously given in two parts. The first part established the correspondence of the unpolarized and polarized logics in an unfocused setting (in an unfocused polarized logic, which is a little weird, the shifts are meaningless and invertible on both the left and right) [3]. The second part established the correspondence of the polarized unfocused logic and the polarized focused logic [4]. Here we compose those two proofs and directly prove the soundness and completeness. Soundness is simple, as focused proofs are essentially a refinement of focused proofs; completeness is a bit more interesting.

4.1 Soundness of the focused, polarized logic

Theorem 9 (Soundness)**.**

- *If* $(\Gamma; \Delta \Rightarrow_\Sigma A^-)$*, then* $(\Gamma^\ominus; \Delta^\oplus \vdash_\Sigma A^{-\ominus})$.
- *If* $(\Gamma; \Delta_L[A^-]\Delta_R \Rightarrow_\Sigma A^-)$*, then* $(\Gamma^\ominus; \Delta_L^\oplus, A^{-\ominus}, \Delta_R^\oplus \vdash_\Sigma A^{-\ominus})$.
- *If* $(\Gamma; \Delta \Rightarrow_\Sigma [A^+])$*, then* $(\Gamma^\ominus; \Delta^\oplus \vdash_\Sigma A^{+\oplus})$.

Proof. By straightforward mutual induction on the given polarized, focused derivation.

- For the $copy$, $place$, $init^-$, $init^+$, $!L$, $!R$, $¡R$, $\mathbf{1}R$, $\mathbf{1}L$, $\bullet R$, $\bullet L$, $\twoheadrightarrow R$, $\twoheadrightarrow L$, $\rightarrowtail R$, $\rightarrowtail L$, $\oplus R_i$, $\oplus L$, $\& R$, $\& L$, $\forall R^x$, $\forall L$, $\exists R$, $\exists L^x$, $\doteq R$, and $\doteq L$ rules, we apply the induction hypothesis to translate all subderivations, and then re-apply the unfocused version of the same rule.
- The $!Rq$ rule translates to a use of $init$ followed by $copy$, and the $¡Rq$ rule translates to a use of $init$ followed by $place$.
- For the $\uparrow R$, $\uparrow L$, $\downarrow R$, $\downarrow L$ rules, the erasure of the premise and conclusion are the same, so we need only apply the induction hypothesis to the premise.

Considering all these cases completes the proof. □

4.2 Completeness of the focused, polarized logic

Theorem 10 (Completeness)**.** *If* $(\Gamma^\ominus; \Delta^\oplus \vdash_\Sigma A^{-\ominus})$*, then* $(\Gamma; \Delta \Rightarrow_\Sigma A^-)$.

Proof. By induction on the given unpolarized, unfocused derivation, along with secondary inductions on the translation of the principal formula. We give a few representative cases; other cases are similar.

Case: $$\dfrac{\Gamma^\ominus; \Delta_L^\oplus, A^{-\ominus}, \Delta_R^\oplus \vdash_\Sigma C^{-\ominus}}{\Gamma^\ominus; \Delta_L^\oplus, \underline{A^{-\ominus}}, \Delta_R^\oplus \vdash_\Sigma C^{-\ominus}}\ place$$

Since $A^{-\ominus} = (\downarrow A^-)^\oplus$, we can apply the IH to the premise to get $\Gamma; \Delta_L, \downarrow A^-, \Delta_R \Rightarrow_\Sigma C^-$. The result then follows from the admissible rule $place'$ (Theorem 8).

Case: $$\dfrac{\Gamma^\ominus; \Delta_L^\oplus \vdash_\Sigma A \quad \Gamma; \Delta_R^\oplus \vdash_\Sigma B}{\Gamma^\ominus; \Delta_L^\oplus, \Delta_R^\oplus \vdash_\Sigma A \bullet B}\ \bullet R \qquad (A^-)^\ominus = A \bullet B.$$

We prove $\Gamma; \Delta_L, \Delta_R \Rightarrow_\Sigma A^-$ by secondary induction on the structure of A^-.

Subcase: $A^- = \uparrow\downarrow B^-$

We have $\Gamma; \Delta_L, \Delta_R \Rightarrow_\Sigma B^-$ by the induction hypothesis, $\Gamma; \Delta_L, \Delta_R \Rightarrow_\Sigma [\downarrow B^-]$ by rule $\downarrow R$, and $\Gamma; \Delta_L, \Delta_R \Rightarrow_\Sigma \uparrow\downarrow B^-$ by rule $\uparrow R$.

Subcase: $A^- = \uparrow(A^+ \bullet B^+)$ (so $A = (A_1^+)^\oplus$ and $B = (A_2^+)^\oplus$)

We also have $A = (\uparrow A_1^+)^\ominus$ and $B = (\downarrow A_2^+)^\oplus$, so we can apply the induction hypothesis to the premises of $\bullet R$ to get $\Gamma; \Delta_L \Rightarrow_\Sigma \uparrow A_1^+$ and $\Gamma; \Delta_R \Rightarrow_\Sigma \uparrow A_2^+$. The result then follows from the admissible rule $\bullet R'$ (Theorem 8).

Case: $\dfrac{\Gamma^{\ominus}; \Delta_L^{\oplus}, A, B, \Delta_R^{\oplus} \vdash_\Sigma C^{-\ominus}}{\Gamma^{\ominus}; \Delta_L^{\oplus}, A \bullet B, \Delta_R^{\oplus} \vdash_\Sigma C^{-\ominus}} \bullet L \qquad (A^+)^{\oplus} = A \bullet B.$

We prove $\Gamma; \Delta_L, A^+, \Delta_R \Rightarrow_\Sigma C^-$ by secondary induction on the structure of A^+.

Subcase: $A^+ = \downarrow\uparrow B^+$

We have $\Gamma; \Delta_L, B^+, \Delta_R \Rightarrow_\Sigma C^-$ by the induction hypothesis, $\Gamma; \Delta_L[\uparrow B^+]\Delta_R \Rightarrow_\Sigma C^-$ by rule $\uparrow L$, and $\Gamma; \Delta_L, \downarrow\uparrow B^+, \Delta_R \Rightarrow_\Sigma C^-$ by rule $\downarrow L$.

Subcase: $A^+ = A_1^+ \bullet B_1^+$ (so $A = (A_1^+)^{\oplus}$ and $B = (A_2^+)^{\oplus}$)

We can apply the induction hypothesis to the premise to get $\Gamma; \Delta_L, A^+, B^+, \Delta_R \Rightarrow_\Sigma C^-$. The result then follows from rule $\bullet L$.

Case: $\dfrac{}{\Gamma^{\ominus}; \cdot \vdash_\Sigma t \doteq t} \doteq R \qquad (A^-)^{\ominus} = t \doteq t$

We prove $\Gamma; \cdot \Rightarrow_\Sigma A^-$ by secondary induction on the structure of A^-.

Subcase: $A^- = \uparrow\downarrow B^-$

We have $\Gamma; \cdot \Rightarrow_\Sigma B^-$ by the induction hypothesis, $\Gamma; \cdot \Rightarrow_\Sigma [\downarrow B^-]$ by rule $\downarrow R$, and $\Gamma; \cdot \Rightarrow_\Sigma \uparrow\downarrow B^-$ by rule $\uparrow R$.

Subcase: $A^- = \uparrow(t \doteq t)$

We have $\Gamma; \cdot \Rightarrow_\Sigma [t \doteq t]$ by rule $\doteq R$, and $\Gamma; \cdot \Rightarrow_\Sigma \uparrow(t \doteq t)$ by rule $\uparrow R$.

Case: $\dfrac{\forall(\Sigma' \vdash \theta : \Sigma): \quad \theta t = \theta s \quad \longrightarrow \quad \theta\Gamma^{\ominus}; \theta\Delta_L^{\oplus}, \theta\Delta_R^{\oplus} \vdash_{\Sigma'} \theta C^{-\ominus}}{\Gamma^{\ominus}; \Delta_L^{\oplus}, t \doteq s, \Delta_R^{\oplus} \vdash_\Sigma C^{-\ominus}} \doteq L \qquad (A^+)^{\oplus} = t \doteq s$

We prove $\Gamma; \Delta_L, A^+, \Delta_R \vdash_\Sigma C^-$ by secondary induction on the structure of A^+.

Subcase: $A^+ = \downarrow\uparrow B^+$

We have $\Gamma; \Delta_L, B^+, \Delta_R \Rightarrow_\Sigma C^-$ by the induction hypothesis, $\Gamma; \Delta_L[\uparrow B^+]\Delta_R \Rightarrow_\Sigma C^-$ by rule $\uparrow L$, and $\Gamma; \Delta_L, \downarrow\uparrow B^+, \Delta_R \Rightarrow_\Sigma C^-$ by rule $\downarrow L$.

Subcase: $A^+ = t \doteq s$

We assume $\Sigma' \vdash \theta : \Sigma$ and $\theta t = \theta s$, and applying them to the premise get $\theta\Gamma^{\ominus}; \theta\Delta_L^{\oplus}, \theta\Delta_R^{\oplus} \vdash_{\Sigma'} \theta C^{-\ominus}$, which by the induction hypothesis gives us $\theta\Gamma; \theta\Delta_L, \theta\Delta_R \Rightarrow_{\Sigma'} \theta C^-$. The result then follows from rule $\doteq L$.

This completes the proof. □

4.3 Metatheory of the unfocused logic

Note that none of the proofs in this paper have been concerned with any properties of the unfocused logic. This means that, as long as we have some arbitrary strategy for polarizing an unpolarized sequent, the results in this section allow us to port all the metatheoretic properties of the focused, polarized logic back to the unfocused, unpolarized logic from Section 1, giving us the internal soundness and completeness of the unfocused logic "for free."

Corollary 1 (Cut admissibility). *If* $(\Gamma; \Delta \vdash_\Sigma A)$ *and* $(\Gamma; \Delta_L, A, \Delta_R \vdash C)$*, then* $(\Gamma; \Delta_L, \Delta, \Delta_R \vdash_\Sigma C)$

Proof. We translate the given derivations into the polarized, focused logic (Theorem 10), apply unfocused cut (Theorem 4), and then translate back (Theorem 9). □

Corollary 2 (Identity). *For all* A*,* $(\Gamma; A \vdash_\Sigma A)$

Proof. We translate A into its polarized version (Theorem 10), apply unfocused identity (Theorem 7), and then translate back (Theorem 9). □

References

[1] Jason Reed and Frank Pfenning. Focus-preserving embeddings of substructural logics in intuitionistic logic, 2010. Unpublished Manuscript. `http://www.cs.cmu.edu/~fp/papers/substruct10.pdf`.

[2] Peter Schroeder-Heister. Rules of definitional reflection. In *Symposium on Logic and Computer Science*, pages 222–232. IEEE Computer Society Press, 1993.

[3] Robert J. Simmons and Frank Pfenning. Lax logic, 2009. On the Twelf Wiki. `http://twelf.plparty.org/wiki/Lax_logic`.

[4] Robert J. Simmons and Frank Pfenning. Weak focusing, 2009. On the Twelf Wiki. `http://twelf.plparty.org/wiki/Weak_focusing`.

[5] Robert J. Simmons and Bernardo Toninho. Constructive provability logic. In *Intuitionstic Modal Logic and Applications*, 2011.