Typed Closure Conversion

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Abstract

We study the typing properties of *closure conversion* for simply-typed and polymorphic λ-calculi. Unlike most accounts of closure conversion, which only treat the untyped λ-calculus, we translate well-typed source programs to well-typed target programs. This allows later compiler phases to take advantage of types for representation analysis and tag-free garbage collection, and it facilitates correctness proofs. Our account of closure conversion for the simply-typed language takes advantage of a simple model of objects by mapping closures to *existentials*. Closure conversion for the polymorphic language requires additional type machinery, namely *translucency* in the style of Harper and Lillibridge's module calculus, to express the type of a closure.

1 Introduction

The usual operational models of programming languages based on the λ-calculus compute by substituting terms for variables in other terms. But substitution is expensive because it requires traversing and copying a term in order to find and replace all occurrences of the given variable. A well-known technique for mitigating these costs is to delay substitution until the binding of the variable is required during evaluation [18, 1]. This is accomplished by pairing an open term with an environment providing values for the free variables in the term. The open term may be thought of as immutable *code* that acts on the environment. Since the code is immutable, it can be generated once and shared among all instances of a function.

Closure conversion [30, 35, 7, 17, 16, 3, 38, 9] is a program transformation that achieves such a separation between code and data. Functions with free variables are replaced by code abstracted on an extra environment parameter. Free variables in the body of the function are replaced by references to the environment. The abstracted code is "partially applied" to an explicitly constructed environment providing the bindings for these variables. This "partial application" of the code to its environment is in fact suspended until the function is actually applied to its argument; the suspended application is called a "closure", a data structure containing pure code and a representation of its environment.

A critical decision in closure conversion is the choice of representation of the environment as a data structure — for example, whether to use a "flat", "linked", or hybrid representation. This decision is influenced by a desire to minimize closure creation time, the space consumed by an environment, and the time to access a given variable in an environment [38, 31]. An important property of closure conversion is that the representation of the environment is *private* to the closure, and is not visible from the outside. This affords considerable flexibility in the representation of environments and is thus exploited to good advantage by Shao and Appel [31] and Wand and Steckler [38].

Most accounts consider closure conversion as a transformation to *untyped* terms, irrespective of whether or not the source term is typed [35, 17, 3, 38]. This is adequate for compilers that make little or no use of types in the back end or at run time. However, when compiling typed languages, it is often advantageous to propagate type information through each stage of the compiler, and to make use of types at link or even run time. For example, Leroy's representation analysis [19, 32] uses types to determine procedure calling conventions, and Ohori's record compilation [26] uses a representation of types at run time to access components of a record. In current compilers, these phases must occur *before* closure conversion because the output of closure conversion is untyped. Compilation strategies for polymorphic languages, such as those proposed by Morrison *et al.* [25] and Harper and Morrisett [14], rely on analyzing types at run time to support unboxed representations and non-parametric operators, including printing and structural equality. Tag-free garbage collection [5, 37, 24] for both monomorphic and polymorphic programming languages also relies upon types at run time to determine the size and the pointers of objects. To support any of these implementation strategies, it is necessary to propagate type information *through* closure conversion and into the generated code. Consequently, the purpose of this paper is to show how closure conversion can be formulated as a *type-preserving transform.*

We are therefore interested in *type-based transformations* as a basis for compiling polymorphic languages. The crucial idea is to define a compiler as a series of transformations on both the program and its type, possibly relying on type information to guide the transformation itself. Each stage of the compiler is thus viewed as a type-preserving translation between typed intermediate languages. Examples of such translations are given by Leroy [19], Ohori [26], Harper and Lillibridge [10], and

Harper and Morrisett [14]. In addition to the practical advantages of propagating type information through the stages of a compiler, type-directed translation also facilitates correctness proofs by defining the invariants of the transformation as a type-indexed family of *logical relations* [36, 8, 28, 33, 34].

We describe closure conversion in two stages. The first stage, *abstract closure conversion*, is a type-based translation from the source language into a target language with explicit closures. The translation is described as a deductive system in which the representation of the environment may be chosen independently for each closure. In this way various environment representations, such as those used by the CAM [7] and the FAM [6], as well as hybrid strategies, such as those suggested by Shao and Appel [31] can be explained in a uniform framework.

The second stage, *closure representation*, is a type-based translation in which the implementation of closures is determined. The main idea is to represent *closures as objects* (in contrast to the proposed representation of *objects as closures* [29]). Following Pierce and Turner [27] we consider objects to be packages of existential type consisting of a single method (the code part of the closure) together with a single instance variable (the environment part) whose type (the environment representation) is held abstract. This captures the critical "privacy" property of environment representations for closures. In the simply-typed case we make direct use of Pierce and Turner's model of objects. In the polymorphic case we must in addition exploit the notion of *translucency* [11] (or *manifest types* [20]) to express the type of a polymorphic closure.

The correctness of both the abstract closure conversion and the closure representation stages are proved using the method of logical relations. The main idea is to define a type-indexed family of simulation relations that establish a correspondence between the source and target terms of the translation. Once a suitable system of relations has been defined, it is relatively straightforward to prove by induction on the definition of the compilation relation that the source and target of the translation are related, from which we may conclude that a closed program and its compilation evaluate to the same result.

Closure conversion is discussed in descriptions of various functional language compilers [35, 17, 4, 3, 31]. It is closely related to λ-lifting [15] in that it eliminates free variables in the bodies of λ-abstractions but differs by making the representation of the environment explicit as a data structure. Making the environment explicit is important because it exposes environment construction and variable lookup to an optimizer. Furthermore, Shao and Appel show that not all environment representations are "safe for space" [31], and thus choosing a good environment representation is an important part of compilation. Wand and Steckler [38] have consider two optimizations of the basic closure conversion strategy, called *selective* and *lightweight* closure conversion, and provide a correctness proof for each of these in an untyped setting. Hannan [9] re-casts Wand's work into a typed setting, and provides correctness proofs for Wand's optimizations. Hannan's translation is given, like ours, as a deductive system, but like λ-lifting, he does not consider the important issue of environment representation (preferring an abstract account), nor does he consider the typing properties of the closure-converted code. Finally, neither Wand nor Hannan consider closure conversion under a type-passing interpretation of polymorphism.

We assume that the reader is familiar with standard λ-calculus presentations, ML language syntax, and various type-theoretic constructs including existential types [23], and the module calculus of Harper and Lillibridge [11]. We have attempted to use standard notation whenever possible. For example, we write $e[v/x]$ to denote capture-avoiding substitution of the value v for the free variable x within e.

The remainder of this paper is organized as follows: In Section 2, we give an overview of closure conversion and the typing issues involved for the simply-typed λ-calculus. In Section 3, we provide the details of our type-preserving transform for the simply-typed case. In Section 4, we give an overview of closure conversion and the typing issues involved for the predicative fragment of the polymorphic λ-calculus. The formal development of this conversion is given in Section 5.

2 Overview of Simply-Typed Closure Conversion

The main ideas of closure conversion may be illustrated by considering the following ML program:

```
let val x = 1
    val y = 2
    val z = 3
    val f = λw. x + y + w
in
    f 100
end
```

The function f contains free variables x and y, but not z. We may eliminate the references to these variables from the body of f by abstracting on an environment env, and replacing x and y by references to the environment. In compensation a suitable environment containing the bindings for x and y must be passed to f before it is applied. This leads to the following translation:

```
let val x = 1
    val y = 2
    val z = 3
    val f = (λenv.λw.(#x env) + (#y env) + w) {x=x, y=y}
in
    f 100
end
```

References to x and y in the body of f are replaced by projections (field selections) #x and #y that access the corresponding component of the environment. Since the code for f is closed, it may be hoisted out of the enclosing definition and defined at the top-level. We ignore this "hoisting" phase and instead concentrate on the process of closure conversion.

In the preceding example the environment contains bindings only for x and y, and is thus as small as possible. Since the body of f *could* contain an occurrence of z, it is also sensible to include z in the environment, resulting in the following code:

```
let val x = 1
    val y = 2
    val z = 3
    val f = (λenv.λw.(#x env) + (#y env) + w) {x=x, y=y, z=z}
in
    f 100
end
```

In the above example we chose a "flat" (FAM-like [6]) representation of the environment as a record with one field for each variable. Alternatively we could choose a "linked" (CAM-like [7]) representation in which, for example, each binding is a separate "frame" attached to the front of the remaining bindings. This idea leads to the following translation:

```
let val x = 1
    val y = 2
    val z = 3
    val f = (λenv.λw.(#x(#link(#link env))) + (#y(#link env)) + w)
              {z=z, link={y=y, link={x=x}}}
in
    f 100
end
```

The linked representation facilitates sharing of environments, but at the expense of introducing link traversals proportional to the nesting depth of the variable in the environment. The linked representation can also support constant-time closure creation, but this requires re-using the current environment and can result in bindings in the environment for variables that do not occur free in the function (such as z above), leading to space leaks.

These simple translations fail to delay the application of the code to its environment under call-by-value evaluation. A natural representation of a delayed application or closure is an ordered pair (code, env) consisting of the code together with its environment. Application of a closure to an argument proceeds by projecting the code part from the closure and applying it simultaneously to the environment and the argument according to some calling convention. For example:

```
let val x = 1
    val y = 2
    val z = 3
    val code = λenv.λw.#x(env) + #y(env) + w
    val env = {x=x, y=y}
    val f = (code, env)
in
    (#1 f) (#2 f) 100
end
```

But since code has a type of the form $\tau_{\rm ve} \to \tau_1 \to \tau_2$, where $\tau_{\rm ve}$ is the type of the environment env, the closure as a whole would have type $(\tau_{\rm ve} \to \tau_1 \to \tau_2) \times \tau_{\rm ve}$, showing the type of the environment explicitly. That violates the "privacy" of the environment representation. As a result, using this translation on a well-typed source program will not, in general, result in a well-typed target program. For example, consider the following ML source program with type int $\to$ int:

```
let val y = 1
in
    if true then
        λx.x+y
    else
        λz.z
end
```

Performing the translation above yields:

$$\frac{\Delta;\Gamma \vdash e : \sigma[\tau/t]}{\Delta;\Gamma \vdash \texttt{pack}\ \tau\ \texttt{with}\ e\ \texttt{as}\ \exists t.\sigma} \qquad \frac{\Delta;\Gamma \vdash e_1 : \exists t.\sigma' \quad \Delta \uplus \{t\};\Gamma \uplus \{x{:}\sigma'\} \vdash e_2 : \sigma}{\Delta;\Gamma \vdash \texttt{open}\ e_1\ \texttt{as}\ t\ \texttt{with}\ x\ \texttt{in}\ e_2 : \sigma}\ (t \notin FTV(\sigma), t \notin \Delta)$$

Figure 1: Typing Rules for Existentials

```
let val y = 1
in
    if true then
       (λenv. λx. x + #y(e), {y=y})
    else
       (λenv. λz. z, {})
end
```

This program fails to type-check because the **then**-arm of the **if**-expression has type $(\{\texttt{y:int}\} \to \texttt{int} \to \texttt{int}) \times \{\texttt{y:int}\}$ while the **else**-arm has type $(\{\} \to \texttt{int} \to \texttt{int}) \times \{\}$.

In order to preserve types in the target language, the representation of the environment may be hidden using existential types [23]. Figure 1 gives the typing rules for existentials. A **pack** operation pairs a type τ with a value e as an existential, holding τ abstract as a type variable, t. An **open** operation takes a package e_1 and opens it, binding the abstract type to t and the value of the package to x within the scope of e_2. The abstract type t is constrained so that it cannot leave the scope of the **open** construct, hence the restriction that t not appear in the free type variables of σ.

Using **pack**, we can hide the type of the environment for a closure value as follows:

$$\texttt{pack}\ \tau_{\text{ve}}\ \texttt{with}\ (\texttt{code}, \texttt{env})\ \texttt{as}\ \exists t_{\text{ve}}.(t_{\text{ve}} \to \tau_1 \to \tau_2) \times t_{\text{ve}}.$$

A closure of type $\tau_1 \to \tau_2$ is represented as a package of type $\exists t_{\text{ve}}.(t_{\text{ve}} \to \tau_1 \to \tau_2) \times t_{\text{ve}}$ where the type of the environment (τ_{ve}) is held abstract as t_{ve}. Under this translation, the example above would be translated to:

```
let val y = 1
in
  if true then
    pack {y:int} with (λenv. λx. x+#y(env),{y=y})
    as ∃t_ve.(t_ve → int → int) × t_ve
  else
    pack {} with (λenv. λz. z, {})
    as ∃t_ve.(t_ve → int → int) × t_ve
end
```

Since the types of the arms of the **if**-expression agree, the target code is well-typed with type $\exists t_{\text{ve}}.(t_{\text{ve}} \to \texttt{int} \to \texttt{int}) \times t_{\text{ve}}$.

An application `e e'` is correspondingly translated to the expression

```
open e as $t_{ve}$ with z : $(t_{ve} \to \tau_1 \to \tau_2) \times t_{ve}$
in
    (#1 z) (#2 z) e'
end
```

which opens the package, extracts the code and environment, and applies the code to the environment and the argument.

This representation of closures bears a striking resemblance to the model of objects suggested by Pierce and Turner [27]. In their model an object has a type of the form $\exists t.t \times \tau[t]$, where t is the type of the instance variable(s) and $\tau[t]$ is the type of the method(s). According to the foregoing account, closures may be thought of as objects with one instance variable (the environment) and one method (the code).

3 A Formal Account of Simply-Typed Closure Conversion

In this section we present the details of closure conversion for the call-by-value, simply-typed λ-calculus. The conversion is described in increasing detail by three stages: The first stage, *abstract closure conversion*, converts each function to a closure but holds the representation of the closure abstract. To simplify the presentation, some freedom is allowed in the construction of environments, but no shared environments are used. The second stage, *environment sharing*, adds more structure to the translation thereby allowing environments to be shared. The third stage, *closure representation*, makes the representation of closures explicit through the use of translucent sums. Each stage is defined as a type-directed translation and the correctness of the translations is established using logical relations.

The syntax of the source language is defined as follows:

$$\begin{array}{lll}
\textit{Types} & \tau ::= b \mid \tau_1 \to \tau_2 \\
\textit{Expressions} & e ::= c \mid x \mid \lambda x{:}\tau.\, e \mid e_1\, e_2 \\
\textit{Values} & v ::= c \mid \lambda x{:}\tau.\, e
\end{array}$$

Types (τ) consist of base types (b) and function types ($\to$)[1]. Expressions (e) consist of constants (c) of base type, variables, abstractions, and applications. We use Γ to denote a *sequence* of type bindings of the form $\{x_1{:}\tau_1, \ldots, x_n{:}\tau_n\}$, $(n \geq 0)$ where the x_i are distinct. The judgement $\Gamma \vdash e : \tau$ asserts that the expression e has type τ under the type assignment Γ, and is derived from the standard typing rules of the simply-typed λ-calculus. We define the dynamic semantics of the language using a judgement of the form $e \hookrightarrow v$ (e evaluates to v). The judgement is derived from the following standard inference rules for call-by-value evaluation:

$$v \hookrightarrow v \qquad\qquad \frac{e_1 \hookrightarrow \lambda x{:}\tau_1.\, e \quad e_2 \hookrightarrow v_2 \quad e[v_2/x] \hookrightarrow v}{e_1\, e_2 \hookrightarrow v}$$

3.1 Abstract Closure Conversion

The target language for abstract closure conversion, λ^{cl} , is defined as follows:

$$\begin{array}{lll}
\textit{Types} & \tau ::= b \mid \tau_1 \to \tau_2 \mid \langle \tau_1 \times \ldots \times \tau_n \rangle \mid \mathsf{code}(\tau_{ve}, \tau_1, \tau_2) \\
\textit{Expressions} & e ::= c \mid x \mid e_1\, e_2 \mid \langle e_1, \ldots, e_n \rangle \mid \pi_i(e) \mid \lambda x_{ve}{:}\tau_{ve}.\, \lambda x{:}\tau_1.\, e \mid \langle\!\langle e_1, e_2 \rangle\!\rangle \\
\textit{Values} & v ::= c \mid \lambda x_{ve}{:}\tau_{ve}.\, \lambda x{:}\tau_1.\, e \mid \langle v_1, \ldots, v_n \rangle \mid \langle\!\langle v_1, v_2 \rangle\!\rangle
\end{array}$$

[1]The results of this paper easily extended to other source types including products and sums.

In the introduction we informally presented closures as partial applications. As noted, we wish to delay this partial application until the closure is applied to an argument, so that the code and environment remain separate and the code can be shared among each instantiation of the closure. Therefore, in this account of closure conversion, we represent the delayed partial application as an abstract closure of the form $\langle\!\langle e, e_{\text{ve}} \rangle\!\rangle$ where e is the code and e_{ve} is the environment. This allows us to distinguish between delayed partial applications (closures) and closure application ($e_1\ e_2$). Code expressions, $\lambda x_{\text{ve}}{:}\tau_{\text{ve}}.\lambda x{:}\tau_1.e$, are a restricted form of closed λ-expressions that abstract both an environment (x_{ve}) and an argument (x). The types of code expressions are also distinguished from the types of closures and are written as $\mathsf{code}(\tau_{\text{ve}}, \tau_1, \tau_2)$ where τ_{ve}, τ_1, and τ_2 are the types of the environment, the argument, and the return value respectively.

The typing rules for λ^{cl} are standard except for code and closures, which are defined as follows:

$$\frac{\{x_{\text{ve}}{:}\tau_{\text{ve}}, x{:}\tau_1\} \vdash e : \tau_2}{\Gamma \vdash \lambda x_{\text{ve}}{:}\tau_{\text{ve}}.\lambda x{:}\tau_1.e : \mathsf{code}(\tau_{\text{ve}}, \tau_1, \tau_2)} \qquad \frac{\Gamma \vdash e : \mathsf{code}(\tau_{\text{ve}}, \tau_1, \tau_2) \quad \Gamma \vdash e_{\text{ve}} : \tau_{\text{ve}}}{\Gamma \vdash \langle\!\langle e, e_{\text{ve}} \rangle\!\rangle : \tau_1 \to \tau_2}$$

Since we require code to be closed in order that it may be hoisted to the top level, only $x_{\text{ve}}{:}\tau_{\text{ve}}$ (the environment) and $x{:}\tau_1$ (the argument) can be assumed when typing the body of the code. A closure consisting of code of type $\mathsf{code}(\tau_{\text{ve}}, \tau_1, \tau_2)$ and an environment of type τ_{ve} has type $\tau_1 \to \tau_2$, corresponding directly to the typing of the partial application of the code to its environment. Tuple types $\langle \tau_1 \times \ldots \times \tau_n \rangle$ and tuples $\langle e_1, \ldots, e_n \rangle$ are introduced to represent environments of closures.

Evaluation of the language is defined using the following inference rules which allow us to conclude a judgement of the form $e \hookrightarrow v$.

$$v \hookrightarrow v \qquad \frac{e_1 \hookrightarrow v_1 \quad \ldots \quad e_n \hookrightarrow v_n}{\langle e_1, \ldots, e_n \rangle \hookrightarrow \langle v_1, \ldots, v_n \rangle} \qquad \frac{e \hookrightarrow \langle v_1, \ldots, v_n \rangle}{\pi_i(e) \hookrightarrow v_i}$$

$$\frac{e_1 \hookrightarrow v_1 \quad e_2 \hookrightarrow v_2}{\langle\!\langle e_1, e_2 \rangle\!\rangle \hookrightarrow \langle\!\langle v_1, v_2 \rangle\!\rangle} \qquad \frac{e_1 \hookrightarrow \langle\!\langle \lambda x_{\text{ve}}{:}\tau_{\text{ve}}.\lambda x{:}\tau_1.e, v_{\text{ve}} \rangle\!\rangle \quad e_2 \hookrightarrow v_2 \quad e[v_{\text{ve}}/x_{\text{ve}}, v_2/x] \hookrightarrow v}{e_1\ e_2 \hookrightarrow v}$$

When a closure is applied to an argument, the environment and the argument are substituted for the corresponding variables and the body of the code is evaluated.

We define abstract closure conversion as a type-directed translation from the source language to λ^{cl} in Figure 2. The translation is formulated as a deductive system with judgements of the form $\Gamma; x{:}\tau \rhd e \leadsto e'$, and $\Gamma; x{:}\tau \rhd \Gamma' \leadsto e'_{\text{ve}}$ where Γ and Γ' are source type assignments, τ is a source type, e is a source expression, and e' and e'_{ve} are target expressions. The variable x is considered as the current argument while the other free variables in a source expression should be in Γ and accessed through the current environment in the translation. The judgement $\Gamma; x{:}\tau \rhd e \leadsto e'$ asserts that e' is the translation of e under the assumption that $\Gamma \uplus \{x{:}\tau\} \vdash e : \tau'$ for some τ'. The judgement $\Gamma; x{:}\tau \rhd \Gamma' \leadsto e'_{\text{ve}}$ asserts that e'_{ve} is an expression that evaluates to the environment corresponding to Γ' under the assumption that each binding in Γ' occurs in $\Gamma \uplus \{x{:}\tau\}$. Note that the order of bindings in Γ is important, and thus it is considered to be a sequence and not a set.

In a translated expression, x_{ve} is always used to hold the current local environment. Consequently, the translation rule (*env*) maps a source variable x_i found in the i^{th} position of type assignment Γ to the i^{th} projection of the environment variable x_{ve}, while the rule (*arg*) translates the argument variable x to itself.

The translation of an abstraction produces a closure consisting of code and an environment. To construct the environment, we choose a type assignment Γ' such that $\Gamma; x'{:}\tau' \rhd \Gamma' \leadsto e_{\text{ve}}$ is derivable via the (*context*) rule and $\Gamma'; x{:}\tau_1 \rhd e \leadsto e'$. These two constraints can be summarized by saying that every binding in Γ' can also be found in $\Gamma \uplus \{x'{:}\tau'\}$. In a more detailed formulation, Γ' would be

$$(\mathit{const})\ \Gamma; x{:}\tau \rhd c \leadsto c \quad (\mathit{arg})\ \Gamma; x{:}\tau \rhd x \leadsto x \quad (\mathit{env})\ \{x_1{:}\tau_1, \ldots, x_n{:}\tau_n\}; x{:}\tau \rhd x_i \leadsto \pi_i(x_{\mathrm{ve}})$$

$$(\mathit{abs})\ \frac{\Gamma; x'{:}\tau' \rhd \Gamma' \leadsto e_{\mathrm{ve}} \quad \Gamma'; x{:}\tau \rhd e \leadsto e'}{\Gamma; x'{:}\tau' \rhd \lambda x{:}\tau.e \leadsto \langle\!\langle \lambda x_{\mathrm{ve}}{:}|\Gamma'|.\, \lambda x{:}\tau_1.\, e', e_{\mathrm{ve}} \rangle\!\rangle} \quad (\mathit{app})\ \frac{\Gamma; x{:}\tau \rhd e_1 \leadsto e_1' \quad \Gamma; x{:}\tau \rhd e_2 \leadsto e_2'}{\Gamma; x{:}\tau \rhd e_1\ e_2 \leadsto e_1'\ e_2'}$$

$$(\mathit{context})\ \frac{\Gamma; x{:}\tau \rhd x_1 \leadsto e_1 \quad \ldots \quad \Gamma; x{:}\tau \rhd x_n \leadsto e_n}{\Gamma; x{:}\tau \rhd \{x_1{:}\tau_1, \ldots, x_n{:}\tau_n\} \leadsto \langle e_1, \ldots, e_n \rangle} \quad (\Gamma \uplus \{x{:}\tau\} \vdash x_i : \tau_i)$$

Figure 2: Simply-Typed Abstract Closure Conversion

obtained from $\Gamma \uplus \{x'{:}\tau'\}$ via the application of *strengthening* and *exchange* rules. Furthermore, Γ' is required to contain bindings for all of the free variables in the original function $\lambda x{:}\tau_1.\, e$. However, Γ' may also contain bindings from $\Gamma \uplus \{x'{:}\tau'\}$ that do not occur free in the function. Therefore, there are many choices for Γ' and we can chose it so as to minimize the running time and/or space consumed by the target code. The environment itself is constructed via the (*context*) rule by translating each of the variables occurring in Γ' (namely $x_1, \cdots, x_n$) to the target expressions $e_1, \cdots, e_n$. The resulting expressions are placed in a tuple ($\langle e_1, \ldots, e_n \rangle$) to form the environment data structure of the closure. The environment has type $\langle \tau_1 \times \cdots \times \tau_n \rangle$ which we summarize by writing $|\Gamma'|$.

To produce the code of the closure, we translate the body of the source function under the strengthened assumptions $\Gamma'; x : \tau$, producing the body of the code, e', and then we abstract the environment and argument, resulting in $\lambda x_{\mathrm{ve}}{:}|\Gamma'|.\, \lambda x{:}\tau.\, e'$.

The derivation of a translation is very closely related to the typing derivation of the source program. In particular, by an examination of the translation rules and by virtue of the source language being explicitly typed, it is clear that if we have a derivation of $\Gamma; x{:}\tau' \rhd e \leadsto e'$, then there exists a unique τ such that we can construct a derivation of $\Gamma \uplus \{x{:}\tau'\} \vdash e : \tau$. Consequently, we can easily show that the translation preserves the type of a source program in the following sense:

Lemma 1 *If* $\Gamma \uplus \{x{:}\tau'\} \vdash e : \tau$ *and* $\Gamma; x{:}\tau' \rhd e \leadsto e'$, *then* $\{x_{\mathrm{ve}} : |\Gamma|, x{:}\tau'\} \vdash e' : \tau$.

Proof. By induction on the derivation of $\Gamma; x{:}\tau' \rhd e \leadsto e'$.

Case (*arg*). $\Gamma; x{:}\tau' \rhd x \leadsto x$ and $\{x_{\mathrm{ve}}{:}|\Gamma|, x{:}\tau'\} \vdash x : \tau'$.

Case (*env*). Let Γ be $\{x_1{:}\tau_1, \ldots, x_n{:}\tau_n\}$. Then $\Gamma \uplus \{x{:}\tau'\} \vdash x_i : \tau_i$. Further, $\{x_{\mathrm{ve}}{:}|\Gamma|, x{:}\tau'\} \vdash x_{\mathrm{ve}} : \langle \tau_1 \times \ldots \times \tau_n \rangle$. Hence, $\{x_{\mathrm{ve}}{:}|\Gamma|, x{:}\tau'\} \vdash \pi_i(x_{\mathrm{ve}}) : \tau_i$.

Case (*abs*). By the second induction hypothesis, $\{x_{\mathrm{ve}}{:}|\Gamma'|, x{:}\tau_1\} \vdash e' : \tau_2$. Thus $\{x_{\mathrm{ve}}{:}|\Gamma|, x'{:}\tau'\} \vdash \lambda x_{\mathrm{ve}}{:}|\Gamma'|.\lambda x{:}\tau_1.e' : \mathtt{code}(|\Gamma'|, \tau_1, \tau_2)$. By the construction of e_{ve}, it is clear that $\Gamma \vdash e_{\mathrm{ve}} : |\Gamma'|$. So by the typing rule for closures, $\{x_{\mathrm{ve}}{:}|\Gamma|, x'{:}\tau'\} \vdash \langle\!\langle \lambda x_{\mathrm{ve}}{:}|\Gamma'|.\lambda x{:}\tau.e', e_{\mathrm{ve}} \rangle\!\rangle : \tau_1 \rightarrow \tau_2$.

Case (*app*). We know that $\Gamma; x{:}\tau' \rhd e_1 \leadsto e_1'$ and for some τ_1, $\Gamma \uplus \{x{:}\tau'\} \vdash e_1 : \tau_1 \rightarrow \tau$, and by induction this implies $\{x_{\mathrm{ve}}{:}|\Gamma|, x{:}\tau'\} \vdash e_1' : \tau_1 \rightarrow \tau$. We also know that $\Gamma; x{:}\tau' \rhd e_2 \leadsto e_2'$ and $\Gamma \uplus \{x{:}\tau'\} \vdash e_2 : \tau_1$, and by induction this implies $\{x_{\mathrm{ve}}{:}|\Gamma|, x{:}\tau'\} \vdash e_2' : \tau_1$. Hence, $\{x_{\mathrm{ve}}{:}|\Gamma|, x{:}\tau'\} \vdash e_1'\ e_2' : \tau$.

□

Using a dummy argument $(x{:}b)$ to translate an entire closed program, it is clear from the previous lemma that the translation preserves the program's type.

Theorem 1 *If* $\emptyset \vdash e{:}\tau$ *and* $\emptyset; x{:}b \rhd e \leadsto e'$, *then* $\emptyset \vdash e' : \tau$.

To prove the operational correctness of the translation, we use a type-indexed family of logical relations relating closed source expressions to closed target expressions ($\sim$) and closed source values to closed target values ($\approx$). The relations are defined by induction on source types as follows:

$$\begin{array}{rll} e \sim_\tau e' & \text{iff} & e \hookrightarrow v \text{ and } e' \hookrightarrow v' \text{ and } v \approx_\tau v' \\ c \approx_b c & & \\ v \approx_{\tau_1 \to \tau_2} v' & \text{iff} & \text{for all } v_1 \approx_{\tau_1} v_1',\ v\ v_1 \sim_{\tau_2} v'\ v_1'. \end{array}$$

We extend the relation to finite source (γ) and target substitutions (γ') mapping variables to their respective class of values. These relations are defined as follows:

$$\begin{array}{rll} \gamma \approx_{\{x_1:\tau_1,\ldots,x_n:\tau_n\}} [\langle v_1,\ldots,v_n\rangle / x_{\text{ve}}] & \text{iff} & \gamma(x_i) \approx_{\tau_i} v_i \text{ for } 1 \leq i \leq n. \\ \gamma \approx_{\Gamma;x:\tau} [v'/x_{\text{ve}}, v/x] & \text{iff} & \gamma \approx_\Gamma [v'/x_{\text{ve}}] \text{ and } \gamma(x) \approx_\tau v. \end{array}$$

The following lemma shows that the translation of a variable and an environment evaluates to the corresponding value.

Lemma 2 *Let* $\gamma \approx_{\Gamma;x:\tau} \gamma'$.

1. *If* $\Gamma; x'{:}\tau' \vdash x{:}\tau$ *and* $\Gamma; x'{:}\tau' \rhd x \leadsto e_0$, *then* $\gamma'(e_0) \hookrightarrow v$ *and* $\gamma(x) \approx_\tau v$.

2. *If* $\Gamma; x{:}\tau \rhd \Gamma' \leadsto e_{\text{ve}}$, *then* $\gamma'(e_{\text{ve}}) \hookrightarrow v$ *and* $\gamma \approx_{\Gamma'} [v/x_{\text{ve}}]$.

Proof. Claim 2 is clear from 1. So we only present the proof for 1.

Case (*arg*). It is clear that $\tau \equiv \tau'$. By definition, $\gamma(x') \approx_{\tau'} \gamma'(x')$. Hence, $\gamma(x') \sim_{\tau'} \gamma'(x')$.

Case (*env*). Let Γ be $\{x_1{:}\tau_1, \ldots, x_n{:}\tau_n\}$. $\gamma'(\pi_i(x_{\text{ve}})) \equiv \pi_i \langle v_1, \ldots, v_n\rangle \hookrightarrow v_i$. By definition, $\gamma(x_i) \approx_{\tau_i} v_i$.

□

With this lemma in hand, we can establish the correctness of the translation.

Theorem 2 (Operational Correctness) *Let* $\gamma \approx_{\Gamma;x':\tau'} \gamma'$. *If* $\Gamma \uplus \{x'{:}\tau'\} \vdash e : \tau$ *and* $\Gamma; x'{:}\tau' \rhd e \leadsto e'$, *then* $\gamma(e) \sim_\tau \gamma'(e')$.

Proof. By induction on the derivation of $\Gamma; x'{:}\tau' \rhd e \leadsto e'$.

Case (*arg*) and (*env*). Clear from Lemma 2.

Case (*abs*). Let $v \approx_{\tau_1} v'$. By the first induction hypothesis, we know that $\Gamma; x'{:}\tau' \rhd \Gamma' \leadsto e_{\text{ve}}$. It is easy to show that for some v_{ve}, $\gamma'(e_{\text{ve}}) \hookrightarrow v_{\text{ve}}$. By Lemma 2, $\gamma[v/x] \approx_{\Gamma';x:\tau_1} [v_{\text{ve}}/x_{\text{ve}}, v'/x]$. By the second induction hypothesis, $\gamma[v/x]e \sim_{\tau_2} [v_{\text{ve}}/x_{\text{ve}}, v'/x]e'$. Thus $\gamma(\lambda x{:}\tau_1.e) \sim_{\tau_1 \to \tau_2} \gamma'(\langle\!\langle \lambda x_{\text{ve}}{:}|\Gamma'|.\lambda x{:}\tau_1.e', e_{\text{ve}}\rangle\!\rangle)$.

Case (*app*). By the induction hypothesis, $\gamma(e_1) \hookrightarrow v_1$, $\gamma'(e'_1) \hookrightarrow v'_1$, and $v_1 \approx_{\tau_1 \to \tau_2} v'_1$ and $\gamma(e_2) \hookrightarrow v_2$, $\gamma'(e'_2) \hookrightarrow v'_2$, and $v_2 \approx_{\tau_1} v'_2$. Then by the definition of $\sim$, $v_1\ v_2 \sim_{\tau_2} v'_1\ v'_2$. Thus $\gamma(e_1\ e_2) \sim_{\tau_2} \gamma'(e'_1\ e'_2)$.

□

This theorem and the definition of the relations imply that for a closed program with a base type, the results of evaluation of the original program and its translation are the same. As a corollary, it is clear that various translations of a program have the same operational behavior.

Corollary 1 (Coherence) *If* $\emptyset \vdash e : b$ *and* $\emptyset; x{:}b \rhd e \rightsquigarrow e_1$ *and* $\emptyset; x{:}b \rhd e \rightsquigarrow e_2$*, then* $e_1 \hookrightarrow c$ *iff* $e_2 \hookrightarrow c$.

3.2 Sharing Environments

Some implementations of functional programming languages use environments with nested structures that may share some portions of the environment with other closures. Sharing environments decreases the amount of space consumed by a closure and decreases the time to construct the closure's environment. However, sharing can also require extra instructions to access a variable's binding in the environment. Furthermore, sharing environments naively can lead to space problems in the presence of a standard tracing garbage collector. In this section we extend our closure conversion to allow for but not require shared environments. We do so by adding extra structure to the typing contexts of the translation judgement and use this extra structure to guide the construction of [possibly] shared environments. We then show how the resulting translation subsumes a wide variety of environment representations used in practice.

In the previous section, translation judgements were of the form $\Gamma; x{:}\tau \rhd e \rightsquigarrow e'$ where Γ was a *flat* type assignment of the form $\{x_1{:}\tau_1, \ldots, x_n{:}\tau_n\}$. Here, we add extra structure to the translation judgement by using *nested* type assignments defined as follows:

$$\Theta ::= \{x{:}\tau\} \mid \langle \Theta_1, \ldots, \Theta_m \rangle$$

A nested type assignment is either a single type binding or a sequence of nested type assignments. The environment corresponding to the type assignment Θ is represented in the target language by type $|\Theta|$ where $|\{x{:}\tau\}| = \tau$ and $|\langle\Theta_1, \ldots, \Theta_m\rangle| = \langle |\Theta_1| \times \ldots \times |\Theta_m| \rangle$. Clearly, we can obtain a non-nested type assignment (Γ) from a nested type assignment (Θ) simply by dropping the extra structure. Hence, we consider Θ to represent a nested type assignment as well as its corresponding flat type assignment.

The relevant translation rules for closure conversion with nested environments are given in Figure 3. The other translation rules are the same as in Figure 2, replacing Γ with Θ.

The (*arg*) rule translates a nested type assignment consisting of only the current argument to the variable itself. The (*env*) rule gives us the current environment directly as x_{ve} allowing us to avoid creating a copy. This rule, coupled with the (*env-tuple*) rule allows us to construct shared environments as nested tuples. If we translate Θ to e under the type assignment Θ_i, then the (*subenv*) rule lets us translate Θ to $e[\pi_i(x_{\text{ve}})/x_{\text{ve}}]$ under a type assignment which contains Θ_i as the i^{th} component. Variable x is translated as a nested type assignment $\{x{:}\tau\}$ by (*var*).

As an example, in the following translation the current environment is reused to construct the environment of the closure:

$$\begin{aligned}&\langle x_1{:}int, x_2{:}int\rangle; x'{:}int \rhd (\lambda x{:}int.x' + x_1 + x_2) \rightsquigarrow \\ &\qquad \langle\!\langle \lambda x_{\text{ve}}{:}\tau.\lambda x{:}int.\pi_1(x_{\text{ve}}) + \pi_1(\pi_2(x_{\text{ve}})) + \pi_2(\pi_2(x_{\text{ve}})), \langle x', x_{\text{ve}}\rangle \rangle\!\rangle\end{aligned}$$

$$(arg)\quad \{x{:}\tau\}; x'{:}\tau' \rhd \{x'{:}\tau'\} \rightsquigarrow x' \qquad (env)\quad \Theta; x{:}\tau \rhd \Theta \rightsquigarrow x_{\mathrm{ve}} \qquad (var)\quad \frac{\Theta; x'{:}\tau' \rhd \{x{:}\tau\} \rightsquigarrow e}{\Theta; x'{:}\tau' \rhd x \rightsquigarrow e}$$

$$(subenv)\quad \frac{\Theta_i; x{:}\tau \rhd \Theta \rightsquigarrow e}{\langle \Theta_1, \ldots, \Theta_n \rangle; x{:}\tau \rhd \Theta \rightsquigarrow e[\pi_i(x_{\mathrm{ve}})/x_{\mathrm{ve}}]}$$

$$(env\text{-}tuple)\quad \frac{\Theta; x{:}\tau \rhd \Theta_1 \rightsquigarrow e_1 \quad \cdots \quad \Theta; x{:}\tau \rhd \Theta_n \rightsquigarrow e_n}{\Theta; x{:}\tau \rhd \langle \Theta_1, \ldots, \Theta_n \rangle \rightsquigarrow \langle e_1, \ldots, e_n \rangle}$$

Figure 3: Simply-Typed Closure Conversion using Nested Environments

where τ is $\langle int \times \langle int \times int \rangle \rangle$. The new environment for the closure is constructed by pairing the current argument, x', and the current environment, x_{ve}. By reusing a portion of an environment we can reduce the cost of creation of a closure. If we use the translation given in Figure 2, then constructing the new environment would require projecting the values for x_1 and x_2 out of the current environment and then these values and the current argument would need to be placed in a newly allocated tuple.

As with flat type assignments, it is easy to prove that a translation of a program using nested type assignments preserves the type of the program and the operational correctness of the translation may be proved by using logical relations.

Nested type assignments are flexible enough to represent various environment representations used in practice. For example, the Categorical Abstract Machine or CAM [7] uses linked lists to represent environments. This is reflected in our framework by restricting the shape of nested type assignments and by restricting the (*env-tuple*) rule to "cons" the current argument onto the current environment:

$$(CAM\ context)\quad \Theta_c ::= \{x{:}\tau\} \mid \langle \{x{:}\tau\}, \Theta_c \rangle$$
$$(env\text{-}tuple)\quad \Theta_c; x{:}\tau \rhd \langle x{:}\tau, \Theta_c \rangle \rightsquigarrow \langle x, x_{\mathrm{ve}} \rangle$$

The advantage of the CAM strategy is that the cost of the construction of a new environment is constant. However, in the worst case accessing values in the environment takes time proportional to the length of the environment.

In contrast, the FAM [6] uses flat environments with no sharing. The closure conversion of Figure 2 accurately models the environment strategy of the FAM if we choose a specific strengthening strategy in the (*abs*) rule where only the free variables of the function are preserved in the resulting closure's environment. The advantage of the FAM environment representation is that the cost of variable lookup is always constant and the representation is "safe for space" [3] according to Appel's definition. However, constructing the environment for a closure takes time proportional to the number of free variables in the function and closures cannot share portions of their environment.

Clearly, there are a variety of other strategies for forming environments. For example, the shared closure strategy described by Appel and Shao [31] that is also safe for space can also be formulated in our framework. However, to determine a good representation for each closure's environment requires a good deal more information including an estimate as to how many times each variable is accessed, when garbage collection can occur, what garbage collection algorithm is used, *etc.*

3.3 Closure Representation

Abstract closure conversion chooses an environment representation for each closure and makes the construction of closures explicit. We have shown how making the environment construction explicit facilitates a variety of strategies that attempt to minimize the space consumed and running time of the resulting program. Furthermore, by making environment construction explicit, we expose operations that are implicit at the source level to an optimizer at the target level. In particular, an optimizer might notice that the same environment is constructed in two places and replace the second construction with a reference to the first. However, abstract closure conversion makes the extraction of the code and environment in an application implicit in the operational semantics. Ideally, these extraction operations should be explicit so that an optimizer can eliminate redundant projections. For instance, if the same closure is repeatedly applied to some arguments in a loop, we should be able to extract the code and environment of the closure one time, name these values, and then use these names within the loop.

A first attempt at making the extraction of the code and environment explicit is to represent closures as pairs (*i.e.*, 2-tuples) in the target language and simply use projection (π). However, we argued in the introduction that this naive translation does not preserve types. The difficulty is that the environment's type is exposed in the translated type. Consequently, two expressions with the same source type will not in general have the same target types when translated. This problem is solved by hiding the type of the environment through the use of existential types [23], as Pierce and Turner did for objects [27].

We therefore define a target language $\lambda^{\exists}$ with existential types as follows:

$$\begin{array}{lll}
\textit{Types} & \tau ::= & b \mid t \mid \langle \tau_1 \times \ldots \times \tau_n \rangle \mid \texttt{code}(\tau_{\rm ve}, \tau_1, \tau_2) \mid \exists t.\tau \\
\textit{Exp's} & e ::= & c \mid e_1(e_2, e_3) \mid \lambda x_{\rm ve}{:}\tau_{\rm ve}.\lambda x{:}\tau_1.e \mid \langle e_1, \ldots, e_n \rangle \mid \pi_i(e) \mid \texttt{pack}\ \tau\ \texttt{with}\ e\ \texttt{as}\ \tau \mid \\
& & \texttt{open}\ e_1\ \texttt{as}\ t\ \texttt{with}\ x{:}\tau\ \texttt{in}\ e_2 \\
\textit{Values} & v ::= & c \mid \lambda x_{\rm ve}{:}\tau_{\rm ve}.\lambda x{:}\tau_1.e \mid \langle v_1, \ldots, v_n \rangle \mid \texttt{pack}\ \tau_1\ \texttt{with}\ v\ \texttt{as}\ \tau_2
\end{array}$$

This language is the same as λ^{cl} except for the addition of package values of type $\exists t.\tau$ that pair an abstract type (t) with a value of type τ. Function types ($\rightarrow$) are no longer necessary because they can be represented using other type constructors (namely $\exists$ and $\texttt{code}(\tau_{\rm ve}, \tau_1, \tau_2)$). In order to prevent the partial application of the code to its environment, we restrict applications to the form $e_1(e_2, e_3)$.

Typing judgements for $\lambda^{\exists}$ are of the form $\Delta; \Gamma \vdash e : \tau$ where Δ is a list of type variables and Γ is a type assignment. We assume that the free type variables of Γ and the free type variables of e and τ are contained in Δ. The typing rules for $\lambda^{\exists}$ are similar to the rules for λ^{cl} except for the introduction and elimination-rules for existentials:

$$\frac{\Delta; \Gamma \vdash e : \tau[\tau'/t]}{\Delta; \Gamma \vdash \texttt{pack}\ \tau'\ \texttt{with}\ e\ \texttt{as}\ \exists t.\tau : \exists t.\tau}$$

$$\frac{\Delta; \Gamma \vdash e : \exists t.\tau \quad \Delta \uplus \{t\}; \Gamma \uplus \{x{:}\tau\} \vdash e' : \tau' \quad (t \notin FTV(\tau'))}{\Delta; \Gamma \vdash \texttt{open}\ e\ \texttt{as}\ t\ \texttt{with}\ x{:}\tau\ \texttt{in}\ e' : \tau'}$$

Similarly, the operational semantics of $\lambda^{\exists}$ is the same as for λ^{cl} except for rules involving existentials and application:

$$\frac{e \hookrightarrow v}{\texttt{pack}\ \tau\ \texttt{with}\ e\ \texttt{as}\ \tau' \hookrightarrow \texttt{pack}\ \tau\ \texttt{with}\ v\ \texttt{as}\ \tau'}$$

$$\frac{e_1 \hookrightarrow \texttt{pack}\ \tau_1\ \texttt{with}\ v\ \texttt{as}\ \tau' \quad e_2[\tau_1/t][v/x] \hookrightarrow v}{\texttt{open}\ e_1\ \texttt{as}\ t\ \texttt{with}\ x{:}\tau\ \texttt{in}\ e_2 \hookrightarrow v}$$

$$(var)\quad \Gamma \rhd x : \tau \leadsto x \quad (x{:}\tau \in \Gamma) \qquad\qquad (const)\quad \Gamma \rhd c : b \leadsto c$$

$$(proj)\ \frac{\Gamma \rhd e : \langle \tau_1, \ldots, \tau_n \rangle \leadsto e'}{\Gamma \rhd \pi_i(e) : \tau_i \leadsto \pi_i(e')} \qquad (tuple)\ \frac{\Gamma \rhd e_i : \tau_i \leadsto e'_i}{\Gamma \rhd \langle e_1, \ldots, e_n \rangle : \langle \tau_1, \ldots, \tau_n \rangle \leadsto \langle e'_1, \ldots, e'_n \rangle}$$

$$(code)\ \frac{\{x_{\mathrm{ve}}{:}\tau_{\mathrm{ve}}, x{:}\tau_1\} \rhd e \leadsto e'}{\Gamma \rhd \lambda x_{\mathrm{ve}}{:}\tau_{\mathrm{ve}}.\lambda x{:}\tau_1.e \leadsto \lambda x_{\mathrm{ve}}{:}|\tau_{\mathrm{ve}}|.\lambda x{:}|\tau_1|.e'}$$

$$(closure)\ \frac{\Gamma \rhd e : \mathtt{code}(\tau_{\mathrm{ve}}, \tau_1, \tau_2) \leadsto e' \qquad \Gamma \rhd e_{\mathrm{ve}} : \tau_{\mathrm{ve}} \leadsto e'_{\mathrm{ve}}}{\Gamma \rhd \langle\!\langle e, e_{\mathrm{ve}} \rangle\!\rangle : \tau_1 \to \tau_2 \leadsto \mathtt{pack}\ |\tau_{\mathrm{ve}}|\ \mathtt{with}\ \langle e', e'_{\mathrm{ve}} \rangle\ \mathtt{as}\ |\tau_1 \to \tau_2|}$$

$$(app)\ \frac{\Gamma \rhd e_1 : \tau_1 \to \tau_2 \leadsto e'_1 \qquad \Gamma \rhd e_2 : \tau_1 \leadsto e'_2}{\begin{array}{c}\Gamma \rhd e_1 e_2 : \tau_2 \leadsto \\ \mathtt{open}\ e'_1\ \mathtt{as}\ t_{\mathrm{ve}}\ \mathtt{with}\ x{:}\langle \mathtt{code}(t_{\mathrm{ve}}, |\tau_1|, |\tau_2|) \times t_{\mathrm{ve}} \rangle\ \mathtt{in}\ (\pi_1 x)(\pi_2 x, e'_2)\end{array}} \quad (x \notin Dom(\Gamma))$$

Figure 4: Simply-Typed Closure Representation

$$\frac{e_1 \hookrightarrow (\lambda y{:}\tau_{\mathrm{ve}}.\lambda x{:}\tau_1.e) \quad e_2 \hookrightarrow v_2 \quad e_3 \hookrightarrow v_3 \quad e[v_2/y][v_3/x] \hookrightarrow v}{e_1(e_2, e_3) \hookrightarrow v}$$

We begin by defining a translation from λ^{cl} to $\lambda^{\exists}$ types, denoted $|\tau|$ and defined as follows:

$$\begin{array}{rcl}
|b| & = & b \\
|\langle \tau_1 \times \ldots \times \tau_n \rangle| & = & \langle |\tau_1| \times \ldots \times |\tau_n| \rangle \\
|\mathtt{code}(\tau_{\mathrm{ve}}, \tau_1, \tau_2)| & = & \mathtt{code}(|\tau_{\mathrm{ve}}|, |\tau_1|, |\tau_2|) \\
|\tau_1 \to \tau_2| & = & \exists t_{\mathrm{ve}}.\langle \mathtt{code}(t_{\mathrm{ve}}, |\tau_1|, |\tau_2|) \times t_{\mathrm{ve}} \rangle.
\end{array}$$

The translation of an arrow type is a pair consisting of code and an environment, with the environment type (t_{ve}) held abstract using an existential.

The translation mapping λ^{cl} terms to $\lambda^{\exists}$ terms is summarized in Figure 4. The translation defines judgements of the form $\Gamma \rhd e : \tau \leadsto e'$ where Γ, e, and τ are a λ^{cl} type assignment, expression, and type respectively, and e' is a $\lambda^{\exists}$ expression. The interesting rules are (*closure*) and (*app*). The other rules simply map the other λ^{cl} constructs to their $\lambda^{\exists}$ counterparts. A closure is translated to a pair of the code and the environment packed with the type of the environment. The translation of an application extracts from a package the pair of a code and an environment and applies the code to the environment and the argument.

It is easy prove that the translation preserves the type of a program up to the translation of the type. We do so by first extending the type translation to type assignments, writing:

$$|\{x_1{:}\tau_1, \ldots, x_n{:}\tau_n\}| = \{x_1{:}|\tau_1|, \ldots, x_n{:}|\tau_n|\}$$

Theorem 3 *If* $\Gamma \vdash e : \tau$ *and* $\Gamma \rhd e : \tau \leadsto e'$, *then* $\emptyset; |\Gamma| \vdash e' : |\tau|$.

Proof. By induction on the derivation of $\Gamma \vdash e : \tau \leadsto e'$.

□

$$
\begin{array}{rll}
e \sim_\tau e' & \text{iff} & e \hookrightarrow v \text{ and } e \hookrightarrow v' \text{ and } v \approx_\tau v'. \\
\\
c \approx_b c & & \\
v \approx_{\mathtt{code}(\tau_{\mathrm{ve}},\tau_1,\tau_2)} v' & \text{iff} & \text{for all } v_{\mathrm{ve}} \approx_{\tau_{\mathrm{ve}}} v'_{\mathrm{ve}} \text{ and } v_1 \approx_{\tau_1} v'_1, \\
 & & \quad \langle\!\langle v, v_0\rangle\!\rangle v_1 \approx_{\tau_2} v'(v'_0, v'_1) \\
v \approx_{\tau_1 \to \tau_2} v' & \text{iff} & \text{for all } v_1 \approx_{\tau_1} v'_1, \\
 & & \quad v\ v_1 \sim_{\tau_2} \mathtt{open}\ v'\ \mathtt{as}\ t_{\mathrm{ve}}\ \mathtt{with}\ x{:}\tau\ \mathtt{in}\ (\pi_1 x)(\pi_2(x), v'_1) \\
 & & \quad \text{where } \tau \equiv \langle(\mathtt{code}(t_{\mathrm{ve}}, \tau_1, \tau_2)) \times t_{\mathrm{ve}}\rangle \\
\langle v_1, \ldots, v_n\rangle \approx_{(\tau_1 \times \ldots \times \tau_n)} \langle v_1, \ldots, v'_n\rangle & \text{iff} & \text{for all } 1 \leq i \leq n,\ v_i \approx_{\tau_i} v'_i. \\
\\
\gamma \approx_\Gamma \gamma' & \text{iff} & \text{for all } x{:}\tau \in \Gamma, \gamma(x) \approx_\tau \gamma'(x)
\end{array}
$$

Figure 5: Logical Relations for Simply-Typed Closure Representation

Operational correctness of the translation is proven using logical relations between λ^{cl} and $\lambda^{\exists}$ expressions, λ^{cl} and $\lambda^{\exists}$ values, and λ^{cl} and $\lambda^{\exists}$ substitutions. The relations are defined in Figure 5.

Theorem 4 (Operational Correctness) *Let* $\gamma \approx_\Gamma \gamma'$. *If* $\Gamma \vdash e : \tau$ *and* $\Gamma \rhd e : \tau \rightsquigarrow e'$, *then* $\gamma(e) \sim_\tau \gamma'(e')$.

Proof. By induction on the derivation of $\Gamma \rhd e : \tau \rightsquigarrow e'$.

Case (*var*). Let $\Gamma \rhd x : \tau \rightsquigarrow x$. From $\gamma \approx_\Gamma \gamma'$, $\gamma(x) \sim_\tau \gamma'(x)$.

Case (*closure*). Let $\Gamma \rhd \langle\!\langle e, e_{\mathrm{ve}}\rangle\!\rangle : \tau_1 \to \tau_2 \rightsquigarrow \mathtt{pack}\ \tau_{\mathrm{ve}}\ \mathtt{with}\ \langle e', e'_{\mathrm{ve}}\rangle\ \mathtt{as}\ |\tau_1 \to \tau_2|$. By the induction hypotheses, $\gamma(e) \sim_{\mathtt{code}(\tau_{\mathrm{ve}},\tau_1,\tau_2)} \gamma'(e')$ and $\gamma(e_{\mathrm{ve}}) \sim_{\tau_{\mathrm{ve}}} \gamma'(e'_{\mathrm{ve}})$. Thus $\gamma(e) \hookrightarrow v$, $\gamma'(e') \hookrightarrow v'$, and $v \sim_{\mathtt{code}(\tau_{\mathrm{ve}},\tau_1,\tau_2)} v'$ and $\gamma(e_{\mathrm{ve}}) \hookrightarrow v_{\mathrm{ve}}$, $\gamma'(e'_{\mathrm{ve}}) \hookrightarrow v'_{\mathrm{ve}}$, and $v_{\mathrm{ve}} \sim_{\tau_{\mathrm{ve}}} v'_{\mathrm{ve}}$. Let $v_1 \approx_{\tau_1} v'_1$. Then $\langle\!\langle v, v_{\mathrm{ve}}\rangle\!\rangle\ v_1 \sim_{\tau_2} v'(v'_{\mathrm{ve}}, v'_1)$. On the other hand, $\mathtt{open}\ \gamma'(\mathtt{pack}\ \tau_{\mathrm{ve}}\ \mathtt{with}\ \langle e', e'_{\mathrm{ve}}\rangle\ \mathtt{as}\ |\tau_1 \to \tau_2|)\ \mathtt{as}\ t_{\mathrm{ve}}\ \mathtt{with}\ x{:}\langle(t_{\mathrm{ve}} \to \tau_1 \to \tau_2) \times t_{\mathrm{ve}}\rangle\ \mathtt{in}\ (\pi_1 x)(\pi_2 x, v'_1) \hookrightarrow v''$ iff $v'(v'_{\mathrm{ve}}, v'_1) \hookrightarrow v''$. Hence $\gamma\langle\!\langle e, e_{\mathrm{ve}}\rangle\!\rangle \sim_{\tau_1 \to \tau_2} \gamma'(\mathtt{pack}\ \tau_{\mathrm{ve}}\ \mathtt{with}\ \langle e', e'_{\mathrm{ve}}\rangle\ \mathtt{as}\ |\tau_1 \to \tau_2|)$.

Case (*code*). Let $\Gamma \rhd (\lambda x_{\mathrm{ve}}{:}\tau_{\mathrm{ve}}.\lambda x{:}\tau_1.e) : \mathtt{code}(\tau_{\mathrm{ve}}, \tau_1, \tau_2) \rightsquigarrow \lambda x_{\mathrm{ve}} : |\tau_{\mathrm{ve}}|.\lambda x{:}|\tau_1|.e'$ and let $v_{\mathrm{ve}} \approx_{\tau_{\mathrm{ve}}} v'_{\mathrm{ve}}$ and $v_1 \approx_{\tau_1} v'_1$. It is clear $[v_{\mathrm{ve}}/x_{\mathrm{ve}}, v_1/x] \approx_{\{x_{\mathrm{ve}}:\tau_{\mathrm{ve}}, x:\tau_1\}} [v'_{\mathrm{ve}}/x_{\mathrm{ve}}, v'_1/x]$. Then by the induction hypothesis, $[v_{\mathrm{ve}}/x_{\mathrm{ve}}, v_1/x]e \approx_{\tau_2} [v'_{\mathrm{ve}}/x_{\mathrm{ve}}, v'_1/x]e'$. Then $\langle\!\langle \lambda x_{\mathrm{ve}}{:}\tau_{\mathrm{ve}}.\lambda x{:}\tau_1, v_{\mathrm{ve}}\rangle\!\rangle\ v_1 \sim_{\tau_2} (\lambda x_{\mathrm{ve}}{:}|\tau_{\mathrm{ve}}|.\lambda x{:}|\tau_1|.e')(v'_{\mathrm{ve}}, v'_1)$. Hence $\lambda x_{\mathrm{ve}}{:}\tau_{\mathrm{ve}}.\lambda x{:}\tau_1.e \sim_{\mathtt{code}(\tau_{\mathrm{ve}},\tau_1,\tau_2)} \lambda x_{\mathrm{ve}}{:}|\tau_{\mathrm{ve}}|.\lambda x{:}|\tau_1|.e'$.

Case (*app*). By the first induction hypothesis, $\gamma(e_1) \hookrightarrow v_1$ and $\gamma'(e'_1) \hookrightarrow v'_1$ and $v_1 \sim_{\tau_1 \to \tau_2} v'_1$. By the second induction hypothesis, $\gamma(e_2) \hookrightarrow v_2$ and $\gamma'(e'_2) \hookrightarrow v'_2$ and $v_2 \sim_{\tau_1} v'_2$. By the definition of the relation, $v_1\ v_2 \sim_{\tau_2} \mathtt{open}\ v'_1\ \mathtt{as}\ t_{\mathrm{ve}}\ \mathtt{with}\ x{:}\langle(t_{\mathrm{ve}} \to \tau_1 \to \tau_2) \times t_{\mathrm{ve}}\rangle\ \mathtt{in}\ (\pi_1 x)(\pi_2 x, v'_2)$. Thus, $\gamma(e_1\ e_2) \sim_{\tau_2} \gamma'(\mathtt{open}\ e'_1\ \mathtt{as}\ t_{\mathrm{ve}}\ \mathtt{with}\ x{:}\langle(t_{\mathrm{ve}} \to \tau_1 \to \tau_2) \times t_{\mathrm{ve}}\rangle\ \mathtt{in}\ (\pi_1 x)(\pi_2 x, e'_2)$.

Case (*tuple*) and (*proj*). By induction.

□

In our operational semantics, **pack** is considered as a pair of a type and an expression. Most implementations would treat the type of the environment of a closure uniformly since it is held abstract and thus expect that the environment value be compiled so as to fit into a single machine word – that is, the environment must be *boxed.* Therefore, most implementations would conceptually erase any type information from **pack** before execution. However, in implementations where abstract types are not treated in a uniform matter, a representation of the type of the environment must remain as part of the data structure at runtime. In particular, the calculus described by Harper and Morrisett [14] supports a **typecase** mechanism that allows the abstract type to be examined and different code can be selected according to this type. This can be used, for example, to support calling conventions where the environment is *unboxed* (*i.e.*, placed in registers). As another example, tag-free garbage collection [5, 2, 37, 24] relies upon type information being associated with closures so that the shape of values in the environment can be reconstructed during garbage collection. In essence, garbage collection, like **typecase**, is a non-parametric operation that is allowed to examine types and select code according to the type. Our type-based closure conversion makes the type information needed to support such non-parametric operations explicit. This provides further evidence that the treatment of closures as existentials is type-theoretically proper.

4 Overview of Polymorphic Closure Conversion

Closure conversion for a language with ML-style (*i.e.*, predicative [13]), explicit polymorphism follows a similar pattern to the simply-typed case, but with the additional complication that we must account for free type variables as well as free value variables in the code of an abstraction, and both value abstractions (λ-terms) and type abstractions (Λ-terms) induce the creation of closures. In this section, we give an overview of the typing difficulties encountered when closure converting value abstractions. The treatment of type abstractions is similar (see Section 5 for details).

To eliminate free occurrences of type variables and ordinary variables from the code, we abstract with respect to a type environment and a value environment, replacing free variables by references to the appropriate environment. By abstracting both free type variables and free value variables, the code becomes closed and can be hoisted to the top level. The abstracted code is then "partially applied" to suitable representations of the type and value environments to form a polymorphic closure. As in the simply-typed case, we need a data structure to represent the delayed partial application of the code to its environments. Also, we need to abstract both the *kind* of the type environment and the *type* of the value environment so that their representations remain private to the closure. Without the abstraction, we run into the same typing problems that we encountered with the simply-typed case.

As a running example, consider the expression:

$\lambda\mathtt{x}{:}t_1.\,(\mathtt{x}{:}t_1,\ \mathtt{y}{:}t_2,\ \mathtt{z}{:}\mathtt{int})$

of type $t_1 \to (t_1 \times t_2 \times \mathtt{int})$ where t_1 and t_2 are free type variables and **y** and **z** are free value variables of type t_2 and **int** respectively. After closure conversion, this expression is translated to the partial application

```
let val code =
      Λtenv :: {t₁::Ω, t₂::Ω}.
        λvenv : {y:#t₂ tenv, z:int}.
          λx : (#t₁ tenv).(x, #y venv, #z venv)
in
    code {t₁=t₁, t₂=t₂} {y=y, z=z}
end
```

The code abstracts type environment (tenv) and value environment (venv) arguments. The actual type environment, $\{t_1=t_1, t_2=t_2\}$, is a constructor record with kind $\{t_1::\Omega, t_2::\Omega\}$ where Ω is the kind of monotypes. The actual value environment, {y=y, z=z} is a record with type $\{y:t_2,\ z:\texttt{int}\}$. However, to keep the code closed so that it may be hoisted, all references to free type variables in the type of venv must come from tenv. Thus, we give venv the type $\{y:\#t_2\ \texttt{tenv},\ z:\texttt{int}\}$. Similarly, the code's argument x is given the type $\#t_1$ tenv. Consequently, the code part of the closure is a closed expression of closed type σ, where

$$\begin{aligned}\sigma = {} & \forall \texttt{tenv}::\{t_1::\Omega,\ t_2::\Omega\}.\\ & \{y:\#t_2\ \texttt{tenv},\ z:\texttt{int}\}\to(\#t_1\ \texttt{tenv})\to((\#t_1\ \texttt{tenv})\times(\#t_2\ \texttt{tenv})\times\texttt{int})\end{aligned}$$

It is easy to check that the entire expression has type $t_1 \to (t_1 \times t_2 \times \texttt{int})$, and thus the type of the original function is preserved.

We must now translate the partial application of the code to it environments into a data structure. The structure must be "mixed-phased" because it needs to hold a type (the type environment) as well as values (the code and value environment). A first attempt is to represent the data structure as a package e, where

$$e = \texttt{pack}\ \{t_1=t_1,\ t_2=t_2\}\ \texttt{with (code, \{y=y, z=z\}) as}\ \exists t_{\mathrm{te}}::\kappa_{\mathrm{te}}.\sigma \times \tau_{\mathrm{ve}}$$

and code is the code of the closure above and

$$\begin{aligned}\kappa_{\mathrm{te}} &= \{t_1::\Omega,\ t_2::\Omega\}\\ \tau_{\mathrm{ve}} &= \{y:\#t_2\ t_{\mathrm{te}},\ z:\texttt{int}\}\end{aligned}$$

It is easy to verify that e is well-typed under the typing rule for pack.

Unfortunately, there is a problem with this approach: an application of e to some argument $e' : t_1$ must open the package to extract the code, type, and value environments prior to the call:

```
open $e$ as $t_{te}::\kappa_{te}$ with z:$\sigma \times \tau_{ve}$
in
    (#1 z) $t_{te}$ (#2 z) $e'$
end
```

Although this is the "obvious" translation of application, it fails to be well-typed! The difficulty is that e' is of type t_1, whereas the expression (#1 z) t_{te} (#2 z) has type:

$$(\#t_1\ t_{\mathrm{te}})\to((\#t_1\ t_{\mathrm{te}})\times(\#t_2\ t_{\mathrm{te}})\times\texttt{int}).$$

Since t_{te} is abstract, t_1 is not provably equivalent to $\#t_1\ t_{\mathrm{te}}$, and this translation of application fails to typecheck.

The problem is that existentials provide a certain kind of mixed-phase data structure where the type portion *must* be abstract. We can use this to hide representations but here, we need to know what the type environment actually is in order to determine the type of the closure. In short, we need a mixed-phase data structure that does not hide its type component.

This same problem has been encountered in the study of the ML-like module systems [12, 21, 22]. Recent solutions are based on the idea of *translucent sums* [11] or *manifest types* [20], which provide the power of both existentials (weak sums), and transparent sums (strong sums). By ascribing the translucent sum type

$$\exists t_{\mathrm{te}} = \{t_1=t_1, t_2=t_2\}.\sigma \times \tau_{\mathrm{ve}}$$

to the closure, the equation t_{te}=$\{t_1$=t_1,t_2=$t_2\}$ is propagated into the scope of the abstraction so that in particular $\#t_1\ \ t_{te} = \#t_1\ \ \{t_1$=$t_1$,$t_2$=$t_2\} = t_1$, and thus the translation of application is type correct.

The next step is to hide the representation of the value environment as we did in the simply-typed case. If we simply abstract the type τ_{ve} from the above type expression we obtain

$$\exists t_{te} = \{t_1{=}t_1, t_2{=}t_2\}.\exists t_{ve}::\Omega.\sigma \times t_{ve}$$

where t_{ve} is the abstract type of the value environment. However, this fails to make type sense because we have abstracted the type of the value environment in the closure, but not the corresponding argument type of the code of the closure. The translation of application is ill-typed because the value environment has abstract type t_{ve}, but the domain type of `(#1 z)` t_{te} is $\{$`y`:t_2, `z:int`$\}$. Since t_{ve} is abstract, these two types are considered distinct. In order to simultaneously abstract the type of the value environment and the corresponding argument type in the code, we need to replace both types with the same abstract type t_{ve}. To do this, we must show that the two types are equivalent. This can be accomplished by requiring that the formal type environment argument (`tenv`) is only instantiated with the type environment t_{te}. One way to achieve this is to perform the application of the code to t_{te}, but the goal of closure conversion is to *delay* such partial applications. An alternative approach is to use translucency again and *coerce* the code so that it has the type σ', where:

$$\sigma' = \forall \texttt{tenv}{=}\ t_{te}::\kappa_{te}.\ \{\texttt{y}:\#t_2\ \texttt{tenv},\ \texttt{z:int}\}{\rightarrow}(\#t_1\ \texttt{tenv}){\rightarrow}((\#t_1\ \texttt{tenv}){\times}(\#t_2\ \texttt{tenv}){\times}\texttt{int})$$

Adding the constraint `tenv`= t_{te} to the type of the code has the effect of performing the type application at the *type-level*, but delays the application at the *term-level*. Note that σ' is a supertype of the original code type σ according to the rules of the translucent sum calculus. Consequently, the code remains the same (*i.e.*, closed) and can still be hoisted to the top level. In contrast, if we had performed the type application, the resulting code would not be closed (containing free references to t_{te}).

Since `tenv`= t_{te} and t_{te} =$\{t_1$=t_1,t_2=$t_2\}$, it follows that `tenv`=$\{t_1$=t_1,t_2=$t_2\}$ and thus `(#`t_i `tenv)`= t_i. Consequently, the data structure holding the components of the closure can be coerced to the equivalent type:

$$\exists t_{te} = \{t_1{=}t_1, t_2{=}t_2\}.\sigma''{\times}\{\texttt{y}:t_1, \texttt{z:int}\}$$

where σ'' is

$$\sigma'' = \forall \texttt{tenv}{=}\ t_{te}::\kappa_{te}.\{\texttt{y}:t_1,\ \texttt{z:int}\}{\rightarrow}t_1 \rightarrow(t_1 \times t_2{\times}\texttt{int})$$

Since this equivalent type makes no mention of the type environment t_{te} except in the constraint for `tenv`, we may drop the constraint on t_{te}, abstract the type of the value environment ($\{$`y`:t_1,`z:int`$\}$), and abstract the kind of the type environment κ_{te} to obtain the closure type:

$$\exists k_{te}.\exists t_{te}::k_{te}.\exists t_{ve}::\Omega.\sigma''' \times t_{ve}$$

where

$$\sigma''' = \forall \texttt{tenv}{=}\ t_{te}::k_{te}.t_{ve} \rightarrow t_1 \rightarrow(t_1 \times t_2{\times}\texttt{int})$$

It is easy to derive a type-preserving translation of application corresponding to this representation of closures. We simply open all of the existentials, and pass the type environment, value environment, and argument to the code.

Careful consideration of the foregoing discussion reveals that only limited use is made of translucency. The equational constraint on t_{te} is dropped from the existential (to ensure privacy of environment representation), and the universally quantified variable tenv does not occur in the scope of the abstraction. This suggests that a substantially simpler mechanism than the full translucent sum calculus is more appropriate for closure conversion. Hence, we introduce a special type, written $\tau \Rightarrow \sigma$, of functions that must be applied to the constructor τ to yield a value of type σ. The following two rules govern this new type constructor:

$$\frac{\Delta \vdash \tau :: \kappa \quad \Delta; \Gamma \vdash e : \forall t{::}\kappa.\sigma}{\Delta; \Gamma \vdash e : \tau \Rightarrow \sigma[\tau/t]} \qquad \frac{\Delta; \Gamma \vdash e : \tau \Rightarrow \sigma}{\Delta; \Gamma \vdash e\,\tau : \sigma}$$

The first rule restricts the domain of type application to the specific constructor τ. This corresponds to restricting the type to $\forall t = \tau.\sigma$ and propagating the equivalence $t = \tau$ into σ. The actual type application for $\tau \Rightarrow \sigma$ is permitted only for constructors equivalent to τ. These two rules naturally come from the necessity of delaying type applications for closure conversion. Using this notation, the type translation of $\tau_1 \to \tau_2$ becomes

$$\exists k_{\mathrm{te}}.\exists t_{\mathrm{te}}{::}k_{\mathrm{te}}.\exists t_{\mathrm{ve}}{::}\Omega.(t_{\mathrm{te}} \Rightarrow t_{\mathrm{ve}} \to \tau_1 \to \tau_2) \times t_{\mathrm{ve}}.$$

The type of closures abstracts the kind of the type environment and the type of the value environment, ensuring that these may be chosen separately for each closure in the system. As in the simply-typed case we have obtained an "object oriented" representation of polymorphic closures by exploiting a combination of the type systems proposed by Pierce and Turner [27] for objects and by Harper and Lillibridge [11] for modules.

5 A Formal Account of Polymorphic Closure Conversion

In this section, we present closure conversion for the predicative subset of the second order λ-calculus. It has been argued that the predicative fragment captures the "essence" of ML-style polymorphism, since there is a stratification between monotypes (types not involving a quantifier) and polytypes, and instantiation of type variables is restricted to monotypes [13]. These restrictions make it easy to use logical relations to argue correctness in the same fashion as we did for the simply-typed λ-calculus.

The syntax of our source language $\lambda^{\forall}$ is defined as follows:

Kinds	$\kappa ::= \Omega$
Constructors	$\tau ::= b \mid t \mid \tau_1 \to \tau_2$
Types	$\sigma ::= \tau \mid \sigma_1 \to \sigma_2 \mid \forall t{::}\kappa.\sigma$
Expressions	$e ::= c \mid x \mid \lambda x{:}\sigma_1.e \mid \Lambda t{::}\kappa.e \mid e_1\, e_2 \mid e\,\tau$
Values	$v ::= c \mid \lambda x{:}\sigma_1.e \mid \Lambda t{::}\kappa.e$

The type constructors (τ) are described by kinds (κ). There is only one kind (Ω) for $\lambda^{\forall}$, but subsequent languages have a richer kind structure, so we introduce kinds here for uniformity. Closed constructors of kind Ω correspond to a subset of types (the monotypes), in particular the types that do not include quantifiers. Thus, constructors of kind Ω can be injected into types. We leave this injection implicit and treat τ as both a constructor and a type.

A kind assignment Δ is a sequence that maps type variables to kinds and is of the form $\{t_1{::}\kappa_1, \ldots, t_n{::}\kappa_n\}$, $(n \geq 0)$. Typing judgements are of the form $\Delta; \Gamma \vdash e : \sigma$ where the free type variables of Γ, e, and σ are contained in the domain of Δ, and the free value variables of e are contained in the domain of Γ. A typing judgement is derived from the standard typing rules of the second-order λ-calculus (see for example [13, 14]). The most interesting rules are the introduction and elimination rules for quantified types:

$$\frac{\Delta \uplus \{t{::}\kappa\}; \Gamma \vdash e : \sigma}{\Delta; \Gamma \vdash \Lambda t{::}\kappa.\, e : \forall t{::}\kappa.\, \sigma} \quad (t \notin Dom(\Delta)) \qquad \frac{\Delta; \Gamma \vdash e : \forall t{::}\kappa.\, \sigma}{\Delta; \Gamma \vdash e\ \tau : \sigma[\tau/t]} \quad (FTV(\tau) \subseteq Dom(\Delta))$$

The introduction rule allows us to conclude that a Λ-expression has a polymorphic type $\forall t{::}\kappa.\sigma$ if, extending the kind assignment Δ with $t{::}\kappa$ allows us to conclude that the body of the Λ-expression has type σ. The elimination rule allows us to conclude that a type application $e\ \tau$ has the type $\sigma[\tau/t]$ if τ is a monotype whose free type variables are contained in the domain of Δ and e is an expression of type $\forall t{::}\Omega.\sigma$.

The operational semantics of $\lambda^{\forall}$ is defined using the following inference rules:

$$v \hookrightarrow v \qquad \frac{e_1 \hookrightarrow \lambda x{:}\tau_1.e \quad e_2 \hookrightarrow v_2 \quad e[v_2/x] \hookrightarrow v}{e_1 e_2 \hookrightarrow v} \qquad \frac{e \hookrightarrow \Lambda t{::}\kappa.e' \quad e'[\tau/t] \hookrightarrow v}{e\tau \hookrightarrow v}$$

5.1 Abstract Closures

As in the simply typed case, we break closure conversion into abstract closure conversion and closure representation stages[2]. The abstract closure conversion stage for $\lambda^{\forall}$ converts both λ-abstractions and Λ-abstractions into abstract closures consisting of code, a type environment and a value environment.

5.1.1 The Target Language

The syntax of the target language $\lambda^{\forall,cl}$ is defined as follows:

$$\begin{array}{lll}
\textit{Kinds} & \kappa ::= & \Omega \mid \langle \kappa_1 \times \ldots \times \kappa_2 \rangle \\
\textit{Con's} & \tau ::= & b \mid t \mid \tau_1 \to \tau_2 \mid \langle \tau_1 \times \ldots \times \tau_n \rangle \mid \\
 & & \langle \tau_1, \ldots, \tau_n \rangle \mid \pi_i\, \tau \mid \\
\textit{Types} & \sigma ::= & \tau \mid \sigma_1 \to \sigma_2 \mid \forall t{::}\kappa.\sigma \mid \langle \sigma_1 \times \ldots \times \sigma_n \rangle \mid \\
 & & \mathtt{vcode}(t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}, \sigma_{\mathrm{ve}}, \sigma_1, \sigma_2) \mid \\
 & & \mathtt{tcode}(t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}, \sigma_{\mathrm{ve}}, t{::}\kappa, \sigma) \\
\textit{Exp's} & e ::= & c \mid x \mid e_1\ e_2 \mid e\ \tau \mid \langle e_1, \ldots, e_n \rangle \mid \pi_i\, e \mid \\
 & & \Lambda t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}.\lambda x_{\mathrm{ve}}{:}\sigma_{\mathrm{ve}}.\lambda x{:}\sigma_1.e \mid \\
 & & \Lambda t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}.\lambda x_{\mathrm{ve}}{:}\sigma_{\mathrm{ve}}.\Lambda t{::}\kappa.e \mid \langle\!\langle e_1, \tau, e_2 \rangle\!\rangle \\
\textit{Values} & v ::= & c \mid \langle v_1, \ldots, v_n \rangle \mid \Lambda t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}.\lambda x_{\mathrm{ve}}{:}\sigma_{\mathrm{ve}}.\lambda x{:}\sigma_1.e \mid \\
 & & \Lambda t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}.\lambda x_{\mathrm{ve}}{:}\sigma_{\mathrm{ve}}.\Lambda t{::}\kappa.e \mid \langle\!\langle v_1, \tau, v_2 \rangle\!\rangle
\end{array}$$

A product kind $\langle \kappa_1 \times \ldots \times \kappa_n \rangle$ is used to specify the shape of type environments just as a product type specifies the shape of value environments. Given constructors τ_i with kind κ_i, the constructor $\langle \tau_1, \ldots, \tau_n \rangle$ has kind $\langle \kappa_1 \times \ldots \times \kappa_n \rangle$ and is used as a type environment consisting of $\tau_1, \ldots, \tau_n$ in a translated program.

[2]The material on environment sharing carries over in a straightforward manner.

There are two types of codes: the code for ordinary abstraction, $\Lambda t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}.\lambda x_{\mathrm{ve}}{:}\sigma_{\mathrm{ve}}.\lambda x{:}\sigma_1.e$, and the code for type abstraction, $\Lambda t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}.\lambda x_{\mathrm{ve}}{:}\sigma_{\mathrm{ve}}.\Lambda t{::}\kappa.e$. Codes take a type environment, a value environment, and a type or value argument respectively. We introduce the types `vcode` and `tcode`, to distinguish the types of codes from the types of closures and to avoid partial applications of codes. Intuitively, they correspond to standard types as follows:

$$\begin{array}{rcl}\mathtt{vcode}(t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}, \sigma_{\mathrm{ve}}, \sigma_1, \sigma_2) & \approx & \forall t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}.\sigma_{\mathrm{ve}} \to \sigma_1 \to \sigma_2 \\ \mathtt{tcode}(t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}, \sigma_{\mathrm{ve}}, t{::}\kappa, \sigma) & \approx & \forall t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}.\sigma_{\mathrm{ve}} \to \forall t{::}\kappa.\sigma\end{array}$$

Only types excluding $\forall$, `vcode`, and `tcode` can be named as a constructor. An abstract closure $\langle\!\langle e_1, \tau, e_2\rangle\!\rangle$ consists of a code e_1, a type environment constructor τ, and a value environment e_2.

For the typing of $\lambda^{\forall,cl}$, kind assignments (Δ) map type variables to kinds while type assignments (Γ) map value variables to types. The judgements of the static semantics are as follows:

$\Delta \vdash \tau :: \kappa$	τ is a well-formed constructor of kind κ.
$\Delta \vdash \sigma$	σ is a well-formed type.
$\Delta \vdash \tau_1 \equiv \tau_2 :: \kappa$	τ_1 and τ_2 are equivalent constructors.
$\Delta \vdash \sigma_1 \equiv \sigma_2$	σ_1 and σ_2 are equivalent types.
$\Delta; \Gamma \vdash e : \sigma$	e is a well-formed expression of type σ.

The typing rules of the language are defined in Figures 6 and 7. We require that code values be closed with respect to both type variables as well as value variables. This allows us to hoist code out of inner definitions to the top level. We remark that the code e_1 of a closure $\langle\!\langle e_1, \tau, e_2\rangle\!\rangle$ with type $\sigma_1 \to \sigma_2$ does not have the type $\mathtt{vcode}(t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}, \sigma_{\mathrm{ve}}, \sigma_1, \sigma_2)$, but rather $\mathtt{vcode}(t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}, \sigma'_{\mathrm{ve}}, \sigma'_1, \sigma'_2)$ where $\sigma_1 \equiv \sigma'_1[\tau/t_{\mathrm{te}}]$ and $\sigma_2 \equiv \sigma'_2[\tau/t_{\mathrm{te}}]$.

The operational semantics for $\lambda^{\forall,cl}$ is given in Figure 8. When a closure expression is evaluated, the code, the type environment, and the value environment are evaluated and then a closure consisting of these components is created. When an argument is applied to a closure, the type environment, the value environment, and the argument are passed to the code of the closure.

Various strategies of evaluation of constructors including call-by-value, call-by-need, or lazy evaluation can be used to reach a normal form. However, for simplicity in this paper we does not evaluate constructors explicitly. The operational correctness for other strategies may be proved with a few changes.

5.1.2 The Translation

Abstract closure conversion for $\lambda^{\forall}$ is formulated as a type-directed translation to $\lambda^{\forall,cl}$ by the deductive system in Figures 9 and 10. The judgement $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}} \triangleright \sigma \rightsquigarrow \sigma'$ means that σ' is the translation of σ where Δ_{env} is a kind assignment corresponding to a type environment and Δ_{arg} is a kind assignment corresponding to a type argument (if any). This judgement also implicitly defines a translation from constructors to constructors, since source-level constructors (τ) are a subset of types (σ) and the translation maps constructors to constructors. In translated programs, the type variable t_{te} is used for type environments. Thus a type variable in a type environment is translated to the appropriate projection of t_{te}.

The judgement $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}}; \Gamma_{\mathrm{env}}; \Gamma_{\mathrm{arg}} \triangleright e \rightsquigarrow e'$ means e' is a translation of e where Δ_{env} and Δ_{arg} are as in the type translation, and Γ_{env} and Γ_{arg} are type assignments corresponding to the value environment and value argument respectively. Depending on whether we are translating a type abstraction or value abstraction, either Γ_{arg} or Δ_{arg} will be empty. A type environment

Well-formedness of constructors:

$$\Delta \vdash b :: \Omega \qquad\qquad \Delta \uplus \{t{::}\kappa\} \vdash t :: \kappa \quad (t \notin Dom(\Delta))$$

$$\frac{\Delta \vdash \tau' :: \Omega \quad \Delta \vdash \tau :: \Omega}{\Delta \vdash \tau' \to \tau :: \Omega} \qquad \frac{\Delta \vdash \tau_1 :: \Omega \quad \cdots \quad \Delta \vdash \tau_n :: \Omega}{\Delta \vdash \langle \tau_1 \times \ldots \times \tau_n \rangle :: \Omega}$$

$$\frac{\Delta \vdash \tau_1 :: \kappa_1 \quad \cdots \quad \Delta \vdash \tau_n :: \kappa_n}{\Delta \vdash \langle \tau_1, \ldots, \tau_n \rangle :: \langle \kappa_1 \times \ldots \times \kappa_n \rangle} \qquad \frac{\Delta \vdash \tau :: \langle \kappa_1 \times \ldots \times \kappa_n \rangle}{\Delta \vdash \pi_i\, \tau :: \kappa_i}$$

Well-formedness of types:

$$\frac{\Delta \vdash \tau{::}\Omega}{\Delta \vdash \tau} \qquad \frac{\Delta \uplus \{t{::}\kappa\} \vdash \sigma}{\Delta \vdash \forall t{::}\kappa.\, \sigma} \quad (t \notin Dom(\Delta))$$

$$\frac{\Delta \vdash \sigma' \quad \Delta \vdash \sigma}{\Delta \vdash \sigma' \to \sigma} \qquad \frac{\Delta \vdash \sigma_1 \quad \cdots \quad \Delta \vdash \sigma_n}{\Delta \vdash \langle \sigma_1 \times \ldots \times \sigma_n \rangle}$$

$$\frac{\{t_{\text{te}}{::}\kappa_{\text{te}}\} \vdash \sigma_{\text{ve}} \quad \{t_{\text{te}}{::}\kappa_{\text{te}}\} \vdash \sigma_1 \quad \{t_{\text{te}}{::}\kappa_{\text{te}}\} \vdash \sigma_2}{\Delta \vdash \mathtt{vcode}(t_{\text{te}}{::}\kappa_{\text{te}}, \sigma_{\text{ve}}, \sigma_1, \sigma_2)}$$

$$\frac{\{t_{\text{te}}{::}\kappa_{\text{te}}\} \vdash \sigma_{\text{ve}} \quad \{t_{\text{te}}{::}\kappa_{\text{te}}, t{::}\kappa\} \vdash \sigma}{\Delta \vdash \mathtt{tcode}(t_{\text{te}}{::}\kappa_{\text{te}}, \sigma_{\text{ve}}, t{::}\kappa, \sigma)} \quad (t_{\text{te}} \neq t)$$

Primary rules for equivalence of constructors:

$$\frac{\Delta \vdash \langle \tau_1, \ldots, \tau_n \rangle :: \langle \kappa_1 \times \ldots \times \kappa_n \rangle}{\Delta \vdash \pi_i(\langle \tau_1, \ldots, \tau_n \rangle) \equiv \tau_i :: \kappa_i} \quad (1 \leq i \leq n)$$

$$\frac{\Delta \vdash \tau :: \langle \kappa_1 \times \ldots \times \kappa_n \rangle}{\Delta \vdash \tau \equiv \{\pi_1\, \tau, \ldots, \pi_n\, \tau\} :: \langle \kappa_1 \times \ldots \times \kappa_n \rangle}$$

Figure 6: Formation and Equivalence of Types and Constructors

$$\Delta;\Gamma \vdash c : b \qquad\qquad \Delta;\Gamma \vdash x : \Gamma(x)$$

$$\frac{\Delta;\Gamma \vdash e_1 : \sigma_1 \quad \cdots \quad \Delta;\Gamma \vdash e_n : \sigma_n}{\Delta;\Gamma \vdash \langle e_1, \ldots, e_n\rangle : \langle \sigma_1 \times \cdots \times \sigma_n\rangle} \qquad \frac{\Delta;\Gamma \vdash e : \langle \sigma_1 \times \cdots \times \sigma_n\rangle}{\Delta;\Gamma \vdash \pi_i\ e : \sigma_i} \quad (1 \leq \imath \leq n)$$

$$\frac{\Delta;\Gamma \vdash e_1 : \sigma_1 \to \sigma_2 \quad \Delta;\Gamma \vdash e_2 : \sigma_1}{\Delta;\Gamma \vdash e_1\ e_2 : \sigma_2} \qquad \frac{\Delta;\Gamma \vdash e : \forall t{::}\kappa.\sigma \quad \Delta \vdash \tau' :: \kappa}{\Delta;\Gamma \vdash e\ \tau : \sigma[\tau/t]}$$

$$\frac{\Delta;\Gamma \vdash e : \sigma' \quad \Delta \vdash \sigma \equiv \sigma'}{\Delta;\Gamma \vdash e : \sigma}$$

$$\frac{\{t_{\text{te}}{::}\kappa_{\text{te}}\}; \{x_{\text{ve}}{:}\sigma_{\text{ve}}, x{:}\sigma_1\} \vdash e : \sigma_2}{\Delta;\Gamma \vdash \Lambda t_{\text{te}}{::}\kappa_{\text{te}}.\lambda x_{\text{ve}}{:}\sigma_{\text{ve}}.\lambda x{:}\sigma_1.\, e : \mathtt{vcode}(t_{\text{te}}{::}\kappa_{\text{te}}, \sigma_{\text{ve}}, \sigma_1, \sigma_2)}$$

$$\frac{\{t_{\text{te}}{::}\kappa_{\text{te}}, t{::}\kappa\}; \{x_{\text{ve}}{:}\sigma_{\text{ve}}\} \vdash e : \sigma}{\Delta;\Gamma \vdash \Lambda t_{\text{te}}{::}\kappa_{\text{te}}.\lambda x_{\text{ve}}{:}\sigma_{\text{ve}}.\Lambda t{::}\kappa.\, e : \mathtt{tcode}(t_{\text{te}}{::}\kappa_{\text{te}}, \sigma_{\text{ve}}, t{::}\kappa, \sigma)}$$

$$\frac{\Delta;\Gamma \vdash e_1 : \mathtt{vcode}(t_{\text{te}}{::}\kappa_{\text{te}}, \sigma_{\text{ve}}, \sigma_1, \sigma_2) \quad \Delta \vdash \tau{::}\kappa_{\text{te}} \quad \Delta;\Gamma \vdash e_2 : \sigma_{\text{ve}}[\tau_{\text{te}}/t_{\text{te}}]}{\Delta;\Gamma \vdash \langle\!\langle e_1, \tau_{\text{te}}, e_2\rangle\!\rangle : (\sigma_1 \to \sigma_2)[\tau_{\text{te}}/t_{\text{te}}]}$$

$$\frac{\Delta;\Gamma \vdash e_1 : \mathtt{tcode}(t_{\text{te}}{::}\kappa_{\text{te}}, \sigma_{\text{ve}}, t{::}\kappa, \sigma) \quad \Delta \vdash \tau_{\text{te}}{::}\kappa_{\text{te}} \quad \Delta;\Gamma \vdash e_2 : \sigma_{\text{ve}}[\tau_{\text{te}}/t_{\text{te}}]}{\Delta;\Gamma \vdash \langle\!\langle e_1, \tau_{\text{te}}, e_2\rangle\!\rangle : (\forall t{::}\kappa.\, \sigma)[\tau_{\text{te}}/t_{\text{te}}]}$$

Figure 7: Typing Rules of $\lambda^{\forall,cl}$

$$v \hookrightarrow v \qquad\qquad \frac{e_1 \hookrightarrow v_1 \quad e_2 \hookrightarrow v_2}{\langle\!\langle e_1, \tau, e_2\rangle\!\rangle \hookrightarrow \langle\!\langle v_1, \tau, v_2\rangle\!\rangle}$$

$$\frac{e_1 \hookrightarrow v_1 \quad \ldots \quad e_n \hookrightarrow v_n}{\langle e_1, \ldots, e_n\rangle \hookrightarrow \langle v_1, \ldots, v_n\rangle} \qquad \frac{e \hookrightarrow \langle v_1, \ldots, v_n\rangle}{\pi_i\ e \hookrightarrow v_i}$$

$$\frac{e_1 \hookrightarrow \langle\!\langle \Lambda t_{\text{te}}{::}\kappa_{\text{te}}.\lambda x_{\text{ve}}{:}\sigma_{\text{ve}}.\lambda x{:}\sigma_1.\, e, \tau_{\text{te}}, v_{\text{ve}}\rangle\!\rangle \quad e_2 \hookrightarrow v' \quad e[\tau_{\text{te}}/t_{\text{te}}, v_{\text{ve}}/x_{\text{ve}}, v'/x] \hookrightarrow v}{e_1\ e_2 \hookrightarrow v}$$

$$\frac{e_1 \hookrightarrow \langle\!\langle \Lambda t_{\text{te}}{::}\kappa_{\text{te}}.\lambda x_{\text{ve}}{:}\sigma_{\text{ve}}.\Lambda t{::}\kappa.\, e, \tau_{\text{te}}, v_{\text{ve}}\rangle\!\rangle \quad e[\tau_{\text{te}}/t_{\text{te}}, v_{\text{ve}}/x_{\text{ve}}, \tau/t] \hookrightarrow v}{e_1\ \tau \hookrightarrow v}$$

Figure 8: Operational Semantics of $\lambda^{\forall,cl}$

$$\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}} \rhd b \leadsto b \qquad \{t_1{::}\Omega,\ldots,t_n{::}\Omega\};\Delta_{\mathrm{arg}} \rhd t_i \leadsto \pi_i\, t_{\mathrm{te}}$$

$$\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}} \rhd t \leadsto t \quad (t \in \mathit{Dom}(\Delta_{\mathrm{arg}}))$$

$$\frac{\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}} \rhd \sigma_1 \leadsto \sigma_1' \quad \Delta_{\mathrm{env}};\Delta_{\mathrm{arg}} \rhd \sigma_2 \leadsto \sigma_2'}{\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}} \rhd \sigma_1 \to \sigma_2 \leadsto \sigma_1' \to \sigma_2'} \qquad \frac{\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}} \uplus \{t{::}\Omega\} \rhd \sigma \leadsto \sigma'}{\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}} \rhd \forall t{::}\kappa.\,\sigma \leadsto \forall t{::}\kappa.\,\sigma'}$$

$$\frac{\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}} \rhd t_1' \leadsto \tau_1 \quad \cdots \quad \Delta_{\mathrm{env}};\Delta_{\mathrm{arg}} \rhd t_n' \leadsto \tau_n}{\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}} \rhd \{t_1'{::}\Omega,\ldots,t_n'{::}\Omega\} \leadsto \langle \tau_1,\ldots,\tau_n\rangle}$$

$$\frac{\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}} \rhd \sigma_1 \leadsto \sigma_1' \quad \cdots \quad \Delta_{\mathrm{env}};\Delta_{\mathrm{arg}} \rhd \sigma_n \leadsto \sigma_n'}{\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}} \rhd \{x_1{:}\sigma_1,\ldots,x_n{:}\sigma_n\} \leadsto \{x_1{:}\sigma_1',\ldots,x_n{:}\sigma_n'\}}$$

Figure 9: Polymorphic Abstract Closure Conversion: Types and Type Assignments

$$(\mathit{const}) \quad \Delta_{\mathrm{env}};\Delta_{\mathrm{arg}};\Gamma_{\mathrm{env}};\Gamma_{\mathrm{arg}} \rhd c \leadsto c$$

$$(\mathit{env}) \quad \Delta_{\mathrm{env}};\Delta_{\mathrm{arg}};\{x_1{:}\sigma_1,\ldots,x_n{:}\sigma_n\};\Gamma_{\mathrm{arg}} \rhd x_i \leadsto \pi_i\, x_{\mathrm{ve}}$$

$$(\mathit{arg}) \quad \Delta_{\mathrm{env}};\Delta_{\mathrm{arg}};\Gamma_{\mathrm{env}};\Gamma_{\mathrm{arg}} \rhd x \leadsto x \qquad (x \in \mathit{Dom}(\Gamma_{\mathrm{arg}}))$$

$$(\mathit{abs}) \quad \frac{\begin{array}{c}\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}};\Gamma_{\mathrm{env}};\Gamma_{\mathrm{arg}} \rhd \Delta_{\mathrm{env}}' \leadsto \tau_{\mathrm{te}} \qquad \Delta_{\mathrm{env}};\Delta_{\mathrm{arg}};\Gamma_{\mathrm{env}};\Gamma_{\mathrm{arg}} \rhd \Gamma_{\mathrm{env}}' \leadsto e_{\mathrm{ve}} \\ \Delta_{\mathrm{env}}';\emptyset \rhd \Gamma_{\mathrm{env}}' \leadsto \Gamma_{\mathrm{env}}'' \qquad \Delta_{\mathrm{env}}';\emptyset \rhd \sigma_1 \leadsto \sigma_1' \\ \Delta_{\mathrm{env}}';\emptyset;\Gamma_{\mathrm{env}}';\{x{:}\sigma_1\} \rhd e \leadsto e'\end{array}}{\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}};\Gamma_{\mathrm{env}};\Gamma_{\mathrm{arg}} \rhd \lambda x{:}\sigma_1.e \leadsto \langle\!\langle \Lambda t_{\mathrm{te}}{::}|\Delta_{\mathrm{env}}'|.\lambda x_{\mathrm{ve}}{:}|\Gamma_{\mathrm{env}}''|.\lambda x{:}\sigma_1'.e', \tau_{\mathrm{te}}, e_{\mathrm{ve}}\rangle\!\rangle}$$

$$(\mathit{tabs}) \quad \frac{\begin{array}{c}\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}};\Gamma_{\mathrm{env}};\Gamma_{\mathrm{arg}} \rhd \Delta_{\mathrm{env}}' \leadsto \tau_{\mathrm{te}} \qquad \Delta_{\mathrm{env}};\Delta_{\mathrm{arg}};\Gamma_{\mathrm{env}};\Gamma_{\mathrm{arg}} \rhd \Gamma_{\mathrm{env}}' \leadsto e_{\mathrm{ve}} \\ \Delta_{\mathrm{env}}';\emptyset \rhd \Gamma_{\mathrm{env}}' \leadsto \Gamma_{\mathrm{env}}'' \qquad \Delta_{\mathrm{env}}';\{t{::}\Omega\};\Gamma_{\mathrm{env}}';\emptyset \rhd e \leadsto e'\end{array}}{\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}};\Gamma_{\mathrm{env}};\Gamma_{\mathrm{arg}} \rhd \Lambda t{::}\Omega.\,e \leadsto \langle\!\langle \Lambda t_{\mathrm{te}}{::}|\Delta_{\mathrm{env}}'|.\lambda x_{\mathrm{ve}}{:}|\Gamma_{\mathrm{env}}''|.\Lambda t{::}\Omega.\,e', \tau_{\mathrm{te}}, e_{\mathrm{ve}}\rangle\!\rangle}$$

$$(\mathit{app}) \quad \frac{\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}};\Gamma_{\mathrm{env}};\Gamma_{\mathrm{arg}} \rhd e_1 \leadsto e_1' \quad \Delta_{\mathrm{env}};\Delta_{\mathrm{arg}};\Gamma_{\mathrm{env}};\Gamma_{\mathrm{arg}} \rhd e_2 \leadsto e_2'}{\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}};\Gamma_{\mathrm{env}};\Gamma_{\mathrm{arg}} \rhd e_1\ e_2 \leadsto e_1'\ e_2'}$$

$$(\mathit{tapp}) \quad \frac{\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}};\Gamma_{\mathrm{env}};\Gamma_{\mathrm{arg}} \rhd e \leadsto e' \quad \Delta_{\mathrm{env}};\Delta_{\mathrm{arg}} \rhd \sigma \leadsto \sigma'}{\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}};\Gamma_{\mathrm{env}};\Gamma_{\mathrm{arg}} \rhd e\ \sigma \leadsto e'\ \sigma'}$$

$$(\mathit{context}) \quad \frac{\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}};\Gamma_{\mathrm{env}};\Gamma_{\mathrm{arg}} \rhd x_i \leadsto e_i'}{\Delta_{\mathrm{env}};\Delta_{\mathrm{arg}};\Gamma_{\mathrm{env}};\Gamma_{\mathrm{arg}} \rhd \{x_1{:}\sigma_1,\ldots,x_n{:}\sigma_n\} \leadsto \langle e_1',\ldots,e_n'\rangle}$$

Figure 10: Polymorphic Abstract Closure Conversion: Terms

corresponding to Δ_{env} and a value environment corresponding to Γ_{env} are implemented in the target language by types of the form $|\Delta_{\text{env}}|$ and $|\Gamma_{\text{env}}|$ respectively, defined below.

$$\begin{aligned}
|\{t_1{::}\kappa_1, \ldots, t_n{::}\kappa_n\}| &= \langle \kappa_1 \times \ldots \times \kappa_n \rangle \\
|\{x_1{:}\sigma_1, \ldots, x_n{:}\sigma_n\}| &= \langle \sigma_1 \times \ldots \times \sigma_n \rangle
\end{aligned}$$

For simplicity, only flat representations of value and type environments are considered in this translation.

The most interesting rules are the term translations of value and type abstractions. In each case, an appropriate type environment and value environment must be constructed as part of the closure. Thus, assignments Δ'_{env} and Γ'_{env} must be chosen as subsets of the current assignments $\Delta_{\text{env}} \uplus \Delta_{\text{arg}}$ and $\Gamma_{\text{env}} \uplus \Gamma_{\text{arg}}$ respectively. These assignments must be chosen so that all of the free value variables of the term are contained in Γ'_{env} and further, all of the free type variables of the term and the value environment must be contained in Δ'_{env}.

The primary subtlety in these rules is that we need *two* type assignments Γ'_{env} and Γ''_{env} to describe the value environment of the closure, depending upon the context. Γ'_{env} is constructed from the context $\Delta_{\text{env}}; \Delta_{\text{arg}}; \Gamma_{\text{env}}; \Gamma_{\text{arg}}$ and is used to build the environment e_{ve} in the context where we are constructing the closure. In contrast, Γ''_{env} is obtained from Γ'_{env} via the translation $\Delta'_{\text{env}}; \emptyset \rhd \Gamma'_{\text{env}} \leadsto \Gamma''_{\text{env}}$ and corresponds to the type of the value environment in the context of the closure itself. This ensures that the code of the closure is closed since the type ascribed to the value environment argument does not refer to free type variables in the context where the closure was created.

5.1.3 Type Correctness

The following lemma shows that type translation commutes with substitution.

Lemma 3 *If* $\Delta_{\text{env}}; \Delta_{\text{arg}} \uplus \{t{::}\Omega\} \rhd \sigma \leadsto \sigma'$, $\Delta_{\text{env}} \uplus \Delta_{\text{arg}} \vdash \tau{::}\Omega$, *and* $\Delta_{\text{env}}; \Delta_{\text{arg}} \rhd \tau \leadsto \tau'$, *then* $\Delta_{\text{env}}; \Delta_{\text{arg}} \rhd \sigma[\tau/t] \leadsto \sigma'[\tau'/t]$.

Proof. By induction on the derivation of $\Delta_{\text{env}}; \Delta_{\text{arg}} \uplus \{t{::}\Omega\} \rhd \sigma \leadsto \sigma'$.

Case $\Delta_{\text{env}}; \Delta_{\text{arg}} \uplus \{t{::}\Omega\} \rhd t \leadsto t$. From the assumption, $\Delta_{\text{env}}; \Delta_{\text{arg}} \rhd \tau \leadsto \tau'$.

Case $\Delta_{\text{env}}; \Delta_{\text{arg}} \uplus \{t{::}\Omega\} \rhd t' \leadsto t'$. From the definition of translation of types, $\Delta_{\text{env}}; \Delta_{\text{arg}} \rhd t' \leadsto t'$.

Case $\Delta_{\text{env}}; \Delta_{\text{arg}} \uplus \{t{::}\Omega\} \rhd t_i \leadsto \pi_i(t_{\text{te}})$. Then $t_i \in \Delta_{\text{env}}$, so $t_i \neq t$. Thus, $\Delta_{\text{env}}; \Delta_{\text{arg}} \rhd t_i[\tau/t] \leadsto \pi_i(t_{\text{te}})$.

Case $\Delta_{\text{env}}; \Delta_{\text{arg}} \uplus \{t{::}\Omega\} \rhd \sigma_1 \rightarrow \sigma_2 \leadsto \sigma'_1 \rightarrow \sigma'_2$. By the induction hypotheses, $\Delta_{\text{env}}; \Delta_{\text{arg}} \uplus \{t{::}\Omega\} \rhd \sigma_1[\tau/t] \leadsto \sigma'_1[\tau'/t]$ and $\Delta_{\text{env}}; \Delta_{\text{arg}} \uplus \{t{::}\Omega\} \rhd \sigma_2[\tau/t] \leadsto \sigma'_2[\tau'/t]$. Thus, by the definition of substitution and the type translation, $\Delta_{\text{env}}; \Delta_{\text{arg}} \uplus \{t{::}\Omega\} \rhd (\sigma_1 \rightarrow \sigma_2)[\tau/t] \leadsto (\sigma'_1 \rightarrow \sigma'_2)[\tau'/t]$.

Case $\Delta_{\text{env}}; \Delta_{\text{arg}} \uplus \{t{::}\Omega\} \rhd \forall t'{::}\Omega.\sigma_1 \leadsto \forall t'{::}\Omega.\sigma'_1$ is derived from $\Delta_{\text{env}}; \Delta_{\text{arg}} \uplus \{t{::}\Omega, t'{::}\Omega\} \rhd \sigma_1 \leadsto \sigma'_1$. Through α-conversion, t' can always be chosen to be different from t. By the assumption and the definition of the translation, $\Delta_{\text{env}}; \Delta_{\text{arg}} \uplus \{t'{::}\Omega\} \rhd \tau \leadsto \tau'$. Then by the induction hypothesis, $\Delta_{\text{env}}; \Delta_{\text{arg}} \uplus \{t'{::}\Omega\} \rhd \sigma_1[\tau/t] \leadsto \sigma_1[\tau'/t]$. Thus by substitution, $\Delta_{\text{env}}; \Delta_{\text{arg}} \rhd \forall t'{::}\Omega.\sigma_1[\tau/t] \leadsto \forall t'{::}\Omega.\sigma_1[\tau'/t]$.

□

The following lemma shows that types are suitably equivalent when translated under the current kind assignment or a kind assignment derived from the current kind assignment.

Lemma 4 *If* $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}} \rhd \Delta'_{\mathrm{env}} \leadsto \tau_{\mathrm{te}}$ *and* $\Delta'_{\mathrm{env}}; \Delta'_{\mathrm{arg}} \rhd \sigma \leadsto \sigma_1$*, then* $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}} \uplus \Delta'_{\mathrm{arg}} \rhd \sigma \leadsto \sigma_2$ *and* $\{t_{\mathrm{te}}{::}|\Delta_{\mathrm{env}}|\} \uplus \Delta_{\mathrm{arg}} \uplus \Delta'_{\mathrm{arg}} \vdash \sigma_2 \equiv \sigma_1[\tau_{\mathrm{te}}/t_{\mathrm{te}}]$.

Proof. By induction on the derivation of $\Delta'_{\mathrm{env}}; \Delta'_{\mathrm{arg}} \rhd \sigma \leadsto \sigma_1$.

Case $\Delta'_{\mathrm{env}}; \Delta'_{\mathrm{arg}} \rhd t \leadsto t$. Clear.

Case $\Delta'_{\mathrm{env}}; \Delta'_{\mathrm{arg}} \rhd t'_i \leadsto \pi_i(t_{\mathrm{te}})$. Let $\Delta_{\mathrm{env}} = \{t_1{::}\Omega, \ldots, t_n{::}\Omega\}$ and $\Delta_{\mathrm{arg}} = \{s_1{::}\Omega, \ldots, s_m{::}\Omega\}$.

Subcase $t'_i = t_j$. Then $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}} \uplus \Delta'_{\mathrm{arg}} \rhd t'_i \equiv t_j \leadsto \pi_j(t_{\mathrm{te}})$ and $\{t_{\mathrm{te}}{::}|\Delta_{\mathrm{env}}|\} \vdash \pi_i(\tau_{\mathrm{te}}) \equiv \pi_j(t_{\mathrm{te}})$.

Subcase $t'_i = s_j$. Then $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}} \uplus \Delta'_{\mathrm{arg}} \rhd t'_i \equiv s_j \leadsto s_j$ and $\{t_{\mathrm{te}}{::}|\Delta_{\mathrm{env}}|\} \vdash \pi_i(\tau_{\mathrm{te}}) \equiv s_j$.

Case $\Delta'_{\mathrm{env}}; \Delta'_{\mathrm{arg}} \rhd \sigma_1 \to \sigma_2 \leadsto \sigma'_1 \to \sigma'_2$. Clear from induction hypotheses.

Case $\Delta'_{\mathrm{env}}; \Delta'_{\mathrm{arg}} \rhd \forall t{::}\Omega.\sigma \leadsto \forall t{::}\Omega.\sigma_1$ is derived from $\Delta'_{\mathrm{env}}; \Delta'_{\mathrm{arg}} \uplus \{t{::}\Omega\} \rhd \sigma \leadsto \sigma_1$. Then $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}} \uplus \Delta'_{\mathrm{arg}} \uplus \{t{::}\Omega\} \rhd \sigma \leadsto \sigma_2$ and $\{t_{\mathrm{te}}{::}|\Delta_{\mathrm{env}}|\} \uplus \Delta_{\mathrm{arg}} \uplus \Delta'_{\mathrm{arg}} \uplus \{t{::}\Omega\} \vdash \sigma_2 \equiv \sigma_1[\tau_{\mathrm{te}}/t_{\mathrm{te}}]$. Thus, $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}} \uplus \Delta'_{\mathrm{arg}} \rhd \forall t{::}\Omega.\sigma \leadsto \forall t{::}\Omega.\sigma_2$ and $\{t_{\mathrm{te}}{::}|\Delta_{\mathrm{env}}|\} \uplus \Delta_{\mathrm{arg}} \uplus \Delta'_{\mathrm{arg}} \vdash \forall t{::}\Omega.\sigma_2 \equiv \forall t{::}\Omega.\sigma_1[\tau_{\mathrm{te}}/t_{\mathrm{te}}]$.

□

As in the simply typed case, the value environment produced from Γ'_{env} has the type obtained by translating Γ'_{env}.

Lemma 5 *If* $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}}; \Gamma_{\mathrm{env}}; \Gamma_{\mathrm{arg}} \rhd \Gamma'_{\mathrm{env}} \leadsto e_{\mathrm{ve}}$ *and* $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}} \rhd \Gamma'_{\mathrm{env}} \leadsto \Gamma'''_{\mathrm{env}}$*, then* $\Delta_{\mathrm{env}} \uplus \Delta_{\mathrm{arg}}; \Gamma_{\mathrm{env}} \uplus \Gamma_{\mathrm{arg}} \vdash e_{\mathrm{ve}} : |\Gamma'''_{\mathrm{env}}|$.

The type correctness of the translation is proved by induction on the derivation of the translation.

Theorem 5 (Type Correctness) *If* $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}}; \Gamma_{\mathrm{env}}; \Gamma_{\mathrm{arg}} \rhd e \leadsto e'$ *and* $\Delta_{\mathrm{env}} \uplus \Delta_{\mathrm{arg}}; \Gamma_{\mathrm{env}} \uplus \Gamma_{\mathrm{arg}} \vdash e : \sigma$*, then* $\{t_{\mathrm{te}}{::}|\Delta_{\mathrm{env}}|\} \uplus \Delta_{\mathrm{arg}}; \{x_{\mathrm{ve}}{:}|\Gamma'_{\mathrm{env}}|\} \uplus \Gamma'_{\mathrm{arg}} \vdash e' : \sigma'$ *where* $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}} \rhd \sigma \leadsto \sigma'$, $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}} \rhd \Gamma_{\mathrm{env}} \leadsto \Gamma'_{\mathrm{env}}$*, and* $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}} \rhd \Gamma_{\mathrm{arg}} \leadsto \Gamma'_{\mathrm{arg}}$.

Proof. By induction on the derivation of $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}}; \Gamma_{\mathrm{env}}; \Gamma_{\mathrm{arg}} \rhd e \leadsto e'$.

Case (*env*). Let Γ_{env} be $\{x_1{:}\sigma_1, \ldots, x_n{:}\sigma_n\}$. Then Γ'_{env} is $\{x_1{:}\sigma'_1, \ldots, x_n{:}\sigma'_n\}$ and $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}} \rhd \sigma_i \leadsto \sigma'_i$. Thus $\{t_{\mathrm{te}}{::}|\Delta_{\mathrm{env}}|\} \uplus \Delta_{\mathrm{arg}}; \{x_{\mathrm{ve}}{:}|\Gamma'_{\mathrm{env}}|\} \uplus \Gamma'_{\mathrm{arg}} \vdash \pi_i(x_{\mathrm{ve}}) : \sigma'_i$.

Case (*arg*). Let Γ_{arg} be $\{x_1{:}\sigma_1, \ldots, x_n{:}\sigma_n\}$. Then Γ'_{arg} is $\{x_1{:}\sigma'_1, \ldots, x_n{:}\sigma'_n\}$ and $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}} \rhd \sigma_i \leadsto \sigma'_i$. Thus $\{t_{\mathrm{te}}{::}|\Delta_{\mathrm{env}}|\} \uplus \Delta_{\mathrm{arg}}; \{x_{\mathrm{ve}}{:}|\Gamma'_{\mathrm{env}}|\} \uplus \Gamma'_{\mathrm{arg}} \vdash x_i : \sigma'_i$.

Case (*abs*). Let $\Delta'_{\mathrm{env}}; \emptyset; \Gamma''''_{\mathrm{env}}; x{:}\sigma_1 \rhd e \leadsto e'$ and $\Delta'_{\mathrm{env}}; \emptyset \rhd \Gamma''''_{\mathrm{env}} \leadsto \Gamma''_{\mathrm{env}}$ and $\Delta'_{\mathrm{env}}; \emptyset \rhd \sigma_1 \leadsto \sigma'_1$. Since $\Delta'_{\mathrm{env}}; \Gamma''''_{\mathrm{env}} \uplus \{x{:}\sigma_1\} \vdash e : \sigma_2$, by the induction hypothesis, $\{t_{\mathrm{te}}{::}|\Delta'_{\mathrm{env}}|\}; \{x_{\mathrm{ve}}{:}|\Gamma''_{\mathrm{env}}|, x{:}\sigma'_1\} \vdash e' : \sigma'_2$ where $\Delta'_{\mathrm{env}}; \emptyset \rhd \sigma_2 \leadsto \sigma'_2$. Let Γ be $\{x_{\mathrm{ve}}{:}|\Gamma'_{\mathrm{env}}|\} \uplus \Gamma'_{\mathrm{arg}}$. Then

$$\{t_{\mathrm{te}}{::}|\Delta_{\mathrm{env}}|\} \uplus \Delta_{\mathrm{arg}}; \Gamma \vdash \Lambda t_{\mathrm{te}}{::}|\Delta'_{\mathrm{env}}|.\lambda x_{\mathrm{ve}}{:}|\Gamma''_{\mathrm{env}}|.\lambda x{:}\sigma'_1.e' : \mathsf{vcode}(t_{\mathrm{te}}{::}|\Delta'_{\mathrm{env}}|, |\Gamma''_{\mathrm{env}}|, \sigma'_1, \sigma'_2)$$

Let $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}} \rhd \sigma_1 \to \sigma_2 \leadsto \sigma'$ and $\Delta_{\mathrm{env}}; \Delta_{\mathrm{arg}} \rhd \Gamma''''_{\mathrm{env}} \leadsto \Gamma'''_{\mathrm{env}}$. Then by Lemma 4, $\{t_{\mathrm{te}}{::}|\Delta_{\mathrm{env}}|\} \uplus \Delta_{\mathrm{arg}} \vdash \sigma' \equiv (\sigma'_1 \to \sigma'_2)[\tau_{\mathrm{te}}/t_{\mathrm{te}}]$ and $\{t_{\mathrm{te}}{::}|\Delta_{\mathrm{env}}|\} \uplus \Delta' \vdash |\Gamma'''_{\mathrm{env}}| \equiv |\Gamma''_{\mathrm{env}}|[\tau_{\mathrm{te}}/t_{\mathrm{te}}]{::}\langle\Omega \times \ldots \times \Omega\rangle$. Since $\{t_{\mathrm{te}}{::}|\Delta_{\mathrm{env}}|\} \uplus \Delta_{\mathrm{arg}}; \Gamma \vdash e_{\mathrm{ve}} : |\Gamma'''_{\mathrm{env}}|$,

$$\{t_{\mathrm{te}}{::}|\Delta_{\mathrm{env}}|\} \uplus \Delta_{\mathrm{arg}}; \Gamma \vdash \langle\!\langle \Lambda t_{\mathrm{te}}{::}|\Delta'_{\mathrm{env}}|.\lambda x_{\mathrm{ve}}{:}|\Gamma''_{\mathrm{env}}|.\lambda x{:}\sigma'_1.e', \tau_{\mathrm{te}}, e_{\mathrm{ve}} \rangle\!\rangle : \sigma'$$

Case (*tabs*). Let $\Delta'_{\text{env}}; \{t{::}\Omega\}; \Gamma''''_{\text{env}}; \emptyset \rhd e \rightsquigarrow e'$ and $\Delta'_{\text{env}}; \emptyset \rhd \Gamma''''_{\text{env}} \rightsquigarrow \Gamma''_{\text{env}}$. Since $\Delta'_{\text{env}} \uplus \{t{::}\Omega\}; \Gamma''''_{\text{env}} \vdash e : \sigma_2$, by the induction hypothesis, $\{t_{\text{te}}{::}|\Delta'_{\text{env}}|, t{::}\Omega\}; \{x_{\text{ve}}{:}|\Gamma''_{\text{env}}|\} \vdash e' : \sigma'_2$ where $\Delta'_{\text{env}}; \{t{::}\Omega\} \rhd \sigma_2 \rightsquigarrow \sigma'_2$. Let Γ be $\{x_{\text{ve}}{:}|\Gamma'_{\text{env}}|\} \uplus \Gamma'_{\text{arg}}$. Then

$$\{t_{\text{te}}{::}|\Delta_{\text{env}}|\} \uplus \Delta_{\text{arg}}; \Gamma \vdash \Lambda t_{\text{te}}{::}|\Delta'_{\text{env}}|.\lambda x_{\text{ve}}{:}|\Gamma''_{\text{env}}|.\Lambda t{::}\Omega.e' : \mathtt{tcode}(t_{\text{te}}{::}|\Delta'_{\text{env}}|, |\Gamma''_{\text{env}}|, t{::}\Omega, \sigma'_2)$$

Let $\Delta_{\text{env}}; \Delta_{\text{arg}} \rhd \forall t{::}\Omega.\sigma_2 \rightsquigarrow \sigma'$ and $\Delta_{\text{env}}; \Delta_{\text{arg}} \rhd \Gamma''''_{\text{env}} \rightsquigarrow \Gamma'''_{\text{env}}$. Then by Lemma 4, $\{t_{\text{te}}{::}|\Delta_{\text{env}}|\} \uplus \Delta_{\text{arg}} \vdash \sigma' \equiv \forall t{::}\Omega.\sigma'_2[\tau_{\text{te}}/t_{\text{te}}]$ and $\{t_{\text{te}}{::}|\Delta_{\text{env}}|\} \uplus \Delta_{\text{arg}} \vdash |\Gamma'''_{\text{env}}| \equiv |\Gamma''_{\text{env}}|[\tau_{\text{te}}/t_{\text{te}}]{::}\langle\Omega \times \ldots \times \Omega\rangle$. Since $\{t_{\text{te}}{::}|\Delta_{\text{env}}|\} \uplus \Delta_{\text{arg}}; \Gamma \vdash e_{\text{ve}} : |\Gamma'''_{\text{env}}|$,

$$\{t_{\text{te}}{::}|\Delta_{\text{env}}|\} \uplus \Delta_{\text{arg}}; \Gamma \vdash \langle\!\langle \Lambda t_{\text{te}}{::}|\Delta'_{\text{env}}|.\lambda x_{\text{ve}}{:}|\Gamma''_{\text{env}}|.\Lambda t{::}\Omega.e', \tau_{\text{te}}, e_{\text{ve}} \rangle\!\rangle : \sigma'$$

Case (*tapp*). Let $\Delta_{\text{env}} \uplus \Delta_{\text{arg}}; \Gamma_{\text{env}} \uplus \Gamma_{\text{arg}} \vdash e : \forall t{::}\Omega.\sigma_1$. By the induction hypothesis, $\{t_{\text{te}}{::}|\Delta_{\text{env}}|\} \uplus \Delta_{\text{arg}}; \{x_{\text{ve}}{:}|\Gamma'_{\text{env}}|\} \uplus \Gamma'_{\text{arg}} \vdash e' : \forall t{::}\Omega.\sigma'_1$ such that $\Delta_{\text{env}}; \Delta_{\text{arg}} \uplus \{t{::}\Omega\} \rhd \sigma_1 \rightsquigarrow \sigma'_1$. Let $\Delta_{\text{env}}; \Delta_{\text{arg}} \rhd \sigma \rightsquigarrow \sigma'$. Then $\{t_{\text{te}}{::}|\Delta_{\text{env}}|\} \uplus \Delta_{\text{arg}}; \Gamma'_{\text{env}} \uplus \Gamma'_{\text{arg}} \vdash e' \; \sigma' : \sigma'_1[\sigma'/t]$. By Lemma 3, $\Delta_{\text{env}}; \Delta_{\text{arg}} \rhd \sigma_1[\sigma/t] \rightsquigarrow \sigma'_1[\sigma'/t]$.

Case (*app*). By the first induction hypothesis, $\{t_{\text{te}}{::}|\Delta_{\text{env}}|\} \uplus \Delta_{\text{arg}}; \{x_{\text{ve}}{:}|\Gamma'_{\text{env}}|\} \uplus \Gamma'_{\text{arg}} \vdash e'_1 : \sigma'_1 \to \sigma'_2$ where $\Delta_{\text{env}}; \Delta_{\text{arg}} \rhd \sigma_i \rightsquigarrow \sigma'_i$ for $i = 1, 2$. By the second induction hypothesis, $\{t_{\text{te}}{::}|\Delta_{\text{env}}|\} \uplus \Delta_{\text{arg}}; \{x_{\text{ve}}{:}|\Gamma'_{\text{env}}|\} \uplus \Gamma'_{\text{arg}} \vdash e'_2 : \sigma'_1$. Hence, $\{t_{\text{te}}{::}|\Delta_{\text{env}}|\} \uplus \Delta_{\text{arg}}; \{x_{\text{ve}}{:}|\Gamma'_{\text{env}}|\} \uplus \Gamma'_{\text{arg}} \vdash e'_1 \; e'_2 : \sigma'_2$.

□

5.1.4 Operational Correctness

We can prove the operational correctness of the translation using logical relations in the same fashion as we did for the simply typed case because the source language is restricted to the predicative polymorphism. First we define the relation between closed source and target constructors.

$$\tau \simeq \tau' \quad \text{iff} \quad \text{for some } \tau'', \emptyset; \emptyset \rhd \tau \rightsquigarrow \tau'' \text{ and } \emptyset \vdash \tau' \equiv \tau''{::}\Omega$$

Then we define the logical relations inductively by the lexicographic order of the number $\forall$-quantifiers in a type σ and by the size of σ. Since instantiation is restricted to types that do *not* contain quantifiers, this measure always decreases at a type application. The relations are defined as follows:

$$\begin{array}{rll}
e \sim_\sigma e' & \text{iff} & e \hookrightarrow v \text{ and } e \hookrightarrow v' \text{ and } v \approx_\sigma v'. \\
c \approx_b c & & \\
v \approx_{\sigma_1 \to \sigma_2} v' & \text{iff} & \text{for all } v_1 \approx_{\sigma_1} v'_1,\ v\ v_1 \sim_{\sigma_2} v'\ v'_1 \\
v \approx_{\forall t::\Omega.\sigma} v' & \text{iff} & \text{for all } \tau \simeq \tau', \text{ then } v\ \tau \sim_{\sigma[\tau/t]} v'\ \tau'
\end{array}$$

We extend the relation to substitutions as follows:

$$\begin{array}{rll}
\gamma \approx_{\{x_1:\sigma_1,\ldots,x_n:\sigma_n\}} [\langle v_1, \ldots, v_n\rangle/x_{\text{ve}}] & \text{iff} & \gamma(x_i) \approx_{\sigma_i} v_i. \\
\gamma \approx_{\Gamma;\{x_1:\sigma_1,\ldots,x_n:\sigma_n\}} [v'/x_{\text{ve}}, v_1/x_1, \ldots, v_n/x_n] & \text{iff} & \gamma \approx_\Gamma [v'/x_{\text{ve}}] \text{ and } \gamma(x_i) \approx_{\sigma_i} v_i.
\end{array}$$

We must also define a relation between constructor substitutions indexed by source kind assignments. Since source kind assignments only bind type variables to the kind Ω, we omit the kinds below:

$$\begin{array}{rll}
\delta \simeq_{\{t_1,\ldots,t_n\}} [\langle \tau_1, \ldots, \tau_n\rangle/t_{\text{te}}] & \text{iff} & \delta(t_i) \simeq \tau_i. \\
\delta \simeq_{\Delta;\{t_1,\ldots,t_n\}} [\tau'/t_{\text{te}}, \tau_1/t_1, \ldots, \tau_n/t_n] & \text{iff} & \delta \simeq_\Gamma [\tau'/t_{\text{te}}] \text{ and } \delta(t_i) \simeq \tau_i{::}\Omega.
\end{array}$$

An important property of the type translation is that by applying related type substitutions, a type and its translation result in types related by $\simeq$.

Lemma 6 *If $\delta \simeq_{\Delta_{\text{env}};\Delta_{\text{arg}}} \delta'$ and $\Delta_{\text{env}};\Delta_{\text{arg}} \rhd \tau \leadsto \tau'$, then $\delta(\tau) \simeq \delta'(\tau')$.*

Proof. By induction on the derivation of $\Delta_{\text{env}};\Delta_{\text{arg}} \rhd \tau \leadsto \tau'$.

Case: τ is t_j where Δ_{env} is $\{t_1,\ldots,t_n\}$. Then $\tau' \equiv \pi_j(t_{\text{te}})$. By definition, $\delta'(t_{\text{te}}) \equiv \langle \tau_1,\ldots,\tau_n\rangle$ such that $\delta(t_i) \simeq \tau_i$ for $1 \le i \le n$. Since $\vdash \pi_j(\delta'(t_{\text{te}})) \equiv \tau_j$, $\delta(t_i) \simeq \delta'(\pi_j(t_{\text{te}}))$.

Case: τ is $t \in \Delta_{\text{arg}}$. Then $\tau' \equiv t$. By definition $\delta(t) \simeq \delta'(t)$.

Case: τ is $\tau_1 \to \tau_2$. By induction.

□

Lemma 7 *If $\delta \simeq_{\Delta_{\text{env}};\Delta_{\text{arg}}} \delta'$ and $\Delta_{\text{env}};\Delta_{\text{arg}} \rhd \Delta'_{\text{env}} \leadsto \tau_{\text{te}}$, then $\delta \simeq_{\Delta_{\text{env}}} [\delta'(\tau_{\text{te}})/t_{\text{te}}]$.*

Proof. Let Δ'_{env} be $\{t_1{::}\Omega,\ldots,t_n{::}\Omega\}$. Then τ_{te} is $\langle \tau_1,\ldots,\tau_n\rangle$ such that $\Delta_{\text{env}};\Delta_{\text{arg}} \rhd t_i \leadsto \tau_i$. Then it is sufficient if we show $\delta(t_i) \simeq \delta'(\tau_i)$. This follows from the previous lemma.

□

Lemma 8 *Let $\delta \simeq_{\Delta_{\text{env}};\Delta_{\text{arg}}} \delta'$ and $\gamma \approx_{\delta(\Gamma_{\text{env}};\Gamma_{\text{arg}})} \gamma'$.*

1. *If $\Delta_{\text{env}} \uplus \Delta_{\text{arg}};\Gamma_{\text{env}} \uplus \Gamma_{\text{arg}} \vdash x : \sigma$ and $\Delta_{\text{env}};\Delta_{\text{arg}};\Gamma_{\text{env}};\Gamma_{\text{arg}} \rhd x \leadsto e_0$, then $\gamma'(e_0) \hookrightarrow v$ and $\gamma(x) \approx_{\delta(\sigma)} v$.*
2. *If $\Delta_{\text{env}};\Delta_{\text{arg}};\Gamma_{\text{env}};\Gamma_{\text{arg}} \rhd \Gamma'_{\text{env}} \leadsto e_{\text{ve}}$, then $\gamma'(e_{\text{ve}}) \hookrightarrow v$ and $\gamma \approx_{\delta(\Gamma'_{\text{env}})} [v/x_{\text{ve}}]$.*

With these lemmas in hand, we may prove that the translation preserves the operational behavior of a program.

Theorem 6 (Operational Correctness)
Let $\delta \simeq_{\Delta_{\text{env}};\Delta_{\text{arg}}} \delta'$ and $\gamma \approx_{\delta(\Gamma_{\text{env}};\Gamma_{\text{arg}})} \gamma'$. If $\Delta_{\text{env}};\Delta_{\text{arg}};\Gamma_{\text{env}};\Gamma_{\text{arg}} \rhd e \leadsto e'$ and $\Delta_{\text{env}} \uplus \Delta_{\text{arg}};\Gamma_{\text{env}} \uplus \Gamma_{\text{arg}} \vdash e : \sigma$ then $\delta(\gamma(e)) \sim_{\delta(\sigma)} \delta'(\gamma'(e'))$.

Proof. By induction of the derivation of $\Delta_{\text{env}};\Delta_{\text{arg}};\Gamma_{\text{env}};\Gamma_{\text{arg}} \rhd e \leadsto e'$.

Case (*arg*) and (*env*). Clear from Lemma 8.

Case (*abs*). Let $v \approx_{\delta(\sigma_1)} v'$ and $\delta'(\gamma'(e_{\text{ve}})) \hookrightarrow v_0$. By Lemma 8, $\gamma[v/x] \approx_{\delta(\Gamma'_{\text{env}};\{x:\sigma_1\})} [v_0/x_{\text{ve}}, v'/x]$. Let $\vdash \delta'(\tau_{\text{te}}) \equiv \tau$. By Lemma 7, $\delta \simeq_{\Delta'_{\text{env}};\emptyset} [\tau/t_{\text{te}}]$. Then by the induction hypothesis, $\delta(\gamma[v/x](e)) \sim_{\delta(\sigma_2)} [\tau/t_{\text{te}}]([v_0/x_{\text{ve}}, v'/x](e'))$. Then

$$\delta(\gamma(\lambda x{:}\sigma_1.e)) \sim_{\delta(\sigma_1\to\sigma_2)} \delta'(\gamma'(\langle\!\langle \Lambda t_{\text{te}}{::}|\Delta'_{\text{env}}|.\lambda x_{\text{ve}}{:}|\Gamma''_{\text{env}}|.\lambda x{:}\sigma'_1.e', \tau, e_{\text{ve}}\rangle\!\rangle)).$$

Case (*tabs*). Let $\delta'(\gamma'(e_{\text{ve}})) \hookrightarrow v_0$. By Lemma 8, $\gamma \approx_{\delta(\Gamma'_{\text{env}};\emptyset)} [v_0/x_{\text{ve}}]$. Let $\vdash \tau_0 \equiv \tau'_0{::}\Omega$ and $\vdash \delta'(\tau_{\text{te}}) \equiv \tau$. By Lemma 7, $\delta[\tau_0/t] \simeq_{\Delta'';t} [\tau/t_{\text{te}}, \tau'_0/t]$. Then by induction hypothesis, $\delta[\tau_0/t](\gamma(e)) \sim_{\delta(\sigma_2)} [\tau/t_{\text{te}}, \tau'_0/t]([v_0/y](e'))$. Then,

$$\delta(\gamma(\Lambda t{::}\Omega.e)) \sim_{\delta(\forall t.\sigma_2)} \delta'(\gamma'(\langle\!\langle \Lambda t_{\text{te}}{::}|\Delta'_{\text{env}}|.\lambda x_{\text{ve}}{:}|\Gamma''_{\text{env}}|.\Lambda t{::}\Omega.e', \tau, e_{\text{ve}}\rangle\!\rangle)).$$

Case (*tapp*). Let $\Delta_{\text{env}};\Delta_{\text{arg}};\Gamma_{\text{env}};\Gamma_{\text{arg}} \vdash e\tau \leadsto e'\tau'$. By induction hypothesis, $\delta\gamma(e) \hookrightarrow v$, $\delta'\gamma'(e') \hookrightarrow v'$, and $v \approx_{\forall t.\sigma_2} v'$. By Lemma 6, $\delta(\tau) \simeq \delta'(\tau')$. Then $v\delta(\tau) \sim_{\delta(\sigma_2[\sigma/t])} v'\tau'$. Hence $\delta\gamma(e\tau) \sim_{\delta(\sigma_2[\tau/t])} \delta'\gamma'(e'\tau')$.

Case (*app*). By induction hypothesis, $\delta\gamma(e_1) \hookrightarrow v_1$, $\delta'\gamma'(e'_1) \hookrightarrow v'_1$, and $v_1 \approx_{\sigma_1\to\sigma_2} v'_1$. By induction hypothesis, $\delta\gamma(e_2) \hookrightarrow v_2$, $\delta'\gamma'(e'_2) \hookrightarrow v'_2$, and $v_2 \approx_{\sigma_1} v'_2$. Then $v_1 v_2 \sim_{\sigma_2} v'_1 v'_2$. Hence, $\delta\gamma(e_1 e_2) \sim_{\sigma_2} \delta'\gamma'(e'_1 e'_2)$.

□

5.2 Closure Representation

In this section we present closure representation for the second order language. We use types with existential *kinds* to abstract the representation of type environments and existential types to abstract the representation of value environments. Further, we introduce the type $\sigma \Rightarrow \sigma'$, derived from translucent types, to solve the typing problems discussed in the overview.

5.2.1 The Target Language

The target language for polymorphic closure representation, called $\lambda^{\forall,\exists}$, is defined as follows:

$$\begin{array}{lll}
\textit{Kinds} & \kappa ::= & k \mid \Omega \mid \langle \kappa_1 \times \ldots \times \kappa_n \rangle \\
\textit{Types} & \sigma ::= & b \mid t \mid \langle \sigma_1 \times \ldots \times \sigma_n \rangle \mid \langle \sigma_1, \ldots, \sigma_n \rangle \mid \pi_i\ \sigma \mid \\
& & \forall t{::}\kappa.\sigma \mid \sigma_1 \Rightarrow \sigma_2 \mid \sigma_1 \rightarrow \sigma_2 \mid \exists t{::}\kappa.\sigma \mid \exists k.\sigma \\
\textit{Exp's} & e ::= & x \mid c \mid \lambda x{:}\sigma.e \mid e_1\ e_2 \mid \Lambda t{::}\kappa.e \mid e\ \sigma \mid \\
& & \langle e_1, \ldots, e_n \rangle \mid \pi_i\ e \mid \\
& & \texttt{pack}\ \sigma\ \texttt{with}\ e\ \texttt{as}\ \sigma' \mid \\
& & \texttt{open}\ e\ \texttt{as}\ t{::}\kappa\ \texttt{with}\ x{:}\sigma\ \texttt{in}\ e' \\
& & \texttt{pack}\ \kappa\ \texttt{with}\ e\ \texttt{as}\ \sigma \mid \\
& & \texttt{open}\ e\ \texttt{as}\ k\ \texttt{with}\ x{:}\sigma\ \texttt{in}\ e'
\end{array}$$

There is no distinction between types and constructors for $\lambda^{\forall,\exists}$ because type application is no longer restricted to just monotypes. $\lambda^{\forall,\exists}$ needs this impredicativity because some monotypes from the source language are translated into types with quantifiers. To simplify the language, we provide full abstractions (λ and Λ) instead of codes which abstract more than one argument at a time.

To provide types with existential kinds, we need to introduce kind variables k and kind contexts for the typing judgements of $\lambda^{\forall,\exists}$. A kind context $\mathcal{K}$ is simply a sequence of kind variables, $\{k_1, \ldots, k_n\}$, $(n \geq 0)$. The typing judgements of the language consist of the following:

$$\begin{array}{ll}
\mathcal{K}; \Delta \vdash \sigma :: \kappa & \sigma \text{ has kind } \kappa. \\
\mathcal{K}; \Delta \vdash \sigma_1 \equiv \sigma_2 :: \kappa & \sigma_1 \text{ and } \sigma_2 \text{ are equal types of kind } \kappa. \\
\mathcal{K}; \Delta; \Gamma \vdash e : \sigma & e \text{ has type } \sigma.
\end{array}$$

The typing rules are defined in Figures 11 and 12. As described in the overview, the typing rule for type applications is split into two rules: (*ins*) and (*tapp*). The rule (*ins*) restricts the domain of type application to the specific type σ'. This corresponds to restricting the type from $\forall t{::}\kappa.\sigma$ to $\forall t = \sigma'{::}\kappa.\sigma$ and propagating the equivalence $t \equiv \sigma'$ into σ. The actual type application for $\sigma' \Rightarrow \sigma$ is permitted only for the type σ' in the rule (*tapp*). Other interesting rules are (*kpk*) and (*kop*) for types with existential kinds. They are almost analogous to the rules for ordinary existential types. However, we have to check that the type abstracted on a kind, $\exists k.\sigma$, is well-formed because this is not necessarily true even if $\sigma[\kappa/k]$ is well-formed. For example, $\forall t{::}\langle \Omega \times \Omega \rangle.\pi_1(t)$ is a well-formed type, but $\exists k.\forall t{::}k.\pi_1(t)$ is not.

The operational semantics of the language is defined in Figure 13.

Lemma 9 (Substitution)

1. *If* $\mathcal{K} \uplus \{k\}; \Delta; \Gamma \vdash e : \sigma$, *then* $\mathcal{K}; \Delta[\kappa/k]; \Gamma[\kappa/k] \vdash e[\kappa/k] : \sigma[\kappa/k]$.
2. *If* $\mathcal{K}; \Delta \uplus \{t{::}\kappa\}; \Gamma \vdash e : \sigma$ *and* $\mathcal{K}; \Delta \vdash \sigma_0 :: \kappa$, *then* $\mathcal{K}; \Delta; \Gamma[\sigma_0/t] \vdash e[\sigma_0/t] : \sigma[\sigma_0/t]$.
3. *If* $\mathcal{K}; \Delta; \Gamma \uplus \{x{:}\sigma_0\} \vdash e : \sigma$ *and* $\mathcal{K}; \Delta; \Gamma \vdash e_0 : \sigma_0$, *then* $\mathcal{K}; \Delta; \Gamma \vdash e[e_0/x] : \sigma$.

Well-formedness of types:

$$\mathcal{K};\Delta \vdash b :: \Omega \qquad \mathcal{K};\Delta \uplus \{t{::}\kappa\} \vdash t :: \kappa \quad (t \notin Dom(\Delta))$$

$$\frac{\mathcal{K};\Delta \vdash \sigma' :: \Omega \quad \mathcal{K};\Delta \vdash \sigma :: \Omega}{\mathcal{K};\Delta \vdash \sigma' \to \sigma :: \Omega} \qquad \frac{\mathcal{K};\Delta \vdash \sigma_1 :: \kappa_1 \quad \cdots \quad \mathcal{K};\Delta \vdash \sigma_n :: \kappa_n}{\mathcal{K};\Delta \vdash \{\sigma_1,\ldots,\sigma_n\} :: \langle \kappa_1 \times \ldots \times \kappa_n\rangle}$$

$$\frac{\mathcal{K};\Delta \vdash \sigma :: \langle \kappa_1 \times \ldots \times \kappa_n\rangle}{\mathcal{K};\Delta \vdash \pi_i\sigma :: \kappa_i} \quad (1 \le i \le n) \qquad \frac{\mathcal{K};\Delta \vdash \sigma_1 \quad \cdots \quad \mathcal{K};\Delta \vdash \sigma_n}{\mathcal{K};\Delta \vdash \langle \sigma_1 \times \ldots \times \sigma_n\rangle :: \Omega}$$

$$\frac{\mathcal{K};\Delta \uplus \{t{::}\kappa\} \vdash \sigma :: \Omega}{\mathcal{K};\Delta \vdash \forall t{::}\kappa.\sigma :: \Omega} \quad (t \notin Dom(\Delta)) \qquad \frac{\mathcal{K};\Delta \vdash \sigma' :: \kappa \quad \mathcal{K};\Delta \vdash \sigma :: \Omega}{\mathcal{K};\Delta \vdash \sigma' \Rightarrow \sigma :: \Omega}$$

$$\frac{\mathcal{K};\Delta \uplus \{t{::}\kappa\} \vdash \sigma :: \Omega}{\mathcal{K};\Delta \vdash \exists t{::}\kappa.\sigma :: \Omega} \quad (t \notin Dom(\Delta)) \qquad \frac{\mathcal{K} \uplus \{k\};\Delta \vdash \sigma :: \Omega}{\mathcal{K};\Delta \vdash \exists k.\sigma :: \Omega} \quad (k \notin \mathcal{K})$$

Primary rules for equivalence of types:

$$\mathcal{K};\Delta \vdash \pi_i(\{\sigma_1,\ldots,\sigma_n\}) \equiv \sigma_i :: \kappa_i \qquad \mathcal{K};\Delta \vdash \{\pi_1(\sigma),\ldots,\pi_n(\sigma)\} \equiv \sigma :: \langle \kappa_1 \times \ldots \times \kappa_n\rangle$$

Figure 11: Formation and Equivalence of Types for $\lambda^{\forall,\exists}$

The following lemma is implicitly used throughout the proof of the Type Preservation Theorem (below).

Lemma 10

1. *If* $\mathcal{K};\Delta \vdash \sigma_1 \to \sigma_2 \equiv \sigma_1' \to \sigma_2' :: \Omega$, *then* $\mathcal{K};\Delta \vdash \sigma_1 \equiv \sigma_1' :: \Omega$ *and* $\mathcal{K};\Delta \vdash \sigma_2 \equiv \sigma_2' :: \Omega$.
2. *If* $\mathcal{K};\Delta \vdash \sigma_1 \Rightarrow \sigma_2 \equiv \sigma_1' \Rightarrow \sigma_2' :: \Omega$, *then* $\mathcal{K};\Delta \vdash \sigma_1 \equiv \sigma_1' :: \Omega$ *and* $\mathcal{K};\Delta \vdash \sigma_2 \equiv \sigma_2' :: \Omega$.

Proof. The lemma is proved by defining reduction of types by $\pi_i\langle\sigma_1,\ldots,\sigma_n\rangle \mapsto \sigma_i$ and $\langle\pi_1(\sigma),\ldots,\pi_n(\sigma)\rangle \mapsto \sigma$. The reduction is Church-Rosser and strong-normalizing. Thus, we have a decision procedure for $\mathcal{K};\Delta \vdash \sigma_1 \equiv \sigma_2{::}\kappa$.

□

Theorem 7 (Type Preservation) *If* $\emptyset;\emptyset;\emptyset \vdash e : \sigma$ *and* $e \hookrightarrow v$, *then* $\emptyset;\emptyset;\emptyset \vdash v : \sigma$.

Proof. By induction on the derivation of $e \hookrightarrow v$. In the proof, we write $\vdash e : \sigma$ instead of $\emptyset;\emptyset;\emptyset \vdash e : \sigma$. We assume the last rule used to drive $\vdash e : \sigma$ is not the rule *(teq)*.

If the last rule is *(teq)*, there exists a derivation of $\vdash e : \sigma'$ such that $\vdash \sigma \equiv \sigma'{::}\Omega$ and the last rule is not (teq). Then, $\vdash v : \sigma'$. Hence, $\vdash v : \sigma$.

Here we show only cases for application, type application, and open expressions.

Case: $e_1\ e_2 \hookrightarrow v$ is derived from $e_1 \hookrightarrow (\lambda x{:}\sigma_1.e)$ and $e_2 \hookrightarrow v_2$ and $e[v_2/x] \hookrightarrow v$. Let $\vdash e_1\ e_2{:}\sigma_2$ be derived from $\vdash e_1 : \sigma_1 \to \sigma_2$ and $\vdash e_2 : \sigma_2$. By the first induction hypothesis, $\vdash (\lambda x{:}\sigma_1.e) : \sigma_1 \to \sigma_2$. and $\vdash v_2 : \sigma_1$. Then $\emptyset;\emptyset;\{x{:}\sigma_1\} \vdash e : \sigma_2$. Hence, by the substitution lemma, $\vdash e[v_2/x] : \sigma_2$. Then by the second induction hypothesis, $\vdash v : \sigma_2$.

$$(ins)\quad \frac{\mathcal{K};\Delta \vdash \sigma'{::}\kappa \quad \mathcal{K};\Delta;\Gamma \vdash e : \forall t{::}\kappa.\sigma}{\mathcal{K};\Delta;\Gamma \vdash e : \sigma' \Rightarrow \sigma[\sigma'/t]} \qquad (tapp)\quad \frac{\mathcal{K};\Delta;\Gamma \vdash e : \sigma' \Rightarrow \sigma}{\mathcal{K};\Delta;\Gamma \vdash e\ \sigma' : \sigma}$$

$$(var)\quad \mathcal{K};\Delta;\Gamma \uplus \{x{:}\sigma\} \vdash x : \sigma \qquad (const)\quad \mathcal{K};\Delta;\Gamma \vdash c : b$$

$$(tuple)\quad \frac{\Delta;\Gamma \vdash e_1 : \sigma_1 \quad \cdots \quad \Delta;\Gamma \vdash e_n : \sigma_n}{\Delta;\Gamma \vdash \langle e_1, \ldots, e_n\rangle : \langle \sigma_1, \ldots, \sigma_n\rangle}$$

$$(proj)\quad \frac{\Delta;\Gamma \vdash e : \langle \sigma_1 \times \ldots \times \sigma_n\rangle}{\Delta;\Gamma \vdash \pi_i\, e : \sigma_i} \quad (1 \leq i \leq n)$$

$$(abs)\quad \frac{\mathcal{K};\Delta;\Gamma \uplus \{x{:}\sigma'\} \vdash e : \sigma}{\mathcal{K};\Delta;\Gamma \vdash \lambda x{:}\sigma'.e : \sigma' \to \sigma} \quad (x \notin Dom(\Gamma))$$

$$(app)\quad \frac{\mathcal{K};\Delta;\Gamma \vdash e_1 : \sigma' \to \sigma \quad \mathcal{K};\Delta;\Gamma \vdash e_2 : \sigma'}{\mathcal{K};\Delta;\Gamma \vdash e_1 e_2 : \sigma}$$

$$(tabs)\quad \frac{\mathcal{K};\Delta \uplus \{t{::}\kappa\};\Gamma \vdash e : \sigma}{\mathcal{K};\Delta;\Gamma \vdash \Lambda t{::}\kappa.e : \forall t{::}\kappa.\sigma} \quad (t \notin Dom(\Delta))$$

$$(teq)\quad \frac{\mathcal{K};\Delta;\Gamma \vdash e : \sigma' \quad \mathcal{K};\Delta \vdash \sigma \equiv \sigma'{::}\Omega}{\mathcal{K};\Delta;\Gamma \vdash e : \sigma}$$

$$(tpk)\quad \frac{\mathcal{K};\Delta \vdash \sigma'{::}\kappa \quad \mathcal{K};\Delta;\Gamma \vdash e : \sigma[\sigma'/t]}{\mathcal{K};\Delta;\Gamma \vdash \texttt{pack}\ \sigma'\ \texttt{with}\ e\ \texttt{as}\ \exists t{::}\kappa.\sigma : \exists t{::}\kappa.\sigma}$$

$$(top)\quad \frac{\mathcal{K};\Delta;\Gamma \vdash e : \exists t{::}\kappa.\sigma \quad \mathcal{K};\Delta \uplus \{t{::}\kappa\};\Gamma \uplus \{x{:}\sigma\} \vdash e' : \sigma'}{\mathcal{K};\Delta;\Gamma \vdash \texttt{open}\ e\ \texttt{as}\ t{::}\kappa\ \texttt{with}\ x{:}\sigma\ \texttt{in}\ e' : \sigma'} \quad (t \notin FTV(\sigma'), x \notin Dom(\Gamma))$$

$$(kpk)\quad \frac{\mathcal{K};\Delta \vdash \exists k.\sigma{::}\Omega \quad \mathcal{K};\Delta;\Gamma \vdash e : \sigma[\kappa/k]}{\mathcal{K};\Delta;\Gamma \vdash \texttt{pack}\ \kappa\ \texttt{with}\ e\ \texttt{as}\ \exists k.\sigma : \exists k.\sigma}$$

$$(kop)\quad \frac{\mathcal{K};\Delta;\Gamma \vdash e : \exists k.\sigma \quad \mathcal{K} \uplus k;\Delta;\Gamma \uplus \{x{:}\sigma\} \vdash e' : \sigma'}{\mathcal{K};\Delta;\Gamma \vdash \texttt{open}\ e\ \texttt{as}\ k\ \texttt{with}\ x{:}\sigma\ \texttt{in}\ e' : \sigma'} \quad (k \notin FKV(\sigma'), x \notin Dom(\Gamma))$$

Figure 12: Typing Rules of $\lambda^{\forall,\exists}$

$$v \hookrightarrow v$$

$$\frac{e_1 \hookrightarrow v_1 \quad \ldots \quad e_n \hookrightarrow v_n}{\langle e_1, \ldots, e_n \rangle \hookrightarrow \langle v_1, \ldots, v_n \rangle} \qquad \frac{e \hookrightarrow \langle v_1, \ldots, v_n \rangle}{\pi_i\, e \hookrightarrow v_i}$$

$$\frac{e_1 \hookrightarrow \lambda x{:}\sigma_1.e \quad e_2 \hookrightarrow v_2 \quad e[v_2/x] \hookrightarrow v}{e_1\ e_2 \hookrightarrow v} \qquad \frac{e_1 \hookrightarrow \Lambda t{::}\kappa.e \quad e[\sigma/t] \hookrightarrow v}{e_1\ \sigma \hookrightarrow v}$$

$$\frac{e \hookrightarrow v}{\texttt{pack}\ \sigma\ \texttt{with}\ e\ \texttt{as}\ \sigma'' \hookrightarrow \texttt{pack}\ \sigma\ \texttt{with}\ v\ \texttt{as}\ \sigma''}$$

$$\frac{e_1 \hookrightarrow \texttt{pack}\ \sigma_1\ \texttt{with}\ v\ \texttt{as}\ \sigma' \quad e_2[\sigma_1/t][v/x] \hookrightarrow v}{\texttt{open}\ e_1\ \texttt{as}\ t{::}\kappa\ \texttt{with}\ x{:}\sigma\ \texttt{in}\ e_2 \hookrightarrow v}$$

$$\frac{e \hookrightarrow v}{\texttt{pack}\ \kappa\ \texttt{with}\ e\ \texttt{as}\ \sigma' \hookrightarrow \texttt{pack}\ \kappa\ \texttt{with}\ v\ \texttt{as}\ \sigma'}$$

$$\frac{e_1 \hookrightarrow \texttt{pack}\ \kappa\ \texttt{with}\ v\ \texttt{as}\ \sigma' \quad e_2[\kappa/k][v/x] \hookrightarrow v}{\texttt{open}\ e_1\ \texttt{as}\ k\ \texttt{with}\ x{:}\sigma\ \texttt{in}\ e_2 \hookrightarrow v}$$

Figure 13: Operational Semantics of $\lambda^{\forall,\exists}$

Case: $e_1\ \sigma_2 \hookrightarrow v$ is derived from $e_1 \hookrightarrow (\Lambda t{::}\kappa.e)$, and $[\sigma_2/t]e \hookrightarrow v$. Let $\vdash e_1\ \sigma_2 : \sigma$ be derived from $\vdash e_1 : \sigma_2 \Rightarrow \sigma$. By the first induction hypothesis, $\vdash \Lambda t{::}\kappa.e : \sigma_2 \Rightarrow \sigma$. Hence, $\vdash \Lambda t{::}\kappa.e : \forall t{::}\kappa.\sigma'$ such that $\sigma'[\sigma_2/t] \equiv \sigma$. Then $\emptyset; \{t{::}\kappa\}; \emptyset \vdash e : \sigma'$. By the substitution lemma, $\vdash e[\sigma_2/t] : \sigma'[\sigma_2/t]$. Hence $\vdash e[\sigma_2'/t] : \sigma$. Then by induction hypothesis, $\vdash v : \sigma$.

Case: `open` e_1 `as` k `with` $x{:}\sigma_1$ `in` $e_2 \hookrightarrow v$ is derived from $e_1 \hookrightarrow$ `pack` κ `with` v_1 `as` $\exists k_1.\sigma_1$ and $e_2[\kappa/k][v_1/x] \hookrightarrow v$. By the definition of the typing derivation, $\vdash e_1 : \exists k_1.\sigma_1$. By the induction hypothesis, $\vdash$ `pack` κ `with` v_1 `as` $\exists k_1.\sigma_1 : \exists k_1.\sigma_1$. Hence, $\vdash v_1 : \sigma[\kappa/k]$. On the other hand, from $\{k\}; \emptyset; \{x{:}\sigma_1\} \vdash e_2 : \sigma$, then by the substitution lemma, $\vdash e_2[\kappa/k][v_1/x] : \sigma$, taking advantage of the fact that $k \notin FKV(\sigma)$. Thus, $\vdash v : \sigma$.

Case: `open` e_1 `as` $t{::}\kappa$ `with` $x{:}\sigma_1$ `in` $e_2 \hookrightarrow v$ is derived from $e_1 \hookrightarrow$ `pack` σ_2 `with` v_1 `as` $\exists k_1.\sigma_1$ and $e_2[\sigma_2/k][v_1/x] \hookrightarrow v$. By the definition of the typing derivation, $\vdash e : \exists t{::}\kappa.\sigma_1$. By the induction hypothesis, $\vdash$ `pack` σ_2 `with` v_1 `as` $\exists t{::}\kappa.\sigma_1 : \exists t{::}\kappa.\sigma_1$. Hence, $\vdash v_1 : \sigma[\sigma_2/t]$. On the other hand, from $\emptyset; \{t{::}\kappa\}; \{x{:}\sigma_1\} \vdash e_2 : \sigma$, then by the substitution lemma, $\vdash e_2[\sigma_2/t][v_1/x] : \sigma$, taking advantage of the fact that $t \notin FTV(\sigma)$. Thus, $\vdash v : \sigma$.

□

5.2.2 The Translation

We define the closure representation stage as a type-directed translation from $\lambda^{\forall,cl}$ to $\lambda^{\forall,\exists}$. We begin by defining a translation from source constructors and types to target type as follows:

$$\begin{array}{rcl}
|t| & = & t \\
|b| & = & b \\
|\langle\sigma_1,\ldots,\sigma_n\rangle| & = & \langle|\sigma_1|,\ldots,|\sigma_n|\rangle \\
|\pi_i\,\sigma| & = & \pi_i\,|\sigma| \\
|\langle\sigma_1\times\ldots\times\sigma_n\rangle| & = & \langle|\sigma_1|\times\ldots\times|\sigma_n|\rangle \\
|\mathtt{vcode}(t{::}\kappa,\sigma_{\mathrm{ve}},\sigma_1,\sigma_2)| & = & \forall t{::}\kappa.|\sigma_{\mathrm{ve}}|\to|\sigma_1|\to|\sigma_2| \\
|\mathtt{tcode}(t{::}\kappa,\sigma_{\mathrm{ve}},s{::}\kappa',\sigma_2)| & = & \forall t{::}\kappa.|\sigma_{\mathrm{ve}}|\to\forall s{::}\kappa'.|\sigma_2| \\
|\sigma_1\to\sigma_2| & = & \exists k.\exists t{::}k.\exists t_0{::}\Omega.\langle(t\Rightarrow t_0\to|\sigma_1|\to|\sigma_2|)\times t_0\rangle \\
|\forall s{::}\kappa.\sigma_2| & = & \exists k.\exists t{::}k.\exists t_0{::}\Omega.\langle(t\Rightarrow t_0\to\forall s{::}\kappa.|\sigma_2|)\times t_0\rangle
\end{array}$$

The code types are translated to the corresponding types described in the previous section. The translation of a function type abstracts the kind of the type environment, k, and the type of the value environment, t_0. The type environment t is paired with the code by using an existential type. Since the type of a code is instantiated by t, only the type environment of the closure can be given to the code. The code and the value environment are paired as in the simply-typed case. The translation of $\forall$ has the same structure as that of an arrow type.

The translation of expressions is given in Figure 14. To simplify the presentation, we introduce the following derived forms for expressions:

```
pack $\kappa,\sigma_1,\sigma_2$ with $e$ as $\exists k.\exists t{::}k.\exists t_0{::}\Omega.\sigma\equiv$
    pack $\kappa$ with
        pack $\sigma_1$ with
            pack $\sigma_2$ with $e$ as $\forall t_0{::}\Omega.\sigma[\kappa/k][\sigma_1/1]$
        as $\exists t{::}\kappa.\exists t_0{::}\Omega.\sigma[\kappa/k]$
    as $\exists k.\exists t{::}k.\exists t_0{::}\Omega.\sigma$

open $e$ as $k,t,t_0$ with $x{:}\sigma$ in $e'\equiv$
    open $e$ as $k$ with $y{:}\exists t{::}k.\exists t_0{::}\Omega.\sigma$
        in open $y$ as $t{::}k$ with $z{:}\exists t_0{::}\Omega.\sigma$
            in open $z$ as $t_0{::}\Omega$ with $x{:}\sigma$ in $e'$
```

The kind of the type environment, the type environment, and the type of the value environment are packed with the pair of the code and the value environment. In the translation of applications, the type environment is obtained from a closure by an **open** expression and the code and the value environment are obtained by projections. Then the type environment, the value environment, and the argument of application are passed to the code.

It is straightforward to show that the type translation preserves the equality of types and commutes with substitutions.

Lemma 11 *1. If $\Delta\vdash\sigma\equiv\sigma'$ in $\lambda^{\forall,cl}$, then $\emptyset;\Delta\vdash|\sigma|\equiv|\sigma'|{::}\Omega$ in $\lambda^{\forall,\exists}$.*

2. If $\Delta\vdash\tau\equiv\tau'{::}\kappa$ in $\lambda^{\forall,cl}$, then $\emptyset;\Delta\vdash|\tau|\equiv|\tau'|{::}\kappa$ in $\lambda^{\forall,\exists}$.

Lemma 12 $|\sigma|[|\tau|/t]\equiv|\sigma[\tau/t]|$.

$$(var)\quad \Delta;\Gamma \triangleright x : \sigma \rightsquigarrow x \qquad (const)\quad \Delta;\Gamma \triangleright c : b \rightsquigarrow c$$

$$(prod)\quad \frac{\Delta;\Gamma \triangleright e_1 : \sigma_1 \rightsquigarrow e_1' \cdots \Delta;\Gamma \triangleright e_n : \sigma_n \rightsquigarrow e_n'}{\Delta;\Gamma \triangleright \langle e_1,\ldots,e_n\rangle : \langle \sigma_1 \times \ldots \times \sigma_n\rangle \rightsquigarrow \langle e_1',\ldots,e_n'\rangle}$$

$$(proj)\quad \frac{\Delta;\Gamma \triangleright e : \langle \sigma_1 \times \ldots \times \sigma_n\rangle \rightsquigarrow e'}{\Delta;\Gamma \triangleright \pi_i\, e : \sigma_i \rightsquigarrow \pi_i\, e'} \quad (1 \leq i \leq n)$$

$$(vcode)\quad \frac{\{t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}\}; \{x_{\mathrm{ve}}{:}\sigma_{\mathrm{ve}}, x{:}\sigma_1\} \triangleright e : \sigma_2 \rightsquigarrow e'}{\begin{array}{c}\Delta;\Gamma \triangleright (\Lambda t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}.\lambda x_{\mathrm{ve}}{:}\sigma_{\mathrm{ve}}.\lambda x{:}\sigma_1.e) : \mathtt{vcode}(t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}, \sigma_{\mathrm{ve}}, \sigma_1, \sigma_2) \rightsquigarrow \\ \Lambda t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}.\lambda x_{\mathrm{ve}}{:}|\sigma_{\mathrm{ve}}|.\lambda x{:}|\sigma_1|.e'\end{array}}$$

$$(tcode)\quad \frac{\{t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}, t{::}\kappa\}; \{x_{\mathrm{ve}}{:}\sigma_{\mathrm{ve}}\} \triangleright e : \sigma \rightsquigarrow e'}{\begin{array}{c}\Delta;\Gamma \triangleright (\Lambda t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}.\lambda x_{\mathrm{ve}}{:}\sigma_{\mathrm{ve}}.\Lambda t{::}\kappa.e) : \mathtt{tcode}(t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}, \sigma_{\mathrm{ve}}, t{::}\kappa, \sigma) \rightsquigarrow \\ \Lambda t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}.\lambda x_{\mathrm{ve}}{:}|\sigma_{\mathrm{ve}}|.\Lambda t{::}\kappa.e'\end{array}}$$

$$(vcl)\quad \frac{\Delta;\Gamma \triangleright e : \mathtt{vcode}(t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}, \sigma_{\mathrm{ve}}', \sigma_1', \sigma_2') \rightsquigarrow e' \quad \Delta;\Gamma \triangleright e_{\mathrm{ve}} : \sigma_{\mathrm{ve}} \rightsquigarrow e_{\mathrm{ve}}'}{\Delta;\Gamma \triangleright \langle\!\langle e, \tau, e_{\mathrm{ve}}\rangle\!\rangle : \sigma_1 \rightarrow \sigma_2 \rightsquigarrow \mathtt{pack}\ \kappa_{\mathrm{te}}, |\tau|, |\sigma_{\mathrm{ve}}|\ \mathtt{with}\ \langle e', e_{\mathrm{ve}}'\rangle\ \mathtt{as}\ |\sigma_1 \rightarrow \sigma_2|}$$

$$(app)\quad \frac{\Delta;\Gamma \triangleright e_1 : \sigma_1 \rightarrow \sigma_2 \rightsquigarrow e_1' \quad \Delta;\Gamma \triangleright e_2 : \sigma_1 \rightsquigarrow e_2'}{\begin{array}{ll}\Delta;\Gamma \triangleright e_1\ e_2 : \sigma_2 \rightsquigarrow & \mathtt{open}\ e_1'\ \mathtt{as}\ k_{\mathrm{te}}, t_{\mathrm{te}}, t_{\mathrm{ve}} \\ & \quad \mathtt{with}\ y : \langle t_{\mathrm{te}} \Rightarrow t_{\mathrm{ve}} \rightarrow |\sigma_1| \rightarrow |\sigma_2| \times t_{\mathrm{ve}}\rangle \\ & \mathtt{in}\ (\pi_1\, y)\ t_{\mathrm{te}}\ (\pi_2\, y)\ e_2'\end{array}}$$

$$(tcl)\quad \frac{\Delta;\Gamma \triangleright e : \mathtt{tcode}(t_{\mathrm{te}}{::}\kappa_{\mathrm{te}}, \sigma_{\mathrm{ve}}', t{::}\kappa, \sigma') \rightsquigarrow e' \quad \Delta;\Gamma \triangleright e_{\mathrm{ve}} : \sigma_{\mathrm{ve}} \rightsquigarrow e_{\mathrm{ve}}'}{\Delta;\Gamma \triangleright \langle\!\langle e, \tau, e_{\mathrm{ve}}\rangle\!\rangle : \forall t{::}\kappa.\sigma_2 \rightsquigarrow \mathtt{pack}\ \kappa_{\mathrm{te}}, |\tau|, |\sigma_{\mathrm{ve}}|\ \mathtt{with}\ \langle e', e_{\mathrm{ve}}'\rangle\ \mathtt{as}\ |\forall t{::}\kappa.\sigma|}$$

$$(tapp)\quad \frac{\Delta;\Gamma \triangleright e : \forall t{::}\kappa.\sigma \rightsquigarrow e'}{\begin{array}{ll}\Delta;\Gamma \triangleright e\ \tau : \sigma[\tau/t] \rightsquigarrow & \mathtt{open}\ e'\ \mathtt{as}\ k_{\mathrm{te}}, t_{\mathrm{te}}, t_{\mathrm{ve}} \\ & \quad \mathtt{with}\ y : \langle t_{\mathrm{te}} \Rightarrow t_{\mathrm{ve}} \rightarrow \forall t{::}\kappa.|\sigma| \times t_{\mathrm{ve}}\rangle \\ & \mathtt{in}\ (\pi_1\, y)\ t_{\mathrm{te}}\ (\pi_2\, y)\ |\tau|\end{array}}$$

Figure 14: Closure Representation

Proof. By induction on the structure of σ.
Case $|\forall s{::}\kappa.\sigma|[|\tau|/t] \equiv \exists k.\exists t{::}k.\exists t_0{::}\Omega..\langle(t \Rightarrow t_0 \rightarrow \forall s{::}\kappa.|\sigma|[|\tau|/t]) \times t_0\rangle$.
By the induction hypothesis, $|\sigma|[|\tau|/t] \equiv |\sigma[\tau/t]|$. Then $|\forall s{::}\kappa.\sigma|[|\tau|/t] \equiv |\forall s{::}\kappa.\sigma[\tau/t]|$.

□

The type correctness of the translation is proven by induction on the derivation of the translation. The typing rules for $\sigma \Rightarrow \sigma'$ are essential to prove the cases for the translations of closures.

Theorem 8 (Type Correctness) *If* $\Delta;\Gamma \rhd e : \sigma \rightsquigarrow e'$, *then* $\emptyset;\Delta;|\Gamma| \vdash e' : |\sigma|$.

Proof. By induction on the derivation of the translation. We only show some important cases below. The other cases are clear from the definition or the induction hypothesis.

Case (*vcl*). By the induction hypothesis. $\emptyset;\Delta;|\Gamma| \vdash e' : \forall t_{\text{te}}{::}\kappa_{\text{te}}.|\sigma'_{\text{ve}}| \rightarrow |\sigma'_1| \rightarrow |\sigma'_2|$. and $\emptyset;\Delta;|\Gamma| \vdash e'_0 : |\sigma_{\text{ve}}|$. Then

$$\emptyset;\Delta;|\Gamma| \vdash e' : |\tau| \Rightarrow (|\sigma'_{\text{ve}}| \rightarrow |\sigma'_1| \rightarrow |\sigma'_2|)[|\tau|/t_{\text{te}}]$$

By Lemma 12, $\emptyset;\Delta;|\Gamma| \vdash e' : |\tau| \Rightarrow |\sigma_{\text{ve}}| \rightarrow |\sigma_1| \rightarrow |\sigma_2|$. Let σ' be $\exists t_0{::}\Omega.\langle(|\tau| \Rightarrow t_0 \rightarrow |\sigma_1| \rightarrow |\sigma_2|) \times t_0\rangle$. Then

$$\emptyset;\Delta;|\Gamma| \vdash \texttt{pack } |\sigma_{\text{ve}}| \texttt{ with } \langle e', e'_0\rangle \texttt{ as } \sigma' : \sigma'$$

Let σ'' be $\exists t{::}\kappa_{\text{te}}.\exists t_0{::}\Omega.\langle(t \Rightarrow t_0 \rightarrow |\sigma_1| \rightarrow |\sigma_2|) \times t_0\rangle$. Then

$$\emptyset;\Delta;|\Gamma| \vdash \texttt{pack } |\tau| \texttt{ with pack } |\sigma_{\text{ve}}| \texttt{ with } \langle e', e'_0\rangle \texttt{ as } \sigma \texttt{ as } \sigma'' : \sigma''$$

Let σ''' be $\exists k.\exists t{::}k.\exists t_0{::}\Omega.\langle(t \Rightarrow t_0 \rightarrow |\sigma_1| \rightarrow |\sigma_2|) \times t_0\rangle$. Then

$$\emptyset;\Delta;|\Gamma| \vdash \texttt{pack } \kappa \texttt{ with pack } |\tau| \texttt{ with pack } |\sigma_{\text{ve}}| \texttt{ with } \langle e', e'_0\rangle \texttt{ as } \sigma' \texttt{ as } \sigma'' \texttt{ as } \sigma''' : \sigma'''$$

Case (*tcl*). By the induction hypothesis. $\emptyset;\Delta;|\Gamma| \vdash e' : \forall t_{\text{te}}{::}\kappa_{\text{te}}.|\sigma'_{\text{ve}}| \rightarrow \forall s{::}\kappa'.|\sigma'_2|$. and $\emptyset;\Delta;|\Gamma| \vdash e'_0 : |\sigma_{\text{ve}}|$. Then

$$\emptyset;\Delta;|\Gamma| \vdash e' : |\tau| \Rightarrow (|\sigma'_{\text{ve}}| \rightarrow \forall s{::}\kappa'.|\sigma'_2|)[|\tau|/t_{\text{te}}]$$

By Lemma 12, $\emptyset;\Delta;|\Gamma| \vdash e' : |\tau| \Rightarrow |\sigma_{\text{ve}}| \rightarrow \forall s{::}\kappa'.|\sigma_2|$. The rest is similar to the case above.

Case (*app*). Let e be $e_1\, e_2$ and $\Delta;\Gamma \rhd e \rightsquigarrow e'$. By induction hypothesis, $\emptyset;\Delta;|\Gamma| \vdash e'_1 : |\sigma_1 \rightarrow \sigma_2|$ and $\emptyset;\Delta;|\Gamma| \vdash e'_2 : |\sigma_1|$. Then $\{k_{\text{te}}\};\Delta \uplus \{t_{\text{te}}, t_{\text{ve}}\};|\Gamma| \uplus \{y{:}\langle t_{\text{te}} \Rightarrow t_{\text{ve}} \rightarrow |\sigma_1| \rightarrow |\sigma_2| \times t_{\text{ve}}\rangle\} \vdash (\pi_1\, y)\; t_{\text{te}}\; (\pi_2\, y)\; e'_2 : |\sigma_2|$. Then $\emptyset;\Delta;|\Gamma| \vdash e' : |\sigma_2|$.

Case (*tapp*). Similar to the case of (*app*).

Case (*teq*). By the induction hypothesis, $\emptyset;\Delta;|\Gamma| \vdash e' : |\sigma'|$. By Lemma 11, $\emptyset;\Delta \vdash |\sigma| \equiv |\sigma'|{::}\Omega$. Hence, $\emptyset;\Delta;|\Gamma| \vdash e' : |\sigma|$.

□

5.2.3 Operational Correctness

Next, we prove that the translation from $\lambda^{\forall,cl}$ to $\lambda^{\forall,\exists}$ preserves the operational behavior of a program. We define logical relations indexed by sources types as before. However, the definition is more complicated due to the presence of types of the form $\pi_i(\sigma)$ in the source language. Consider the reduction of types generated by orienting the type equivalences as follows: $\pi_i\langle\sigma_1,\ldots,\sigma_n\rangle \mapsto \sigma_i$ and $\langle\pi_1(\sigma),\ldots,\pi_n(\sigma)\rangle \mapsto \sigma$. It is clear that this reduction is Church-Rosser and strong normalizing. Thus for all $\vdash \sigma$, there exists a unique σ' in normal form such that $\vdash \sigma \equiv \sigma' ::$. Further, if $\vdash \sigma \equiv \sigma'$, then the normal forms of σ and σ' are syntactically equal. Thus, we write $NF(\sigma)$ for the normal form of σ.

The logical relation for type σ that is not in formal form is defined by the relation for $NF(\sigma)$. In the definition of the relations, we make use of the fact that if σ is a normal form, then σ is not of the form $\pi_i(\sigma')$. The relations are defined inductively using the lexicographic order of the number of $\forall$, `vcode`, and `tcode` constructors in σ and the size of σ.

$$e \sim_\sigma e' \quad \text{iff} \quad e \hookrightarrow v \text{ and } e \hookrightarrow v' \text{ and } v \approx_\sigma v'.$$

$$\begin{array}{rll}
v \approx_\sigma v & \text{iff} & v \approx_{NF(\sigma)} v' \text{ where } \sigma \text{ is not in normal form.} \\
c \approx_b c & & \\
v \approx_{\texttt{vcode}(t::\kappa,\sigma_{\text{ve}},\sigma_1,\sigma_2)} v' & \text{iff} & \text{for all } \vdash \tau : \kappa,\ v_0 \approx_{\sigma_{\text{ve}}[\tau/t]} v_0' \text{ and } v_1 \approx_{\sigma_1[\tau/t]} v_1', \\
 & & \qquad \langle\!\langle v,\tau,v_0\rangle\!\rangle\ v_1 \sim_{\sigma_2[\tau/t]} v'\ |\tau|\ v_0'\ v_1' \\
v \approx_{\texttt{tcode}(t::\kappa,\sigma_{\text{ve}},s::\kappa',\sigma_2)} v' & \text{iff} & \text{for all } \vdash \tau : \kappa, \vdash \tau_1 : \kappa' \text{ and } v_0 \approx_{\sigma_{\text{ve}}[\tau/t]} v_0', \\
 & & \langle\!\langle v,\tau,v_0\rangle\!\rangle\ \tau_1 \sim_{\sigma_2[\tau/t,\tau_1/s]} v'\ |\tau|\ v_0'\ |\tau_1| \\
v \approx_{\sigma_1\to\sigma_2} v' & \text{iff} & \text{for all } v_1 \approx_{\sigma_1} v_1', \\
 & & \quad v\ v_1 \sim_{\sigma_2} \ \ \texttt{open } v' \texttt{ as } k,t,t_0 \\
 & & \qquad\qquad\quad \texttt{with } x{:}\langle(t \Rightarrow t_0 \to \sigma_1 \to \sigma_2)\times t\rangle \\
 & & \qquad\qquad\quad \texttt{in } (\pi_1\,x)\ t\ (\pi_2\,x)\ v_1' \\
v \approx_{\forall s::\kappa.\sigma_2} v' & \text{iff} & \text{for all } \vdash \tau :: \kappa, \\
 & & \quad v\ \tau \sim_{\sigma_2[\tau/s]} \ \ \texttt{open } v' \texttt{ as } k,t,t_0 \\
 & & \qquad\qquad\qquad \texttt{with } x{:}\langle(t \Rightarrow t_0 \to \forall s{::}\Omega.\sigma_2)\times t\rangle \\
 & & \qquad\qquad\qquad \texttt{in } (\pi_1\,x)\ t\ (\pi_2\,x)\ |\tau| \\
\langle v_1,\ldots,v_n\rangle \approx_{\langle\sigma_1\times\ldots\times\sigma_n\rangle} \langle v_1',\ldots,v_n'\rangle & \text{iff} & \text{for all } 1 \le i \le n,\ v_i \approx_{\sigma_i} v_i'.
\end{array}$$

From the definition, it is clear that if $\vdash \sigma \equiv \sigma'$, then $\approx_\sigma = \approx_{\sigma'}$ and $\sim_\sigma = \sim_{\sigma'}$. The value relation extends to substitutions (γ) in a straightforward manner.

We write $\vdash \delta :: \Delta$ if for all $t{::}\kappa \in \Delta$, $\emptyset \vdash \delta(t) :: \kappa$. The operational correctness of the translation is proved by induction on the derivation of the translation.

Theorem 9 (Operational Correctness) *Let* $\vdash \delta :: \Delta$ *and* $\gamma \approx_{\delta(\Gamma)} \gamma'$. *If* $\Delta;\Gamma \rhd e : \sigma \leadsto e'$ *then* $\delta(\gamma(e)) \sim_{\delta(\sigma)} |\delta|(\gamma'(e'))$.

Proof. By induction on the derivation of $\Delta;\Gamma \rhd e : \sigma \leadsto e'$.

Case (*var*) and (*const*). Clear from the definition.

Case (*prod*) and (*proj*). Clear from the induction hypothesis.

Case (*teq*). By induction hypothesis, $\delta\gamma(e) \sim_{\sigma'} |\delta|\gamma'(e')$. Since $\sim_\sigma = \sim_{\sigma'}$, $\delta\gamma(e) \sim_\sigma |\delta|\gamma'(e')$.

Case (*vcl*). By induction hypothesis, $\delta(\gamma(e)) \sim_{\delta(\mathtt{vcode}(t::\kappa,\sigma'_{\mathrm{ve}},\sigma'_1,\sigma'_2))} |\delta|(\gamma(e'))$ and $\delta(\gamma(e_{\mathrm{ve}})) \sim_{\delta(\sigma_{\mathrm{ve}})} |\delta|(\gamma'(e'_{\mathrm{ve}}))$. Then $\delta\gamma(e) \hookrightarrow v$, $|\delta|\gamma'(e') \hookrightarrow v'$, and $v \approx_{\delta(\mathtt{vcode}(t::\kappa,\sigma'_{\mathrm{ve}},\sigma'_1,\sigma'_2))} v'$ and $\delta\gamma(e_{\mathrm{ve}}) \hookrightarrow v_0$, $|\delta|\gamma'(e'_{\mathrm{ve}}) \hookrightarrow v'_0$, and $v_0 \approx_{\delta(\sigma_{\mathrm{ve}})} v'_0$. Let $v_1 \approx_{\delta(\sigma_1)} v'_1$. Then $\langle\!\langle v, \tau, v_0\rangle\!\rangle v_1 \sim_{\delta(\sigma_2)} v'|\tau|v_0 v_1$.

On the other hand, let e_1 be $\mathtt{pack}\ \kappa, |\tau|, |\sigma_{\mathrm{ve}}|\ \mathtt{with}\ \langle e', e'_{\mathrm{ve}}\rangle\ \mathtt{as}\ |\sigma_1 \to \sigma_2|$ and σ be $\langle t \Rightarrow t_0 \to |\sigma_1| \to |\sigma_2| \times t_0\rangle$. Then,

$$\mathtt{open}\ |\delta|(\gamma'(e_1))\ \mathtt{as}\ k, t, t_0\ \mathtt{with}\ x{:}\sigma\ \mathtt{in}\ \pi_1(x)t\pi_2(x)v'_1 \hookrightarrow v'_3 \text{ iff } v'|\tau|v'_0 v' \hookrightarrow v'_3.$$

Hence $\delta\gamma(\langle\!\langle e, \tau, e_{\mathrm{ve}}\rangle\!\rangle) \sim_{\delta(\sigma_1 \to \sigma_2)} |\delta|\gamma'(e_1)$.

Case (*vcode*). Let $\vdash \tau::\kappa$, δ' be $[\tau/t_{\mathrm{te}}]$, $v_0 \approx_{\sigma_{\mathrm{ve}}[\tau/t_{\mathrm{te}}]} v'_0$, and $v_1 \approx_{\sigma_1[\tau/t_{\mathrm{te}}]} v'_1$. Then, $[v_0/x_{\mathrm{ve}}, v_1/x] \approx_{\delta'(\{x_{\mathrm{ve}}:\sigma_{\mathrm{ve}}, x:\sigma_1\})} [v'_0/x_{\mathrm{ve}}, v'_1/x]$. Then by induction hypothesis, $\delta'([v_0/x_{\mathrm{ve}}, v_1/x]e) \sim_{\delta'(\sigma_2)} |\delta'|([v'_0/x_{\mathrm{ve}}, v'_1/x]e')$. Since $FV(e) \subseteq \{x, x_{\mathrm{ve}}\}$ and $FTV(e) \subseteq \{t_{\mathrm{te}}\}$, $\langle\!\langle \delta(\gamma(\Lambda t_{\mathrm{te}}::\kappa_{\mathrm{te}}.\lambda x_{\mathrm{ve}}{:}\sigma_{\mathrm{ve}}.\lambda x{:}\sigma_1.e)), \tau, v_0\rangle\!\rangle v_1 \sim_{\delta(\sigma_2)[\tau/t_{\mathrm{te}}]} |\delta|\gamma'(\Lambda t_{\mathrm{te}}::\kappa_{\mathrm{te}}.\lambda x_{\mathrm{ve}}{:}|\sigma_{\mathrm{ve}}|.\lambda x{:}|\sigma_1|.e')|\tau|v'_0 v'_1$. Then, $\delta(\gamma(\Lambda t_{\mathrm{te}}::\kappa_{\mathrm{te}}.\lambda x_{\mathrm{ve}}{:}\sigma_{\mathrm{ve}}.\lambda x{:}\sigma_1.e)) \sim_{\delta(\mathtt{vcode}(t_{\mathrm{te}}::\kappa_{\mathrm{te}},\sigma_{\mathrm{ve}},\sigma_1,\sigma_2))} |\delta|\gamma'(\Lambda t_{\mathrm{te}}::\kappa.\lambda x_{\mathrm{ve}}{:}|\sigma_{\mathrm{ve}}|.\lambda x{:}|\sigma_1|.e')$.

Case (*app*). By induction hypothesis, $\delta(\gamma(e_1)) \hookrightarrow v_1$ and $|\delta|(\gamma'(e'_1)) \hookrightarrow v'_1$ and $v_1 \approx_{\delta(\sigma_1 \to \sigma_2)} v'_1$. By induction hypothesis, $\delta(\gamma(e_2)) \hookrightarrow v_2$ and $|\delta|(\gamma'(e'_2)) \hookrightarrow v'_2$ and $v_2 \approx_{\delta(\sigma_2)} v'_2$.

By the definition, $v_1 v_2 \sim_{\delta(\sigma_2)} \mathtt{open}\ v'_1\ \mathtt{as}\ k_{\mathrm{te}}, t_{\mathrm{te}}, t_{\mathrm{ve}}\ \mathtt{with}\ y{:}\langle(t_{\mathrm{te}} \Rightarrow t_{\mathrm{ve}} \to \sigma_2 \to \sigma_2) \times t_{\mathrm{ve}}\rangle\ \mathtt{in}\ \pi_1(y)t_{\mathrm{te}}\pi_2(y)v_2$. Thus, $\delta\gamma(e_1 e_2) \sim_{\delta(\sigma_2)} |\delta|\gamma'(\mathtt{open}\ e'_1\ \mathtt{as}\ k_{\mathrm{te}}, t_{\mathrm{te}}, t_{\mathrm{ve}}\ \mathtt{with}\ y{:}\langle(t_{\mathrm{te}} \Rightarrow t_{\mathrm{ve}} \to \sigma_1 \to \sigma_2) \times t_{\mathrm{ve}}\rangle\ \mathtt{in}\ \pi_1(y)t_{\mathrm{te}}\pi_2(y)e_2)$.

Case (*tcl*). By induction hypothesis, $\delta(\gamma(e)) \sim_{\delta(\mathtt{tcode}(t::\kappa,\sigma'_{\mathrm{ve}},s::\kappa',\sigma'_2))} |\delta|(\gamma(e'))$ and $\delta(\gamma(e_{\mathrm{ve}})) \sim_{\delta(\sigma_{\mathrm{ve}})} |\delta|(\gamma'(e'_{\mathrm{ve}}))$. Then $\delta\gamma(e) \hookrightarrow v$, $|\delta|\gamma'(e') \hookrightarrow v'$, and $v \approx_{\delta(\mathtt{tcode}(t::\kappa,\sigma'_{\mathrm{ve}},s::\kappa',\sigma'_2))} v'$ and $\delta\gamma(e_{\mathrm{ve}}) \hookrightarrow v_0$, $|\delta|\gamma'(e'_{\mathrm{ve}}) \hookrightarrow v'_0$, and $v_0 \approx_{\delta(\sigma_{\mathrm{ve}})} v'_0$. Then $\langle\!\langle v, \tau, v_0\rangle\!\rangle \tau' \sim_{\sigma_2} v'|\tau|v_0|\tau'|$.

On the other hand, let e_1 be $\mathtt{pack}\ \kappa, |\tau|, |\sigma_{\mathrm{ve}}|\ \mathtt{with}\ \langle e', e'_0\rangle\ \mathtt{as}\ |\forall s\kappa'.\sigma_2|$ and σ be $\langle t \Rightarrow t_0 \to \forall s::\kappa'.|\sigma_2| \times t_0\rangle$. Then

$$\mathtt{open}\ |\delta|(\gamma'(e_1))\ \mathtt{as}\ k, t, t_0\ \mathtt{with}\ x{:}\sigma\ \mathtt{in}\ \pi_1(x)t\pi_2(x)|\tau'| \hookrightarrow v'_3 \text{ iff } v'|\tau|v'_0|\tau'| \hookrightarrow v'_3.$$

Hence $\delta\gamma(\langle\!\langle e, \tau, e_{\mathrm{ve}}\rangle\!\rangle) \sim_{\delta(\forall s::\kappa'.\sigma_2)} |\delta|\gamma'(e_1)$.

Case (*tcode*). Let $\vdash \tau::\kappa$, $\vdash \tau'::\kappa'$, δ' be $[\sigma/t_{\mathrm{te}}, \sigma'/s]$, and $v_0 \approx_{\sigma_{\mathrm{ve}}[\tau/t_{\mathrm{te}}]} v'_0$. Then, $[v_0/x_{\mathrm{ve}}] \approx_{\delta'(\{x_{\mathrm{ve}}:\sigma_{\mathrm{ve}}\})} [v'_0/x_{\mathrm{ve}}]$. Then by induction hypothesis, $\delta'([v_0/x_{\mathrm{ve}}]e) \sim_{\delta'(\sigma_2)} |\delta'|([v'_0/x_{\mathrm{ve}}]e')$. Since $FV(e) \subseteq \{x_{\mathrm{ve}}\}$ and $FTV(e) \subseteq \{t_{\mathrm{te}}, s\}$, $\langle\!\langle \delta(\gamma(\Lambda t_{\mathrm{te}}::\kappa_{\mathrm{te}}.\lambda x_{\mathrm{ve}}{:}\sigma_{\mathrm{ve}}.\Lambda s::\kappa'.e)), \tau, v_0\rangle\!\rangle \tau' \sim_{\delta(\sigma_2)[\tau/t_{\mathrm{te}},\tau'/s]} |\delta|\gamma'(\Lambda t_{\mathrm{te}}::\kappa.\lambda x_{\mathrm{ve}}{:}|\sigma_{\mathrm{ve}}|.\Lambda s::\kappa'.e')|\tau|v'_0|\tau'|$. Then, $\delta(\gamma(\Lambda t_{\mathrm{te}}::\kappa.\lambda x_{\mathrm{ve}}{:}\sigma_{\mathrm{ve}}.\Lambda s::\kappa'.e)) \sim_{\delta(\mathtt{tcode}(t_{\mathrm{te}}::\kappa,\sigma_{\mathrm{ve}},s::\kappa',\sigma_2))} |\delta|\gamma'(\Lambda t_{\mathrm{te}}::\kappa.\lambda x_{\mathrm{ve}}{:}|\sigma_{\mathrm{ve}}|.\Lambda s::\kappa'.e')$.

Case (*tapp*). By induction hypothesis, $\delta(\gamma(e)) \hookrightarrow v$ and $|\delta|(\gamma'(e')) \hookrightarrow v'$ and $v \approx_{\delta(\forall t::\kappa.\sigma)} v'$. By the definition, $v\delta(\tau) \sim_{\delta(\sigma)[\delta(\tau)/s]} \mathtt{open}\ v'\ \mathtt{as}\ k_{\mathrm{te}}, t_{\mathrm{te}}, t_{\mathrm{ve}}\ \mathtt{with}\ y{:}\langle(t_{\mathrm{te}} \Rightarrow t_{\mathrm{ve}} \to \forall s::\kappa.\sigma) \times t_{\mathrm{ve}}\rangle\ \mathtt{in}\ \pi_1(y)t_{\mathrm{te}}\pi_2(y)|\delta(\tau)|$. By Lemma 12, $|\delta(\tau)| \equiv |\delta|(|\tau|)$. Thus, $\delta\gamma(e\tau) \sim_{\delta(\sigma[\tau/s])} |\delta|\gamma'(\mathtt{open}\ e'\ \mathtt{as}\ k_{\mathrm{te}}, t_{\mathrm{te}}, t_{\mathrm{ve}}\ \mathtt{with}\ y{:}\langle(t_{\mathrm{te}} \Rightarrow t_{\mathrm{ve}} \to \forall s::\kappa.\sigma) \times t_{\mathrm{ve}}\rangle\ \mathtt{in}\ \pi_1(y)t_{\mathrm{te}}\pi_2(y)|\tau|)$.

□

6 Summary and Conclusions

We have presented a type-theoretic account of closure conversion for the simply-typed and polymorphic λ-calculi. Our translations are unique in that they map well-typed source terms to well-typed target terms. This facilitates correctness proofs, allows other type-directed transforms such as CPS conversion or unboxing to be applied after closure conversion, and supports run-time examination of types for tag-free garbage collection.

We have put the ideas in this paper to practical use in two separate compilers for ML: one compiler is being used to study novel approaches to tag-free garbage collection and the other compiler provides a general framework for analyzing types at run time to determine the shapes of objects. Propagating types through closure conversion is necessary for both compilers so that types can be examined at run time. We have also found that typed closure conversion, along with our other type-preserving translations, made it possible to find and eliminate compiler bugs since we can automatically typecheck the output of each compiler phase.

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