Computational Perspectives on Democracy

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For Grandpa and Harabeoji.

Abstract

Democracy is a natural approach to large-scale decision-making that allows people affected by a potential decision to provide input about the outcome. However, modern implementations of democracy are based on outdated information technology and must adapt to the changing technological landscape. This thesis explores the relationship between computer science and democracy, which is, crucially, a two-way street—just as principles from computer science can be used to analyze and design democratic paradigms, ideas from democracy can be used to solve hard problems in computer science.

Question 1: What can computer science do for democracy?

To explore this first question, we examine the theoretical foundations of three democratic paradigms: liquid democracy, participatory budgeting, and multiwinner elections. Each of these paradigms broadly redistributes power from the few to the many: For instance, liquid democracy allows people to choose delegates more flexibly and participatory budgeting enables citizens to directly influence government spending toward public projects. However, because these paradigms are relatively new, their theoretical properties are relatively unexplored. We analyze each of these three settings from the point of view of computational social choice, which is a mathematical framework for collective decision-making. In particular, we focus on a combination of robustness, fairness, and efficiency with the end goal of providing actionable advice for future iterations of these paradigms.

Question 2: What can democracy do for computer science?

Toward this end, we explore two settings in which democratic principles can be used to augment approaches to making difficult decisions—in our case, automating ethical decision-making and hiring in online labor markets. Both of these problems are difficult in the sense that there is no universally agreed-upon function to optimize, making them a poor fit for traditional approaches in computer science. Instead, we try to emulate a world in which we can get input from people in order to arrive at a "societal" decision. In each of these settings, we first propose and analyze a theoretical approach that leads a single decision, and then, in collaboration with HCI researchers, run experiments in the real world to test the efficacy and practicability of our approaches in the real world.

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Contents

List of Figures

List of Tables

Democracy is the worst form of government except for all those other forms that have been tried from time to time.

Winston Churchill

0 Introduction

Group decision-making is a fundamental challenge in human society. How should groups of people (e.g., nations, states, cities, or neighborhoods) make collective decisions based on heterogeneous opinions?

Democracy offers a compelling answer to this question: Let the people themselves decide. Societies over the years have ascribed to this philosophy to varying degrees. The ancient Greeks were early champions of democracy, and Athenians notably used democratic processes like sortition (random selection from the public) and voting (soliciting structured input from the public) at all levels of society. Since then, democracy has ebbed and flowed, but mostly flowed. Global powers up until the 17th century were largely monarchies or oligarchies, but John Locke's liberal democratic framework espoused in his seminal work, *Two Treatises of Government* [162], set in motion a democratic wave that has continued into the 21st century.

However, although many countries have ostensibly embraced democratic principles, the world is currently in a state of democratic retrograde, otherwise known as democratic backsliding. Implementations of modern democracy largely adhere to the model of representative democracy, where the general public may directly elect representatives in legislative bodies, but these representatives may make decisions that do not accurately reflect the will of the people. This, among other factors, has led to a marked decrease in trust in democratic systems in recent years. Notably, advances in technology have not been blameless in this erosion of trust. In fact, social networks have led to increased division and greater discord online by facilitating the creation of echo chambers, an increase in polarization, and the proliferation of fake news.

All this is to say, humanity still has significant work to do in figuring out how to make democracy "work" as a stable, harmonious decision-making paradigm in which people affected by large-scale decisions get a say in what these decisions are. Luckily, despite the rather negative picture painted by the foregoing paragraphs, not all hope is lost for demo-

cratic ideals. In particular, as new technologies develop, academics and practitioners alike have turned to the new frontiers of digital democracy, or e-democracy, where technology is explicitly used to promote and strengthen democratic practices. Throughout this thesis, we hope to contribute to work in this vein, and the first part of this thesis focuses on the theoretical analysis of democratic paradigms that redistribute power into the hands of the people by allowing them to directly influence decisions.

However, the relationship between computer science and democracy is not a one-way street: Just as we can ask how tools from computer science can help evaluate and design new democratic paradigms, we can also ask how principles of democracy can help address difficult problems in computer science. This line of research is related to participatory artificial intelligence, wherein human values are taken into consideration in order to design more democratic and ethical machine learning systems.

Overall, we hope to address the following broad questions.

> **Question 1**: How can we formally analyze existing democratic paradigms with an eye toward creating better iterations of these paradigms in the future?
>
> **Question 2**: How can we apply democratic principles to augment traditional computer science-based approaches in order to solve difficult real-world problems?

Background: Computational Social Choice

Computational social choice is an interdisciplinary discipline that combines tools from social choice theory and theoretical computer science. Below, we provide a short overview of relevant topics; please see [47; 211; 197; 94; 114; 199] for a significantly more detailed introduction.

Social choice theory formalizes the problem of aggregating individual preferences to make a collective decision. In general, most problems in social choice can be thought of as preference aggregation, in which a mechanism receives a collection of opinions (e.g., ordinal rankings of alternatives) from agents and must return a single outcome, which may take the form of a single winner, a subset of winners, or a complete ranking, among other formats. Within this framework, notable branches of social choice include voting theory, which concerns the design and analysis of voting rules that take opinions as input and apply a well-specified mechanism (which can equivalently be thought of as a function) to make a final decision; resource allocation and fair division, where a common resource must be divided among a set of agents with different utility functions; and ranking systems, in which the set of agents and alternatives coincide, i.e., agents provide ranking information among themselves. Each of these topics will be touched upon in this thesis: Chapters 1, 3 and 4 primarily concern voting theory; Chapter 2 considers resource allocation; and Chapter 5 involves ranking systems.

Computational social choice introduces tools from theoretical science like complexity theory, approximation algorithms, and worst-case analysis to traditional social choice theory. Early work in computational social choice focused on the computational complexity

of evaluating and manipulating voting rules; however, in this thesis, we focus on a combination of the following topics, each of which is often a desideratum of mechanisms in practice.

- *Robustness*: For our purposes, a voting rule is robust if it is resistant to noise. In other words, we desire rules with guarantees that hold with high probability even in the presence of perturbations. Chapters 1, 4 and 5 analyze mechanisms from the perspective of formal robustness guarantees.
- *Fairness and representation*: Fairness has many definitions, depending on the context. In the context of this thesis, we primarily consider proportionality, which maintains that groups of people deserve some amount of representation in the outcome of a decision in accordance with their size and/or cohesiveness. Chapters 2 and 3 explore notions of fairness and proportionality in participatory budgeting and multiwinner elections, respectively.
- *Strategyproofness*: A voting rule is strategyproof if each agent is (weakly) best-off reporting her true beliefs to the mechanism, i.e., dishonest manipulation has no benefits. We often desire strategyproofness in order to incentivize good behavior among agents. The work in Chapter 5 concerns the design of strategyproof peer ranking mechanisms, both in theory and practice.
- *Computability*: Some problems we wish to solve are computationally intractable, and voting rules must be computationally tractable in order to be useful. Therefore, in some cases, we must turn our attention to polynomial-time approximation algorithms. The work in Chapters 1 and 2 involves analysis of this flavor.

As an introductory note on voting theory, we note that many seminal results in the field are negative. In particular, there are three fundamental impossibility results in the field of voting theory: Condorcet's paradox, Arrow's Impossibility Theorem, and the Gibbard-Satterthwaite Theorem. Condorcet's paradox [88] states that, on a population level, majority judgments between pairs of alternatives may not be transitive; i.e., that there can exist settings where alternative a is preferred to alternative b by at least half of the voters, b is preferred to c by at least half of the voters, but over half of the voters prefer c to a, causing a cycle. Arrow's Impossibility Theorem [11] states that, under mild assumptions (i.e., the rule is not a dictatorship and satisfies unanimity) and in settings with at least three alternatives, every deterministic voting rule must violate a property known as independence of irrelevant alternatives (IIA). Finally, the Gibbard-Satterthwaite Theorem [116; 203] maintains that any deterministic, non-dictatorship voting rule for three or more candidates is not strategyproof; i.e., that agents can benefit by misreporting their preferences. These fundamental impossibility results mean that, in our work, we must restrict either the domain of (strategic) voter behavior or weaken the theoretical results we hope to prove.

Finally, we note that computational social choice has limitations inherent in its model of decision-making. For instance, it is generally assumed that voters' utilities do not change throughout the process, i.e., that they possess the same values before, during, and after voting. This is not necessarily the case in practice, and work on *deliberation* explicitly

takes this into account. Additionally, computational social choice does not provide guidance about how to choose the set of alternatives over which voters opine, and the voting format—e.g., approval, veto, knapsack, or ranked votes—necessarily restricts the amount of information that voters can provide to any voting mechanism.

0.1 Structure

Part I: How computer science can help democracy

The first part of this thesis asks how computer science can help democracy. In particular, we explore the theoretical foundations of three new democratic paradigms: liquid democracy, participatory budgeting, and proportional multiwinner elections. On a high level, both liquid democracy and participatory budgeting allow people to more directly influence decision-making processes, and multiwinner elections allow people to choose committees of winners instead of only a single winner.

For each of the three paradigms in this part, we focus primarily on one or more of three theoretical desiderata of voting rules: *robustness*, *fairness*, and *efficiency*. In the context of this thesis, the robustness of a voting rule is its resilience against worst-case noise, fairness broadly refers to notions of proportionality and representation, and efficiency will be measured in terms of utilitarian social welfare, or the sum of all agents' utilities.

Liquid Democracy (Chapter 1) *Liquid democracy* is a collective decision making paradigm that allows voters to transitively delegate their votes. Our first work on this subject studied liquid democracy through an algorithmic lens [137]. In our model, there are two alternatives, one correct and one incorrect, and we are interested in the probability that the majority opinion is correct. Our main question is whether there exist delegation mechanisms that are *guaranteed* to outperform direct voting, in the sense of being always at least as likely, and sometimes more likely, to make a correct decision. Even though we assume that voters can only delegate their votes to more informed voters, we show that *local* delegation mechanisms, which only take the local neighborhood of each voter as input (and, arguably, capture the spirit of liquid democracy), cannot provide the foregoing guarantee. By contrast, we design a non-local delegation mechanism that does provably outperform direct voting under mild assumptions about voters.

The above result corroborates a common critique of liquid democracy: Often, a small subset of agents may gain massive influence. To address this, we propose to change the current practice by allowing agents to specify multiple delegation options instead of just one. Then, we seek to control the flow of votes in a way that balances influence as much as possible. Specifically, we analyze the problem of choosing delegations to approximately minimize the maximum number of votes entrusted to any agent, by drawing connections to the literature on confluent flow. We also introduce a random graph model for liquid democracy, and draw on prior work on the *power of choice*, which allows us to establish a doubly exponential separation between the maximum weight of a voter in the case where each voter provides a single delegation and the maximum weight of a voter in the case

where each voter provides even two possible delegations [119].

District-Fair Participatory Budgeting (Chapter 2) *Participatory budgeting* is a democratic paradigm in which local governments solicit input from their constituents to make budget decisions about public funds. In practice, it often takes the form of cities asking their residents to vote over a list of public projects to fund. Furthermore, cities often use participatory budgeting on a district level, where each district in a city holds a separate election to spend their portion of the budget (generally allocated proportionally to population). However, district-level elections may yield poor social welfare because no single district has enough money to fund large, widely beneficial projects. On the other hand, making decisions solely based on global social welfare may be unfair to some districts: A social-welfare-maximizing solution may not fund any projects preferred by a particular district.

Thus, we study how to fairly maximize social welfare in participatory budgeting subject to a *district fairness*, which is a fairness constraint that promises each district at least as much utility as it would have received under a district-level participatory budgeting process. We show that, although optimizing social welfare subject to district fairness is NP-hard, we can efficiently construct a lottery over welfare-optimal outcomes that is district-fair in expectation. Moreover, we show that, when we are allowed to slightly relax fairness, we can efficiently compute a deterministic, almost-district-fair solution that is welfare-maximizing, but which may overspend the budget by a small constant factor [125].

Proportionality in Multiwinner Elections (Chapter 3) Finally, we analyze mechanisms for multiwinner elections, where a group of agents, or voters, selects a committee from a set of candidates based on the agents' preferences. In our setting, each agent expresses her preferences through an approval vote, where she designates a subset of candidates she approves for the committee, and all votes are then aggregated to select a winning committee from the pool of candidates.

The property we would like to satisfy is that of proportionality, which intuitively says that a $c \leq 1$ fraction of voters who agree on a c fraction of the alternatives should be able to guarantee themselves control over a c fraction of the committee. We propose a measure of proportionality for elections where the size of the committee is not fixed beforehand and hope to establish a clear separation between the theoretical guarantees of deterministic and randomized rules under this notion of proportionality [108].

Part II: How democracy can help computer science

In the second part of this thesis, we ask how traditional democratic principles can help solve hard problems in computer science. In particular, we explore virtual democracy, which provides a principled approach to automating ethical decision-making, and impartial peer ranking, in which we design mechanisms to leverage individual expertise in settings with conflicts of interest.

In both of these settings, we try to solve problems that do not have clear objective functions to optimize. For instance, when trying to automate ethical decision-making, the

very concept of morality differs from person to person depending on their worldview; the same phenomenon holds when evaluating individual expertise and fit for a job. Therefore, standard optimization techniques are not a good fit for these types of problems because we often cannot even identify a widely agreed-upon metric over which to optimize. Instead, we will design mechanisms that directly ask participants for their opinions and then use these opinions in a principled way to make a final decision.

Virtual Democracy (Chapter 4) *Virtual democracy* is an approach to automate decisions by learning models of the preferences of individual people, and, at runtime, aggregating the *predicted* preferences of those people on the dilemma at hand. One of the key questions is which aggregation method — or *voting rule* — to use; we offer a novel statistical viewpoint that provides guidance. Specifically, we seek voting rules that are *robust* to prediction errors, in that their output on people's true preferences is likely to coincide with their output on noisy estimates thereof. We prove that the classic Borda count rule is robust in this sense, whereas any voting rule belonging to the wide family of pairwise-majority consistent (PMC) rules is not [138].

In concert with our theoretical results, we have worked closely with a local nonprofit organization in Pittsburgh, 412 Food Rescue, to build a framework for virtual democracy that enables people to build algorithmic policy for their communities. Through this framework, we study how to design algorithmic policy in order to balance varying interests in a moral, legitimate way. Our findings suggest that the framework successfully enabled participants to build models that they felt confident represented their own beliefs. Participatory algorithm design also improved both procedural fairness and the distributive outcomes of the algorithm, raised participants' algorithmic awareness, and helped identify inconsistencies in human decision-making in the governing organization [155].

Impartial Aggregation (Chapter 5) We study rank aggregation algorithms that take as input the opinions of players over their peers, represented as rankings, and output a social ordering of the players (which reflects, e.g., relative contribution to a project or fit for a job). To prevent strategic behavior, these algorithms must be *impartial*; that is, players should not be able to influence their own position in the output ranking. We design several randomized algorithms that are impartial and closely emulate given (non-impartial) rank aggregation rules in a rigorous sense [136].

We complement these theoretical results with a study of impartial algorithms applied to online labor markets through *HirePeer*, a novel alternative approach to hiring at scale we created that leverages peer assessment to elicit honest assessments of fellow workers' job application materials, which it then aggregates using an impartial ranking algorithm [148]. Surprisingly, we find that applying peer assessment to online hiring—even without impartial ranking algorithms to remove conflicts of interest—was an accurate and pedagogically beneficial practice.

If Internet is the new printing press, then what is democracy for the Internet era? ... How can we get our representatives, our elected representatives, to represent us?

Pia Mancini.

1 Liquid Democracy

In this chapter, we examine theoretical properties of liquid democracy, a democratic paradigm that allows for transitive vote delegation. First, in a model with a ground truth, we demonstrate that a large class of decentralized delegation mechanisms for liquid democracy is susceptible to the concentration of voting power in relatively few voters. We also demonstrate that a simple centralized delegation mechanism for liquid democracy can avoid this problem. Second, informed by the findings above, we turn our sights to centralized delegation mechanisms that effectively reduce the maximum weight of any voter. In particular, we show that allowing voters to give multiple possible delegations leads to a significant reduction in the maximum weight of any voter.

1.1 An Algorithmic Perspective on Liquid Democracy

Liquid democracy is a modern approach to voting in which voters can either vote directly or delegate their vote to other voters. In contrast to the classic *proxy voting* paradigm [173], the key innovation underlying liquid democracy is that proxies—who were selected by voters to vote on their behalf—may delegate their own vote to a proxy, and, in doing so, further delegate all the votes entrusted to them. Put another way (to justify the liquid metaphor), votes may freely flow through the directed delegation graph until they reach a sink, that is, a vertex with outdegree 0. When the election takes place, each voter who did not delegate his vote, but rather voted, is weighted by the total number of votes delegated to him, including his own. In recent years, this approach has been implemented and used on a large scale, notably by eclectic political parties such as the German Pirate Party (Piratenpartei) and Sweden's Demoex (short for Democracy Experiment).

One reason for the success of liquid democracy is that it is seen as a practical com-

promise between *direct democracy* (voters vote directly on every issue) and *representative democracy*, and, in a sense, is the best of both worlds. Direct democracy is particularly problematic, as nicely articulated by Green-Armytage [121]:

> "Even if it were possible for every citizen to learn everything they could possibly know about every political issue, people who did this would be able to do little else, and massive amounts of time would be wasted in duplicated effort. Or, if every citizen voted but most people did not take the time to learn about the issues, the results would be highly random and/or highly sensitive to overly simplistic public relations campaigns."

Another example is polling the audience in *Who Wants to Be a Millionaire*, in which the audience would like to help but sometimes gets the question wrong because people who don't know the correct answer systematically favor a specific incorrect answer.

By contrast, under liquid democracy, voters who did not invest an effort to learn about the issue at hand (presumably, most voters) would ideally delegate their votes to well-informed voters. This should intuitively lead to collective decisions that are less random, and more likely to be correct, than those that would be made under direct democracy.

Our goal is to rigorously investigate the intuition that liquid democracy "outperforms" direct democracy from an algorithmic viewpoint. Indeed, we are interested in *delegation mechanisms*, which decide how votes should be delegated based on how relatively informed voters are, and possibly even based on the structure of an underlying social network. Our main research question is:

> *Are there delegation mechanisms that are* guaranteed *to yield more accurate decisions than direct voting?*

1.1.1 Overview of the Model and Results

We focus on a (common) setting where a decision is to be made on a binary issue, i.e., one of two alternatives must be selected. To model the idea of accuracy, we assume that one alternative is correct, and the other is incorrect. Each voter i has a competence level p_i, which is the probability he would vote correctly if he cast a ballot himself.

Voters may delegate their votes to neighbors in a social network, represented as a directed graph. At the heart of our model is the assumption that voters may only delegate their votes to strictly more competent neighbors (and, therefore, there can be no delegation cycles). Specifically, we say that voter i *approves* voter j if $p_j > p_i + \alpha$, for a parameter $\alpha \geq 0$; voters may only delegate to approved neighbors. In defense of this strong assumption, we note that the first of our two theorems—arguably the more interesting of the two—is an *impossibility* result, so assuming that delegation necessarily boosts accuracy only *strengthens* it.

As mentioned above, we are interested in studying *delegation mechanisms*, which decide how votes are delegated (possibly randomly), based on the underlying graph and the approval relation between voters. We pay special attention to *local* delegation mechanisms, which make delegation decisions based only on the neighborhood of each voter. Local mechanisms capture the spirit of liquid democracy in that voters make independent del-

egation decisions based solely on their own viewpoint, without guidance from a central authority. By contrast, non-local mechanisms intuitively require a centralized algorithm that coordinates delegations.

Recall that our goal is to design delegation mechanisms that are *guaranteed* to be more accurate than direct voting. To this end, we define the *gain* of a mechanism with respect to a given instance as the difference between the probability that it makes a correct decision (when votes are delegated and weighted majority voting is applied) and the probability that direct voting makes a correct decision on the same instance. The desired guarantee can be formalized via two properties of mechanisms: *positive gain (PG)*, which means that there are some sufficiently large instances in which the mechanism has positive gain that is bounded away from 0; and *do no harm (DNH)*, which requires that the loss (negative gain) of the mechanism goes to 0 as the number of voters grows. These properties are both weak; in particular, PG is a truly minimal requirement which, in a sense, mainly rules out direct voting itself as a delegation mechanism.

In Section 1.1.4, we study local delegation mechanisms and establish an impossibility result: such mechanisms cannot satisfy both PG and DNH. In a nutshell, the idea is that for any local delegation mechanism that satisfies PG we can construct an instance where few voters amass a large number of delegated votes, that is, delegation introduces significant *correlation* between the votes. The instance is such that, when the high-weight voters are incorrect, the weighted majority vote is incorrect; yet direct voting is very likely to lead to a correct decision.

In Section 1.1.5, we show that non-local mechanisms can circumvent the foregoing impossibility. Specifically, we design a delegation mechanism, GreedyCap, that satisfies the PG and DNH properties under mild assumptions about voter competencies. It does so by imposing a cap on the number of votes that can be delegated to any particular voter, thereby avoiding excessive correlation.

In conclusion, our work highlights the significance, and potential dangers, of delegating many votes to few voters. Importantly, there is evidence that this can happen in practice. For example, Der Spiegel reported[1] that one member of the German Pirate Party, a linguistics professor at the University of Bamberg, amassed so much weight that his "vote was like a decree." Although recent work by Kling et al. [146] highlights the fact that, in practice, high-weight voters vote reasonably and do not abuse their power, our results corroborate the intuition that this situation should ideally be avoided.

1.1.2 Related Work

There is a significant body of work on delegative democracy and proxy voting [173; 221; 5]. In particular, Cohensius et al. [74] study a model where voters' positions on an issue are points in a metric space. In their version of direct democracy, a small subset of active voters report their positions, and an aggregation method (such as the median or mean when the metric space is the real line) outputs a single position. Under proxy voting,

[1] http://www.spiegel.de/international/germany/liquid-democracy-web-platform-makes-professor-most-powerful-pirate-a-818683.html

each inactive voter delegates his vote to the closest active voter. Cohensius et al. identify conditions under which proxy voting gives a more accurate outcome than direct voting, where the measure is proximity of the outcome to the aggregation method applied to all voters' positions.

To the best of our knowledge, there are only two papers prior to the initial publication of our work that provide theoretical analyses of liquid democracy. The first is the aforementioned paper by Green-Armytage [121]. He considers a setting where, similarly to Cohensius et al. [74], voters are identified with points on the real line, but in his model votes are noisy estimates of those positions. Green-Armytage defines the *expressive loss* of a voter as the squared distance between his vote and his position and proves that delegation (even transitive delegation) can only decrease the expressive loss in his model. He also defines *systematic loss* as the squared distance between the median vote and the median position, but discusses this type of loss only informally (interestingly, he does explicitly mention that correlation can lead to systematic loss in his model).

The second paper is by Christoff and Grossi [72]. They introduce a model of liquid democracy based on the theory of binary aggregation (i.e., their model has a mathematical logic flavor). Their results focus on two problems: the possibility of delegation cycles, and logical inconsistencies that can arise when opinions on interdependent propositions are expressed through proxies. Both are nonissues in our model (although the need to avoid cycles is certainly a concern in practice).

Further afield, there is a rich body of work in computational social choice [47] on the aggregation of objective opinions [76; 78; 96; 236; 235; 163; 192; 14; 15; 169; 64; 65; 193]. As in our work, the high-level goal is to pinpoint the correct outcome based on noisy votes. However, previous work in this area does not encompass any notion of vote delegation.

One seminal result in the aggregation of objective opinions—in particular, when deciding between two options, one of which is correct and the other of which is incorrect—is the Condorcet Jury Theorem [123], which states that if voters are independent and each have probability greater than $1/2$ of choosing the correct outcome, then the probability of choosing correctly approaches one as the size of the electorate increases. Note that the Condorcet Jury Theorem is directly applicable to the setting of direct democracy, but not immediately to the (weighted) setting of liquid democracy. Researchers have also studied voting rules in a networked setting, but without delegation, from the perspective of maximum likelihood estimation [81; 80].

There are also several papers that have explored the theoretical foundations of liquid democracy. Notably, a paper by Brill and Talmon [51] considers liquid democracy in the setting of ordinal elections in which the electorate wishes to construct a complete ordering over alternatives, as opposed to deciding a binary issue as in this work. In this framework, each voter may specify a partial ordering over the alternatives and delegate to others in order to construct a complete ranking. However, decisions made by delegates may violate transitivity with respect to each voter's partial ordering, and even checking whether delegated votes satisfy transitivity is NP-hard. In order to circumvent issues of transitivity, they introduce a novel class of voting rules for liquid democracy based on distance rationalization, which take as input (perhaps intransitive) delegation graphs and output the "closest" consensus profile.

Bloembergen et al. [36] consider a game-theoretic version of liquid democracy in which voters must determine whether or not it is rational to delegate their votes to others. They introduce a *delegation game* in which each voter has a hidden true "type" that she knows imperfectly, and the goal of each voter is to communicate her true type to the mechanism either directly (by voting) or indirectly (by delegating). While this setting distills the problem of finding delegates that represent one's own opinion, it focuses on proving the existence of Nash equilibria under certain assumptions and provides only weak performance bounds in the setting we consider.

Finally, Abramowitz and Mattei [2] propose a variant of representative democracy that incorporates ideas of liquid democracy by allowing voters to alter the voting weights of their representatives depending on the issue at hand. Although this circumvents some issues of liquid democracy—for instance, because delegations are no longer transitive, delegation cycles cannot occur—the proposed system is considerably more constrained than general liquid democracy.

1.1.3 The Model

We represent an instance of our problem using a directed, labeled graph $G = (V, E, \vec{p})$. $V = \{1, \ldots, n\}$ is a set of n *voters*, also referred to as *vertices* (we use the two terms interchangeably). E represents a (directed) social network in which the existence of an edge (i, j) means that voter i knows (of) voter j. We denote the neighborhood of voter i to be the set of neighbors that i knows of, or $N_G(i) = \{j \in V : i \text{ knows of } j\}$.

We assume that the voters vote on a binary issue; there is a correct alternative and an incorrect alternative. Each voter $i \in V$ is labeled by his *competence level* p_i. This is the probability that i has the correct opinion about the issue at hand, i.e., the probability that i will vote correctly.

Our setting is also parameterized by $\alpha \in (0, 1)$. Given this parameter and a labeled graph $G = (V, E, \vec{p})$, we define an approval relation between voters: $i \in V$ *approves* $j \in V$ if $(i, j) \in E$ and $p_j \geq p_i + \alpha$. In words, i approves his neighbor j if the difference in their competence levels is at least than α. Note that the approval relation is acyclic because $\alpha > 0$. Denote

$$A_G(i) = \{j \in V : i \text{ approves } j\}.$$

Delegation Mechanisms

The liquid democracy paradigm is implemented through a *delegation mechanism* M, which takes as input a labeled graph G, and outputs, for each voter i, a delegation probability distribution over $A_G(i) \cup \{i\}$ that represents the probability that i will delegate his vote to each of his approved neighbors, or to himself (which means he does not delegate his vote).

To determine whether a delegation mechanism M makes a correct decision on a labeled graph $G = (V, E, \vec{p})$, we use the following 4-step process (which is described in words to avoid introducing notation that will not be used again):

1. Apply M to G to output a delegation probability distribution for each voter i.

2. Sample the probability distribution for each vertex to obtain an acyclic *delegation graph*. Each sink i of the delegation graph (i.e., vertex with no outgoing edges) has weight equal to the number of vertices with directed paths to i, including i itself.
3. Each sink i votes for the correct alternative with probability p_i, and for the incorrect alternative with probability $1 - p_i$.
4. A decision is made based on the weighted majority vote.[2]

We denote the probability that the mechanism M makes a correct decision on graph G via this 4-step process by $P_M(G)$.

Local Mechanisms

We are particularly interested in a special class of delegation mechanisms that we call *local* mechanisms. Intuitively, local mechanisms capture the natural setting where each voter makes an independent delegation decision without central coordination or knowledge of global properties about the delegation graph. Formally, a *local delegation mechanism* is a delegation mechanism such that the probability distribution of each vertex i depends only on (a) the subset $A_G(i)$ of neighbors that i approves, (b) an arbitrary ranking π_i over $A_G(i)$, and (c) $N_G(i)$, or i's neighborhood. Note that the ranking π_i does not have any inherent meaning; it is simply a way to distinguish specific neighbors. In particular, local mechanisms assume that each voter has knowledge of the identities of his approved and non-approved neighbors; a local delegation mechanism is applied to π_i and $N_G(i)$ in order to output a delegation probability distribution for voter i.

For instance, say that in a setting with $\alpha = 0.15$, Alice ($p_{Alice} = 0.6$) has four neighbors: Bob ($p_{Bob} = 0.8$), Carla ($p_{Carla} = 0.9$), Dean ($p_{Dean} = 0.5$), and Evelyn ($p_{Evelyn} = 0.7$). Alice approves of Bob and Carla, and let $\pi_{Alice} = \text{Carla} \succ \text{Bob}$. Then, the local delegation mechanism takes π_{Alice} and the set of Alice's neighbors, and returns a probability distribution over delegating to Bob, delegating to Carla, and voting directly.

Let us give some examples of local delegation mechanisms:

- Voters do not delegate their votes. This *direct voting* mechanism plays a special role in our model, and we denote it by D.
- Each voter delegates his vote to a random approved neighbor, if he has any.
- Each voter delegates his vote to a random approved neighbor, if he has approved neighbors but has even more non-approved neighbors.
- Each voter delegates his vote deterministically to a single approved neighbor (e.g., the first in his local ordering π_i), if he has any. The ranking π_i is needed only in order to enable this type of mechanism.

By contrast, the following delegation mechanisms are not local:

- Each voter delegates his vote to his most competent approved neighbor. (Voters cannot distinguish between their approved neighbors, except through the information given by the "arbitrary" ranking π_i.)

[2] Ties can be broken arbitrarily.

- Let there exist a distinguished voter with global identifier V_1. If V_1 appears in the approval set of any voter, that voter delegates to V_1 with probability 1.
- Each voter delegates his vote only if all agents in his approval set have global identifiers that are even integers.

Desiderata

Recall that we are interested in comparing the likelihood of making correct decisions via delegative voting with that of direct voting. To this end, define the *gain* of delegation mechanism M on labeled graph G as

$$\text{gain}(M, G) = P_M(G) - P_D(G).$$

We would like to design delegation mechanisms that have positive gain (bounded away from zero) in some situations, and which never lose significantly to direct voting. Formally, we are interested in the following two desirable axioms:

- A mechanism M satisfies the *positive gain (PG)* property if there exist $\gamma > 0, n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there exists a graph G_n on n vertices such that $\text{gain}(M, G_n) \geq \gamma$.
- A mechanism M satisfies the *do no harm (DNH)* property if for all $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that for all graphs G_n on $n \geq n_1$ vertices, $\text{gain}(M, G_n) \geq -\varepsilon$.

The choice of quantifiers here is of great significance. PG asks for the *existence* of (large enough) instances where the gain is at least γ, for a constant γ. By contrast, DNH essentially requires that any loss would go to 0 as the size of the graph goes to infinity. That is, there may certainly be small instances where delegative voting loses out to direct voting, but that should not be the case in the large.

We note that PG and DNH are defined over worst-case instances. Another natural question to ask is about the expected gain of delegation mechanisms: For a random graph and choice of competence levels, is a given mechanism expected to outperform direct voting? However, we leave this to future work.

1.1.4 Impossibility for Local Mechanisms

In our model, we make the strong assumption that voters can only delegate their vote to other voters who are more competent than they are, and, in particular, delegation chains can significantly boost the competence of any particular vote. Under this assumption, it seems natural to expect that delegative voting will always do at least as well as direct voting in every situation, and strictly better in some situations. This should intuitively be true under local mechanisms, say, when each voter delegates his vote to an arbitrary approved neighbor (if he has any). The following example helps build intuition for what can go wrong.

Example 1. *Consider the labeled graph $G_n = (V, E, \vec{p})$ over n vertices, where $E = \{(i, 1) : i \in V \setminus \{1\}\}$, i.e., G is a star with 1 at the center. Moreover, $p_1 = 4/5$, $p_i = 2/3$ for all*

$i \in V \setminus \{1\}$, and $\alpha = 1/10$. Then, as n grows larger, $P_D(G_n)$ goes to 1 by the Law of Large Numbers, or, equivalently, by the Condorcet Jury Theorem [123]. By contrast, all leaves approve the center, and a naïve local delegation mechanism M would delegate all their votes. In that case, the decision depends only on the vote of the center, so $P_M(G_n) = 4/5$ for all $n \in \mathbb{N}$, and gain(M, G_n) converges to $-1/5$. We conclude that M violates the DNH property.

One might hope that there are "smarter" local delegation mechanisms, that, say, recognize that when a voter only has one approved neighbor, his vote should not be delegated. However, our first result shows that this is not the case: local delegation mechanisms cannot even satisfy the two minimal requirements of PG and DNH.

Theorem 1.1. *For any $\alpha_0 \in (0, 1)$ such that $i \in V$* approves *$j \in V$ if $(i, j) \in E$ and $p_j > p_i + \alpha_0$, there is no local mechanism that satisfies the PG and DNH properties.*

The first step in the proof is to better understand the way in which local mechanisms are constrained. This is captured by the following lemma.

Lemma 1.2. *Let M be a local mechanism. Then M satisfies the PG property only if there exist $k, m, \rho > 0$ such that, if a voter approves k out of his m total neighbors, then the total probability of delegation to any of these approved neighbors is exactly ρ.*

Proof. Suppose that PG holds. Let $\gamma > 0$ and fix a labeled graph G such that gain$(M, G) \geq \gamma > 0$. In order for this to be the case, there must exist some vertex i that delegates with positive probability ρ. Let k be the number of neighbors in G that i approves, and let m be his total number of neighbors in G; this yields the desired tuple (k, m, ρ).[3] □

The crux of the theorem's proof is the construction of a graph that, from the local viewpoint of many of the vertices, looks like the neighborhood prescribed by Lemma 1.2. Specifically, a *k-center m-uniform star* consists of vertices called *leaves* that are each connected to k central vertices (the centers) as well as $m - k$ other leaves. Each leaf vertex has competence level p_ℓ, and each center vertex has competence level p_c, such that $p_c > p_\ell + \alpha$. We set the value of k and m to be the values whose existence is guaranteed by Lemma 1.2, which means that the construction of a k-center m-uniform star satisfies the property that each leaf delegates to some center vertex with probability ρ. Throughout the proof, we will let $n_c = k$ be the number of centers, and n_ℓ will denote the number of leaves.

At a high level, we show that the loss of any local mechanism can approach $(1 - p_c)^k$, which is constant given k. We do this by constructing a graph that consists of a k-center m-uniform star with an independent disconnected component consisting of n_d vertices of competence level p_d. We set the parameters so that the direct voting mechanism D decides correctly with high probability. By contrast, under the local delegation mechanism M, enough leaves delegate their votes to the centers so that if all centers were to vote incorrectly, which happens with probability $(1 - p_c)^k$, then M would decide incorrectly. While the basic idea is simple enough, the formal construction is quite delicate, as many different parameters must be carefully balanced.

[3] Note that the conclusion is invariant to the ranking π_i.

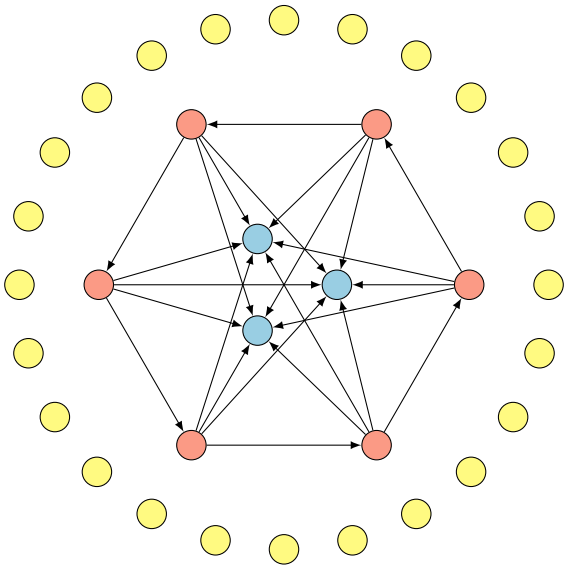


Figure 1.1: Graph G for $n_\ell = 6$ leaves (shown in red), $n_c = 3$ centers (shown in blue), $n_d = 24$ disconnected vertices (shown in yellow), and $m = 4$.

Proof of Theorem 1.1. Let M be a local mechanism that satisfies PG. By Lemma 1.2, there must exist at least one (k, m, ρ) tuple for M that satisfies the lemma's conclusion. For any n_1 prescribed by DNH and any α_0, we can construct a graph G_n with $n \geq n_1$ such that DNH does not hold.

Let G be a graph of size $n = n_c + n_\ell + n_d$ that consists of a k-center m-uniform star and a disconnected component containing n_d disconnected points (see Figure 1.1). Each center has competence level p_c, each leaf in the star has competence level p_ℓ, and each point in the disconnected component has competence level p_d. Given (k, m, ρ), n_1, and α_0, note that the following constraints must hold.

$$n_\ell \geq m - n_c \tag{1.1}$$

$$n = n_\ell + n_c + n_d \geq n_1 \tag{1.2}$$

$$p_c > p_\ell + \alpha_0 \tag{1.3}$$

We will prove that the construction above instantiated with the following parameter values violates DNH for any input of (k, m, ρ), n_1, and $\alpha = \alpha_0 + \varepsilon'$ for $\varepsilon' = \frac{1-\alpha_0}{2} > 0$, for sufficiently small δ (i.e., as $\delta \to 0$).

We begin by defining the sizes of each component: n_c, n_ℓ, and n_d.

$$n_c = k \tag{1.4}$$

$$n_\ell = \frac{n_1 m}{\alpha\delta} \tag{1.5}$$

$$n_d = C_1 \frac{n_1 m}{\alpha\delta} \tag{1.6}$$

Note that n_d depends on a constant C_1, which, along with another constant σ, is defined next.

$$C_1 = \frac{\left(\frac{\left(\frac{p_\ell \rho}{\sigma} n_\ell - p_\ell \sqrt{n_\ell}\right)}{2}\right)^2 - n_c}{n_\ell} - 1 \tag{1.7}$$

$$\sigma = \sqrt{-\frac{\ln\left(\frac{\delta}{2}\right)}{2}} \tag{1.8}$$

Now, we define the competency values for each component, p_c, p_ℓ, and p_d. Note that there is a range of acceptable compentency values for p_d.

$$p_c = \frac{1+\alpha}{2} \tag{1.9}$$

$$p_\ell = \frac{1-\alpha}{2} \tag{1.10}$$

$$p_d \in \left[\left(\frac{n/2 - n_\ell p_\ell}{n_d}\right) + \frac{\sigma\sqrt{n}}{n_d}, \left(\frac{n/2 - n_\ell p_\ell}{n_d}\right) + \frac{(n_\ell \rho - \tau)p_\ell - \sigma\sqrt{n}}{n_d}\right] \tag{1.11}$$

Finally, we define τ, another constant that will be useful for establishing concentration guarantees in the proof.

$$\tau = \sqrt{-\frac{\left(\ln\frac{\delta}{2}\right) n_\ell}{2}} \tag{1.12}$$

The following claim asserts that the construction is feasible.

Claim 1. *$C_1 > 0$ and the range of values for p_d in (1.11) is nonempty.*

Proof. From above, we have

$$C_1 = \frac{\left(\frac{\left(\frac{p_\ell \rho}{\sigma} n_\ell - p_\ell \sqrt{n_\ell}\right)}{2}\right)^2 - k}{n_\ell} - 1$$

and rearranging terms yields

$$2\sqrt{(C_1+1)n_\ell + k} = \frac{p_\ell \rho}{\sigma} n_\ell - p_\ell \sqrt{n_\ell}.$$

Now, note that $n_d = C_1 n_\ell$ and therefore $(C_1+1)n_\ell + k = n_d + n_\ell + k = n$. Additionally, note that $\sqrt{n_\ell} = \frac{\tau}{\sigma}$. Substituting this in, we have

$$2\sqrt{n} = \frac{p_\ell \rho n_\ell - p_\ell \tau}{\sigma}$$

and therefore

$$\sigma\sqrt{n} = p_\ell(\rho n_\ell - \tau) - \sigma\sqrt{n}. \tag{1.13}$$

Now, we note that $\sigma\sqrt{n} - kp_c < \sigma\sqrt{n}$ and

$$\sqrt{n} > \sqrt{n - k - (\rho n_\ell - \tau)}$$

because both k and $\rho n_\ell - \tau = 2\sigma\sqrt{n}/p_\ell$ are greater than 0, and $n - k - (\rho n_\ell - \tau) > n - k - n_\ell \geq 0$. Now, from (1.13), we can conclude that

$$\begin{aligned}\sigma\sqrt{n} - kp_c < \sigma\sqrt{n} &= p_\ell(\rho n_\ell - \tau) - \sigma\sqrt{n} \\ &< (\rho n_\ell - \tau)p_\ell - \sigma\sqrt{n - k - (\rho n_\ell - \tau)},\end{aligned}$$

which means

$$p_d \in \left[\left(\frac{n/2 - n_\ell p_\ell}{n_d}\right) + \frac{\sigma\sqrt{n} - kp_c}{n_d}, \left(\frac{n/2 - n_\ell p_\ell}{n_d}\right) + \frac{(n_\ell\rho - \tau)p_\ell - \sigma\sqrt{n - k - (n_\ell\rho - \tau)}}{n_d}\right]$$

is non-empty, and our value for p_d is admissible.

Lastly, we have to show that C_1 is itself admissible; i.e., that the following holds:

$$\frac{\left(\frac{\left(\frac{p_\ell\rho}{\sigma}n_\ell - p_\ell\sqrt{n_\ell}\right)}{2}\right)^2 - k}{n_\ell} - 1 > 0.$$

Rearranging and expanding, we obtain

$$\frac{p_\ell n_\ell \rho}{\sigma} - p_\ell\sqrt{n_\ell} \geq 2\sqrt{n_\ell + k}.$$

Now, note that both sides are positive as $\delta \to 0$. Indeed, the right hand side consists of positive terms and the left hand side simplifies to $p_\ell\sqrt{n_\ell}(\rho\sqrt{n_\ell}/\sigma - 1)$, which is positive iff $\rho\sqrt{n_\ell} > \sigma$, which is true as $\delta \to 0$ because $1/\delta$ grows more quickly than $\ln(2/\delta)$. Therefore, squaring both sides yields

$$\left(\frac{p_\ell n_\ell \rho}{\sigma}\right)^2 + (p_\ell)^2 n_\ell - 2\frac{(p_\ell)^2\rho(n_\ell)^{3/2}}{\sigma} \geq 4(n_\ell + k).$$

Now, substituting in our value for n_ℓ, we obtain

$$\left[\left(\frac{p_\ell\rho}{\sigma}\right)\left(\frac{n_1 m}{\alpha\delta}\right)\right]^2 + (p_\ell)^2\left(\frac{n_1 m}{\alpha\delta}\right) - \frac{2(p_\ell)^2\rho}{\sigma}\left(\frac{n_1 m}{\alpha\delta}\right)^{3/2} - 4\left(\frac{n_1 m}{\alpha\delta}\right) - 4k. \tag{1.14}$$

As $\delta \to 0$, (1.14) becomes dominated by the highest-order $1/\delta$ term, and therefore is always positive for any assignment to the other variables because the rest of them are constrained to be strictly positive. □

Because $\alpha, \delta \in (0, 1)$, the value of n_ℓ in (1.5) is greater than both n_1 and m, hence constraints (1.1) and (1.2) are immediately satisfied. Moreover, constraint (1.3) is satisfied by (1.9) and (1.10).

Turning to the proof that DNH is violated, let cor_D, del_M, and $nondel_M$ be the random variables corresponding to the number of correct votes under D, the number of delegated correct votes under M, and the number of non-delegated correct votes under M. Additionally, let ε, τ (defined again below), and ξ be as follows.

$$\varepsilon = \sqrt{-\frac{\left(\ln \frac{\delta}{2}\right) n}{2}},$$
$$\tau = \sqrt{-\frac{\left(\ln \frac{\delta}{2}\right) n_\ell}{2}}, \text{ and}$$
$$\xi = \sqrt{-\frac{\left(\ln \frac{\delta}{2}\right) (n - n_c - (\rho n_\ell - \tau))}{2}}.$$

Our goal is to bound the expectations of cor_D, del_M, and $nondel_M$. First, we examine $\mathbb{E}[cor_D]$. We would like to show that

$$\mathbb{E}[cor_D] \geq n/2 + \varepsilon. \tag{1.15}$$

Expanding out the expected value, this is equivalent to

$$p_c n_c + p_\ell n_\ell + p_d n_d \geq n/2 + \varepsilon.$$

From (1.11), we have

$$p_d \geq \frac{n/2 - p_\ell n_\ell + \varepsilon}{n_d},$$

so it is sufficient to show that

$$p_c n_c + p_\ell n_\ell + n_d \left(\frac{n/2 - p_\ell n_\ell + \varepsilon}{n_d}\right) \geq n/2 + \varepsilon,$$

and simplifying yields $p_c n_c \geq 0$. This is true by Equation (1.9), because α and k are both constrained to be strictly positive.

Next, we examine $\mathbb{E}[del_M]$. We would like to show that

$$\mathbb{E}[del_M] = n_\ell \rho. \tag{1.16}$$

This is trivial to see, as del_M is a sum of n_ℓ Bernoulli random variables with "success" probability ρ.

Finally, we examine the "typical case" over $nondel_M$, or $\mathbb{E}[nondel_M | del_M = v]$ for all integers $v \in [n_\ell \rho - \tau, n_\ell \rho + \tau]$. Intuitively, this case considers the number of correct votes cast by still-independent vertices after "enough" leaf vertices have delegated their votes. If these votes do not make up a majority, then all centers voting incorrectly will cause the entire graph to vote incorrectly. We would like to show that

$$\mathbb{E}[nondel_M | del_M = v] \leq n/2 - \xi. \tag{1.17}$$

for all integers $v \in [n_\ell\rho - \tau, n_\ell\rho + \tau]$. Conditionally on del_M being in the prescribed range above, we see that in the worst case, $del_M = n_\ell\rho - \tau$, meaning the fewest possible voters delegate under this assumption. Given this, we would like to show that

$$p_d n_d + p_\ell(n_\ell - (\rho n_\ell - \tau)) \leq n/2 - \xi.$$

From Equation (1.11) we have

$$p_d \leq \frac{n/2 - p_\ell n_\ell + (n_\ell\rho - \tau)p_\ell - \xi}{n_d},$$

which yields

$$\left(\frac{n/2 - p_\ell n_\ell + (n_\ell\rho - \tau)p_\ell - \xi}{n_d}\right) n_d + p_\ell(n_\ell - (\rho n_\ell - \tau)) \leq n/2 - \xi.$$

Simplifying yields $0 \leq 0$—a tautology. This establishes Equation (1.17).

We now wish to bound the probability of cor_D, del_M, and $nondel_M$ deviating by too much. We use Hoeffding's inequality [127], which states that given n independent Bernoulli random variables $X_i \in [0, 1]$ and $X = \sum_i X_i$, the following concentration bound holds:

$$\Pr\left[|X - \mathbb{E}[X]| \geq \varepsilon\right] \leq 2\exp\left(\frac{-2\varepsilon^2}{n}\right). \tag{1.18}$$

First, we examine cor_D. From (1.18) and a straightforward substitution for ε, we obtain

$$\begin{aligned}\Pr\left(|cor_D - \mathbb{E}[cor_D]| \geq \varepsilon\right) &\leq 2\exp\left(\frac{-2\varepsilon^2}{n}\right) \\ &= 2\exp\left(-\frac{2\left[\sqrt{-\frac{\left(\ln\frac{\delta}{2}\right)n}{2}}\right]^2}{n}\right) \\ &= \delta.\end{aligned} \tag{1.19}$$

Likewise, for del_M, from (1.18) and a straightforward substitution for τ, we obtain

$$\begin{aligned}\Pr\left[|del_M - \mathbb{E}[del_M]| \geq \tau\right] &\leq 2\exp\left(\frac{-2\tau^2}{n_\ell}\right) \\ &= 2\exp\left(-\frac{2\left[\sqrt{-\frac{\left(\ln\frac{\delta}{2}\right)n_\ell}{2}}\right]^2}{n_\ell}\right) \\ &= \delta.\end{aligned} \tag{1.20}$$

Finally, for $nondel_M$, we are interested in upper-bounding

$$\Pr[|nondel_M - \mathbb{E}[nondel_M | del_M = v]| \geq \xi \mid del_M = v],$$

for every integer $v \in [n_\ell\rho - \tau, n_\ell\rho + \tau]$. As before, we apply Equation (1.18), and, as it turns out, we can derive an upper bound when $del_M = n_\ell\rho - \tau$. Therefore, we obtain that for every $v \in [n_\ell\rho - \tau, n_\ell\rho + \tau]$,

$$\begin{aligned}
&\Pr\left[|nondel_M - \mathbb{E}[nondel_M | del_M = v]| \geq \xi \mid del_M = v\right] \\
&\quad \leq 2\exp\left(\frac{-2\xi^2}{n - n_c - (\rho n_\ell - \tau)}\right) \\
&\quad = 2\exp\left(\frac{2\left[\sqrt{-\frac{\left(\ln\frac{\delta}{2}\right)(n-n_c-(\rho n_\ell-\tau))}{2}}\right]^2}{n - n_c - (\rho n_\ell - \tau)}\right) \\
&\quad = \delta,
\end{aligned} \tag{1.21}$$

where the denominator comes from the (worst-case) total number of non-delegated votes under M.

From the above, we see that

$$\begin{aligned}
\Pr[cor_D > n/2] &\geq 1 - \delta, && \text{(by (1.15) and (1.19))} \\
\Pr[del_M \in (n_\ell\rho - \tau, n_\ell\rho + \tau)] &\geq 1 - \delta, && \text{(by (1.16) and (1.20))} \\
\Pr[nondel_M < n/2 \mid del_M = v] &\geq 1 - \delta, && \text{(by (1.17) and (1.21))}
\end{aligned}$$

where the last inequality holds for all integers $v \in [n_\ell\rho - \tau, n_\ell\rho + \tau]$.

Therefore, the lower bound on the probability of D deciding correctly is $p_d(G) \geq 1 - \delta$. We can lower-bound the probability of M deciding incorrectly in order to upper-bound $P_M(G)$. We slightly overload notation and let M be the event that M decides correctly, and $\neg M$ be the event that M decides incorrectly. Moreover, denote by V the event that $del_M \in [n_\ell\rho - \tau, n_\ell\rho + \tau]$. By definition, we have

$$\Pr[\neg M] = \Pr[\neg M | V] \Pr[V] + \Pr[\neg M | \neg V] \Pr[\neg V],$$

and because probabilities cannot be negative,

$$\Pr[\neg M] \geq \Pr[\neg M | V] \Pr[V].$$

Now, because $\Pr[V] \geq 1 - \delta$,

$$\Pr[\neg M] \geq \Pr[\neg M | V](1 - \delta).$$

Furthermore, we know that $\Pr[\neg M | V]$ is also lower-bounded by $(1 - p_c)^{n_c}(1 - \delta)$ because one setting under which M decides incorrectly is exactly when all centers vote incorrectly and $nondel_M < n/2$. It follows that

$$\Pr[\neg M] \geq (1 - p_c)^{n_c}(1 - \delta)(1 - \delta).$$

Therefore, taking the complement, we have an upper bound on the probability of M voting correctly of

$$\Pr[M] \leq 1 - (1 - p_c)^{n_c}(1 - \delta)^2,$$

and the total loss can be lower-bounded by

$$(1 - \delta) - (1 - (1 - p_c)^{n_c}(1 - \delta)^2) = (1 - p_c)^{n_c}(1 - \delta)^2 - \delta.$$

As $\delta \to 0$, this tends to $(1 - p_c)^{n_c} = (1 - p_c)^k$, which is constant and bounded away from 0. We conclude that M violates the DNH property. □

We note that even if each voter had access to a ranking of his approved neighbors by competence, this impossibility still holds because the construction is such that all approved vertices have equal competence.

1.1.5 Possibility for Non-Local Mechanisms

The main idea underlying Theorem 1.1 is that liquid democracy can correlate the votes to the point where the mistakes of a few popular voters tip the scales in the wrong direction. As we show in the theorem's proof, this is unavoidable under local delegation mechanisms, which, intuitively, cannot identify situations in which certain voters amass a large number of votes. However, non-local delegation mechanisms can circumvent this issue. Indeed, consider the following delegation mechanism.

input: labeled graph G with n vertices, cap $C : \mathbb{N} \to \mathbb{N}$

$V' \leftarrow V$
while $V' \neq \emptyset$ **do**
  let $i \in \text{argmax}_{j \in V'} |A_G^{-1}(j) \cap V'|$
  $J \leftarrow A_G^{-1}(i) \cap V'$
  if $|J| \leq C(n) - 1$ **then**
    $J' \leftarrow J$
  else
    let $J' \subseteq J$ such that $|J'| = C(n) - 1$
  end if
  vertices in J' delegate to i
  $V' \leftarrow V' \setminus (\{i\} \cup \{J'\})$
end while

Algorithm 1: GreedyCap

In words, the mechanism GreedyCap, given as Algorithm 1, receives as input a labeled graph G, and a *cap* C which is a function of n. It iteratively selects a voter with maximum approvals, and delegates votes to him, so that no more than $C(n)-1$ votes are delegated to a single voter (that is, no voter can have weight more than $C(n)$). All voters involved in the current iteration are then eliminated from further consideration, which is why delegations under this mechanism are only 1-hop.

It is obvious that GreedyCap satisfies the PG property. Intuitively, for any value of α, it is always possible to construct large instances of graphs where a few voters delegate to more competent voters in a way that increases the probability of making the correct decision overall. However, although it seems at first glance that it should satisfy DNH as well (as it solves the excessive correlation problem), the following example shows that, without further assumptions, it does not.

Example 2. *Assume for ease of exposition that $\alpha < 1/3$. For any odd $n = 2k+1$, consider the labeled graph $G_n = (V, E, \vec{p})$ on n vertices, defined as follows: $E = \{(1, 2)\}$ (i.e., the only edge in the graph is from 1 to 2), $p_1 = 1/3$, $p_2 = 2/3$, there are k vertices with $p_i = 1$, and $k - 1$ vertices with $p_i = 0$. Even if $C(n) \equiv 2$, GreedyCap would delegate the vote of voter 1 to voter 2. Therefore, the mechanism decides correctly if and only if voter 2 votes correctly, which happens with probability $2/3$. By contrast, under direct voting, it is enough for either voter 1 or voter 2 to vote correctly, which happens with probability $7/9$. It follows that the loss of GreedyCap is $1/9$—a constant. We conclude that GreedyCap violates DNH.*

The reason the example works is that the outcome completely depends on voters 1 and 2, as the others vote deterministically (competence level 0 or 1). To avoid this problem, we make the natural assumption that competence levels are bounded away from 0 and 1, i.e., voters are never horribly misinformed or perfectly informed. It turns out that this additional assumption is sufficient to guarantee that GreedyCap satisfies the DNH property.

Theorem 1.3. *Assume that there exists $\beta \in (0, 1/2)$ such that all competence levels are in $[\beta, 1 - \beta]$. Then for any difference in competencies $\alpha \in (0, 1 - 2\beta)$, GreedyCap with cap $C : \mathbb{N} \to \mathbb{N}$ such that $C(n) \in \omega(1)$ and $C(n) \in o(\sqrt{\log n})$ satisfies the PG and DNH properties.*

We begin with a proof sketch, focusing on the DNH property (as PG is rather simple). Given n voters, we denote the number of correct votes under direct voting and GreedyCap by X_D and X_M, respectively, and consider two cases.

1. $|\mathbb{E}[X_D] - \frac{n}{2}| > \frac{n}{\log n}$.
2. $|\mathbb{E}[X_D] - \frac{n}{2}| \leq \frac{n}{\log n}$.

In Case 1, the direct voting mechanism has mean far away from $n/2$. When $\mathbb{E}[X_D] < n/2 - n/\log n$, we can show that P_D goes to 0 as n goes to infinity. This means that DNH is satisfied for any value of P_M. In the case where $\mathbb{E}[X_D] > n/2 + n/\log n$, we can show that P_M goes to 1 as n goes to infinity, which means that DNH is satisfied for any value of P_D.

In Case 2, the direct voting setting has mean close to $n/2$. From here, we consider two subcases.

1. The number of voters who delegate is greater than $n/g(n)$, where $g(n) \in o(\log n)$ and $g(n) \in \omega(C(n)^2)$. Note that this yields—hence the upper bound on $C(n)$ in the statement of Theorem 1.3.
2. The number of voters who delegate is at most $n/g(n)$.

In Subcase 1, because a relatively large fraction of voters delegate their votes to more competent neighbors, $\mathbb{E}[X_M] - \mathbb{E}[X_D]$ is large enough to offset the simultaneous increase in the variance of X_M, and, in the limit, P_M goes to 1. In Subcase 2, we again have $\mathbb{E}[X_M] \geq \mathbb{E}[X_D]$ due to delegation. Additionally, because so few voters delegate, the ratio of the variance of X_M and that of X_D converges to 1 as n approaches infinity, which means that (in the worst case) the difference between P_D and P_M converges to 0.

Before presenting the theorem's detailed proof, we establish three useful lemmas that establish uniform convergence as n grows large.

The first lemma is the Lindeberg Central Limit Theorem [161], reproduced below.

Lemma 1.4 (Lindeberg Central Limit Theorem [161])**.** *Let $\{X_{ni} : n \geq 1; i = 1, \ldots, k_n\}$ be a triangular array of random variables with $X_{n1}, \ldots, X_{nk_n}$ independent for each n, $\mathbb{E}[X_{ni}] = 0$, and $\sum_{i=1}^{k_n} \mathbb{E}[X_{ni}^2] = 1$. If for each fixed $\epsilon > 0$,*

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} \mathbb{E}\left[X_{ni}^2 \mathbb{1}\{|X_{ni}| > \epsilon\}\right] = 0,$$

where $\mathbb{1}$ is an indicator random variable, then

$$Z_n = \sum_{i=1}^{k_n} X_{ni} \to N(0, 1).$$

In our proof, we will use the Lindeberg Central Limit Theorem to prove the following key lemma, which states that we can treat arbitrary instances of liquid democracy like normal distributions as the number of voters increases.

Lemma 1.5. *For all $\beta \in (0, 1/2)$ and $C(n) \in o(\sqrt{n})$, for all $\epsilon > 0$, there is a constant $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$, given arbitrary competencies $p_1, \ldots, p_n$ where each $p_i \in (\beta, 1-\beta)$ for $\beta \in (0, 1/2)$ and weights $b_1, \ldots, b_n$ such that each $b_i \in [0, C(n)]$ and $\sum_{i=1}^n b_i = n$, we have*

$$\left| \Pr[X > n/2] - \Phi\left(\frac{\mathbb{E}[X] - n/2}{\sqrt{\mathrm{Var}[X]}}\right) \right| < \epsilon,$$

where X represents the number of correct votes in liquid democracy with these competencies and weights, and $\mathbb{E}[X] = \sum_{i=1}^n b_i \cdot p_i$ and $\mathrm{Var}[X] = \sum_{i=1}^n b_i^2 p_i(1 - p_i)$.

Proof. Let $\{Y_{nk} : 1 \leq k \leq n\}$ be a triangular array of independent Bernoulli random variables where Y_{ni} has success probability $p_{nk} \in [\beta, 1 - \beta]$ for $\beta \in (0, 1/2)$. Furthermore, define $\{b_{nk} : 1 \leq k \leq n\}$ be a triangular array of nonnegative integers such that $0 \leq b_{nk} \leq C(n)$ for all $1 \leq k \leq n$ and $\sum_{k=1}^n b_{nk} = n$. Lastly, define the triangular array $X_{nk} \coloneqq b_{nk} \cdot Y_{nk}$.

We will first show that the sum of all the X_{nk} random variables is approximately normal, or

$$Z_n = \sum_{k=1}^n X_{nk} \to Z \sim N\left(\sum_{k=1}^n b_{nk}\,\mathbb{E}[Y_k], \sum_{k=1}^n b_{nk}^2\,\mathrm{Var}[Y_k]\right). \tag{1.22}$$

In order to show Equation (1.22), define $s_n^2 = \sum_{k=1}^n \operatorname{Var}[X_{nk}] = \sum_{k=1}^n b_{nk}^2 \operatorname{Var}[Y_{nk}]$. Let

$$W_{nk} \coloneqq \frac{X_{nk} - \mathbb{E}[X_{nk}]}{s_n} = \frac{X_{nk} - \mathbb{E}[X_{nk}]}{\sqrt{\sum_{k=1}^n b_{nk}^2 \operatorname{Var}[Y_{nk}]}}.$$

We must first show that the triangular array W_{nk} satisfies the preconditions of Lemma 1.4. Note that $X_{n1}, \ldots, X_{nn}$ are independent because the $\{Y_{nk}\}$ are independent. Also, by definition, $\mathbb{E}[W_{nk}] = 0$. Furthermore, we have

$$\begin{aligned}
\sum_{k=1}^n \mathbb{E}[W_{nk}^2] &= \sum_{k=1}^n \mathbb{E}\left[\frac{(X_{nk} - \mathbb{E}[X_{nk}])^2}{s_n^2}\right] \\
&= \frac{1}{s_n^2} \sum_{k=1}^n \mathbb{E}[(X_{nk} - \mathbb{E}[X_{nk}])^2] \\
&= \frac{1}{s_n^2} \sum_{k=1}^n b_{nk}^2 \, \mathbb{E}[(Y_{nk} - \mathbb{E}[Y_{nk}])^2] \\
&= \frac{1}{\sum_{k=1}^n b_{nk}^2 \operatorname{Var}[Y_{nk}]} \sum_{k=1}^n b_{nk}^2 \operatorname{Var}[Y_{nk}] \\
&= 1.
\end{aligned}$$

Now, we must check whether the last precondition holds, i.e., whether

$$\lim_{n\to\infty} \sum_{k=1}^n \mathbb{E}\left[W_{nk}^2 \mathbb{1}\{|W_{nk}| > \epsilon\}\right] = 0.$$

We have

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^n \mathbb{E}\left[W_{nk}^2 \mathbb{1}\{|W_{nk}| > \epsilon\}\right] &= \lim_{n\to\infty} \sum_{k=1}^n \mathbb{E}\left[\frac{(X_{nk} - \mathbb{E}[X_{nk}])^2}{s_n^2} \mathbb{1}\left\{|b_{nk}(Y_{nk} - \mathbb{E}[Y_{nk}])| > \epsilon \cdot s_n\right\}\right] \\
&= \lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^n b_{nk}^2 \, \mathbb{E}\left[(Y_{nk} - \mathbb{E}[Y_{nk}])^2 \mathbb{1}\left\{|b_{nk}(Y_{nk} - \mathbb{E}[Y_{nk}])| > \epsilon \cdot s_n\right\}\right].
\end{aligned}$$

Let us now examine the $\mathbb{1}\left\{|b_{nk}(Y_{nk} - \mathbb{E}[Y_{nk}])| > \epsilon \cdot s_n\right\}$ term. On the left side of the inequality, we have that

$$|b_{nk}(Y_{nk} - \mathbb{E}[Y_{nk}])| \in o(\sqrt{n})$$

because $b_{nk} \in o(\sqrt{n})$, and $|Y_{nk} - \mathbb{E}[Y_{nk}]| \leq \max(\beta, 1-\beta) \leq 1 - \beta$ because we consider $\beta \in (0, 1/2)$. Furthermore, on the right side, we have

$$\begin{aligned}
s_n &= \sqrt{\sum_{k=1}^n b_{nk}^2 p_{nk}(1 - p_{nk})} \\
&\geq \sqrt{\sum_{k=1}^n b_{nk}^2 \beta(1-\beta)} && (p_{nk} \in (\beta, 1-\beta)) \\
&\geq \sqrt{n\beta(1-\beta)} && (\textstyle\sum_{k=1}^n b_{nk} = n) \\
&\in \Omega(\sqrt{n}).
\end{aligned}$$

Therefore, as n approaches infinity, the probability that the left hand side exceeds the right hand side goes to 0, satisfying the final precondition. We may now apply the Lindeberg Central Limit Theorem to see that

$$Z'_n = \sum_{k=1}^{n} W_{nk} \to N(0,1) \text{ as } n \to \infty.$$

By straightforward scaling and shifting arguments, as above, we see that this implies that

$$Z_n = \sum_{k=1}^{n} X_{nk} \to N\left(\sum_{k=1}^{n} b_{nk}\,\mathbb{E}[Y_k], \sum_{k=1}^{n} b_{nk}^2\,\mathrm{Var}[Y_k]\right) \text{ as } n \to \infty,$$

as desired.

This means that all instances with arbitrary competence and weight values approach a normal distribution as the size of the instance grows. We now must show uniform convergence. In order to do so, first assume toward a contradiction that there exists an ϵ such that for infinitely many n, there exist b_n and p_n such that the resulting instance is not within ϵ of normal. Then, we may construct a sequence including all of these failing instances, which would violate Equation (1.22). □

We also require the following simple lemma.

Lemma 1.6. *Given a normally distributed variable $X \sim \mathcal{N}(\mathbb{E}[X], \mathrm{Var}[X])$ with $\mathbb{E}[X] \in [\mu_{min}, \mu_{max}]$ and $\mathrm{Var}[X] \in [\sigma^2_{min}, \sigma^2_{max}]$, then the following is true.*

Case 1: if $\mu_{max} > k$:

$$\Pr[X > k] \leq \Pr[Y \sim \mathcal{N}(\mu_{max}, \sigma^2_{min}) > k]$$
$$\Pr[X > k] \geq \Pr[Y \sim \mathcal{N}(\mu_{min}, \sigma^2_{max}) > k]$$

Case 2: if $\mu_{max} < k$:

$$\Pr[X > k] \leq \Pr[Y \sim \mathcal{N}(\mu_{max}, \sigma^2_{max}) > k]$$
$$\Pr[X > k] \geq \Pr[Y \sim \mathcal{N}(\mu_{min}, \sigma^2_{min}) > k]$$

Proof. For both upper bounds, we want to minimize the value of $\Phi\left(\frac{k-\mathbb{E}[X]}{\mathrm{Var}[X]}\right)$. Because Φ is monotonically increasing, this is equivalent to minimizing the value of $\frac{k-\mathbb{E}[X]}{\mathrm{Var}[X]}$. It is clear that $k - \mu_{max} < k - \mu_{min}$. Now, if $k - \mu_{max} < 0$, then

$$\frac{k-\mu_{max}}{\sigma_{min}} < \frac{k-\mu_{max}}{\sigma_{max}}.$$

However, if $k - \mu_{max} > 0$, then

$$\frac{k-\mu_{max}}{\sigma_{max}} < \frac{k-\mu_{max}}{\sigma_{min}}.$$

For both lower bounds, we want to maximize the value of $\Phi\left(\frac{k-\mathbb{E}[X]}{\mathrm{Var}[X]}\right)$. Because Φ is monotonically increasing, this is equivalent to maximizing the value of $\frac{k-\mathbb{E}[X]}{\mathrm{Var}[X]}$. As in the above case, it is clear that $k - \mu_{min} > k - \mu_{max}$. Now, if $k - \mu_{min} < 0$, then

$$\frac{k - \mu_{min}}{\sigma_{max}} > \frac{k - \mu_{min}}{\sigma_{min}}.$$

However, if $k - \mu_{min} > 0$, then

$$\frac{k - \mu_{min}}{\sigma_{min}} > \frac{k - \mu_{min}}{\sigma_{max}}.$$

□

Finally, by definition, $\mathrm{Erf}(\infty) = 1$ and $\mathrm{Erf}(-\infty) = -1$, where $\mathrm{Erf}(\cdot)$ denotes the (Gauss) error function,

$$\mathrm{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt.$$

We will use this fact repeatedly throughout the proof of the theorem, which we now turn to.

Proof of Theorem 1.3. Given a total number of voters n, let us define two random variables, X_D and X_M, where X_D denotes the number of correct votes under the direct voting mechanism D, and X_M represents the (weighted) number of correct votes under GreedyCap. We are interested in comparing $P_D = \Pr[X_D > n/2]$ and $P_M = \Pr[X_M > n/2]$.

Let $V = \{V_1, \ldots, V_n, \ldots\}$ be a sequence of independent Bernoulli random variables in which V_i represents the vote of voter i; i.e., each V_i has success probability $p_i \in [\beta, 1-\beta]$ for $\beta \in (0, 1/2)$. Using V, we define a sequence of instances indexed by n, where each instance consists of the first n voters in V. Let $\{b_{ni}^D : 1 \le i \le n\}$ and $\{b_{ni}^M : 1 \le i \le n\}$ be triangular arrays of nonnegative integers that denote the weight of each voter under direct voting and GreedyCap, respectively. Under direct voting, $b_{ni}^D = 1$ for all $1 \le i \le n$. Note that, in this case, $0 \le b_{ni}^D \le C(n)$ for all voters i, and $\sum_{i=1}^n b_{ni}^D = n$. Now, in the delegative case, let $b_{ni}^M = w_{ni}$ for all $1 \le i \le n$, where $w_{ni} \in \mathbb{Z}_{\ge 0}$ is the total weight accumulated by voter i in instance n (note that voters who choose to delegate have weight zero). Because each voter cannot accumulate weight greater than $C(n)$, we have that $0 \le w_{ni} \le C(n)$ for all voters i, and $\sum_{i=1}^n w_{ni} = n$.

Note that, given a population of voters of size n, $X_D = \sum_{i=1}^n b_{ni}^D V_i = \sum_{i=1}^n V_i$ and $X_M = \sum_{i=1}^n b_{ni}^M V_i = \sum_{i=1}^n w_{ni} V_i$. Now, because X_D and X_M both satisfy the conditions under which Lemma 1.1.5 holds, we may apply Lemma 1.1.5 to establish that X_D and X_M are approximately normally distributed as n goes to infinity; i.e.,

$$X_D = \sum_{i=1}^n V_i \to N\left(\sum_{i=1}^n \mathbb{E}[V_i], \sum_{i=1}^n \mathrm{Var}[V_i]\right) \text{ as } n \to \infty$$

and

$$X_M = \sum_{i=1}^n w_{ni} V_i \to N\left(\sum_{i=1}^n w_{ni}\, \mathbb{E}[V_i], \sum_{i=1}^n w_{ni}^2\, \mathrm{Var}[V_i]\right) \text{ as } n \to \infty.$$

Therefore, we can use the following formulas.

$$P_D \approx \int_{n/2}^{n} \frac{1}{\sqrt{2\pi \operatorname{Var}[X_D]}} \exp\left(\frac{-(x - \mathbb{E}[X_D])^2}{2 \operatorname{Var}[X_D]}\right) dx \tag{1.23}$$

$$P_M \approx \int_{n/2}^{n} \frac{1}{\sqrt{2\pi \operatorname{Var}[X_M]}} \exp\left(\frac{-(x - \mathbb{E}[X_M])^2}{2 \operatorname{Var}[X_M]}\right) dx \tag{1.24}$$

Indeed, throughout this proof, we will assume that P_D and P_M are exactly equal to these quantities; this is because Lemma 1.1.5 says that as n goes to infinity, this approximation becomes arbitrarily accurate.

Note that, from above, the PG property means that there exists ε such that $P_M - P_D > \varepsilon$ for at least one graph G_n on n vertices for all suitably large n. Similarly, the DNH property corresponds to $P_D - P_M < \varepsilon$ for all graphs G_n on n vertices for suitably large n and all values of ε. We show that these two properties hold.

For the PG property, we construct a simple family of examples where the property is satisfied. Let the social graph G be composed of pairs of nodes with one competent voter and one incompetent voter with an edge pointing to the competent voter. The competent voters have competence $1-\beta$ and the incompetent voters have competence β. If the voters vote independently, the symmetry between the competent and incompetent voters makes it clear that $P_D = 1/2$. Under Algorithm 1, the incompetent voters all delegate to the competent voters. We now have $\frac{n}{2}$ independent voters who each have one vote of weight two and competence $1 - \beta$. By the Condorcet Jury Theorem [123], it follows that P_M approaches 1.

In the remainder of the proof, therefore, we focus on establishing the DNH property. We first show that

$$\operatorname{Var}[X_D] \in [\beta(1-\beta)n, n/4]. \tag{1.25}$$

Indeed, $X_D = \sum_{i=1}^{n} V_i$, where V_i is the Bernoulli random variable representing the vote of voter i. In particular, $V_i \sim \text{Bernoulli}(p_i)$, where $p_i \in [\beta, 1-\beta]$ is the competence level of voter i. Because all voters vote independently, $\operatorname{Var}[X_D] = \sum_{i=1}^{n} \operatorname{Var}[V_i]$, and

$$\operatorname{Var}[V_i] = p_i(1-p_i) \in [\beta(1-\beta), (1/2)^2].$$

This establishes Equation (1.25).

Now, let us separate the instances into two cases:

1. $|\mathbb{E}[X_D] - \frac{n}{2}| > \frac{n}{\log n}$.
2. $|\mathbb{E}[X_D] - \frac{n}{2}| \leq \frac{n}{\log n}$.

Case 1. In this case, we can give strong lower bounds on both P_D and P_M.

Subcase 1: $\mathbb{E}[X_D] < n/2 - n/\log n$. By Equation (1.25), $\operatorname{Var}[X_D] \leq n/4 < n$. Because $\mathbb{E}[X_D] < n/2$, by Lemma 1.6 we have

$$P_D < \int_{\frac{n}{2}}^{n} \frac{1}{\sqrt{2\pi n^2}} e^{\frac{-\left(x - \frac{n}{2} + \frac{n}{\log n}\right)^2}{2n}} dx. \tag{1.26}$$

This is equivalent to

$$P_D < \frac{1}{2}\left(\text{Erf}\left(\frac{\sqrt{n}(2+\log n)}{2\sqrt{2}\log n}\right) - \text{Erf}\left(\frac{\sqrt{n}}{\sqrt{2}\log n}\right)\right).$$

As n approaches infinity, both arguments go to infinity, and therefore (as $\text{Erf}(\infty) = 1$) P_D approaches 0. This means that, no matter the value of P_M, DNH is satisfied.

Subcase 2: $\mathbb{E}[X_D] > n/2 + n/\log n$. We now examine the maximum possible value of $\text{Var}[X_M] = \sum_{i=1}^{n} w_{ni}^2 \text{Var}[V_i]$, where w_{ni} is the total weight accumulated by voter i and, again, V_i is the Bernoulli random variable representing the vote of voter i. Additionally, $\text{Var}[V_i] \in [\beta(1-\beta), 1/4]$, and applying this yields

$$\text{Var}[X_M] \leq \frac{1}{4} \cdot \sum_{i=1}^{n} w_{ni}^2.$$

Because each voter can accumulate at most weight $C(n)$, by the convexity of x^2, we can see that this is maximized when the maximum number of voters have weight exactly $C(n)$. Therefore, we have

$$\text{Var}[X_M] \leq \frac{1}{4} \cdot \sum_{i=1}^{\lceil n/C(n) \rceil} C(n)^2 < nC(n).$$

Because $\mathbb{E}[X_D] > n/2$, by Lemma 1.6 we have

$$P_M > \int_{\frac{n}{2}}^{n} \frac{1}{\sqrt{2\pi n C(n)}} e^{\frac{-\left(x - \frac{n}{2} + \frac{n}{\log n}\right)^2}{2nC(n)}}\, dx. \tag{1.27}$$

This simplifies to

$$P_M > \frac{1}{2}\left(\text{Erf}\left(\frac{\sqrt{n}(\log n - 2)}{2\sqrt{2C(n)}\log n}\right) + \text{Erf}\left(\frac{\sqrt{n}}{\sqrt{2C(n)}\log n}\right)\right).$$

As n approaches infinity, both arguments go to infinity, and P_M approaches 1. Therefore, no matter what the value of P_D, DNH is satisfied.

Case 2. In this case, we split the argument into two further subcases:

1. The number of voters who delegate is greater than $n/g(n)$, where $g(n)$ is $o(\log n)$ and $\omega(C(n)^2)$.
2. The number of voters who delegate is less than or equal to $n/g(n)$.

Subcase 1: Due to delegation, we have $\mathbb{E}[X_M] - \mathbb{E}[X_D] \geq n\alpha/g(n)$. We can now bound the mean by

$$\mathbb{E}[X_M] \geq \frac{n}{2} - \frac{n}{\log n} + \frac{n\alpha}{g(n)}.$$

Therefore, because $g(n) = o(\log n)$, $\mathbb{E}[X_M] > n/2$ as n increases. As before, we also know that $\text{Var}[X_M]$ is bounded from above by $nC(n)$, and therefore, by Lemma 1.6,

$$P_M \geq \int_{\frac{n}{2}}^{n} \frac{1}{\sqrt{2\pi n C(n)}} e^{\frac{-\left(x - \frac{n}{2} - \frac{n}{\log n} - \frac{n\alpha}{g(n)}\right)^2}{2nC(n)}} \, dx. \tag{1.28}$$

We would like to show that this integral goes to 1 as n goes to infinity.

This is equivalent to

$$\frac{1}{2}\left(\text{Erf}\left(\frac{\frac{n}{2} - \frac{n\alpha}{g(n)} + \frac{n}{\log n}}{\sqrt{2nC(n)}}\right) - \text{Erf}\left(\frac{\sqrt{n}\left(-\frac{\alpha}{g(n)} + \frac{1}{\log n}\right)}{\sqrt{2C(n)}}\right)\right).$$

Note that as n goes to infinity, the first argument goes to infinity and the second argument goes to negative infinity when $g(n) = o(\log n)$. Therefore, P_M goes to 1, satisfying DNH.

Subcase 2: In this case, most voters remain independent. We will argue that although the delegation does impact the variance, this impact will get arbitrarily small as n grows larger, implying that the loss will get arbitrarily small.

Let us index the voters according to what happens in the delegation scheme. Let the first n_1 indexed voters represent those who remain independent and do not get delegated a vote. Let the next n_2 indexed voters be those who got delegated at least one vote. Finally, the last $n - n_1 - n_2$ indexed voters are those who delegated their vote to another voter. Based on our assumption above, we know that $\lim_{n\to\infty} \frac{n_1}{n} = 1$; most voters remain independent and unaffected by the delegation scheme.

Additionally, note that the mean will be slightly different in the two schemes, but this to our advantage because the mean will improve in the delegation scheme due to "uphill" delegation.

Therefore, given

$$P_D = \int_{\frac{n}{2}}^{n} \frac{1}{\sqrt{2\pi \, \text{Var}[X_D]}} e^{\frac{-(x - \mathbb{E}[X_D])^2}{2\,\text{Var}[X_D]}} \, dx$$

and

$$P_M = \int_{\frac{n}{2}}^{n} \frac{1}{\sqrt{2\pi \, \text{Var}[X_M]}} e^{\frac{-(x - \mathbb{E}[X_M])^2}{2\,\text{Var}[X_M]}} \, dx,$$

because $\mathbb{E}[X_M] \geq \mathbb{E}[X_D]$, we can say that

$$P_M \geq \int_{\frac{n}{2}}^{n} \frac{1}{\sqrt{2\pi \, \text{Var}[X_M]}} e^{\frac{-(x - \mathbb{E}[X_D])^2}{2\,\text{Var}[X_M]}} \, dx.$$

Now, we have to relate $\text{Var}[X_M]$ and $\text{Var}[X_D]$. Ideally, we want to show that they are multiplicatively close to each other.

We can decompose the variance of X_D.

$$\mathrm{Var}[X_D] = \sum_{i=1}^{n_1} p_i(1-p_i) + \sum_{i=n_1+1}^{n} p_i(1-p_i).$$

Likewise, we can decompose the variance of X_M.

$$\mathrm{Var}[X_M] = \sum_{i=1}^{n_1} p_i(1-p_i) + \sum_{i=n_1+1}^{n_1+n_2} w_{ni}^2 p_i(1-p_i) + \sum_{i=n_1+n_2+1}^{n} 0.$$

Therefore, we have

$$\begin{aligned}
\mathrm{Var}[X_M] - \mathrm{Var}[X_D] &= \sum_{i=n_1+1}^{n_1+n_2} (w_{ni}^2 - 1) p_i(1-p_i) \\
&\quad - \sum_{i=n_1+n_2+1}^{n} p_i(1-p_i) \\
&\leq \sum_{i=n_1+1}^{n_1+n_2} (w_{ni}^2 - 1) p_i(1-p_i) \\
&\leq \frac{n_2}{4} (\max {w_{ni}}^2 - 1) \\
&\leq \frac{1}{4} \cdot \frac{n}{g(n)} (C(n)^2 - 1),
\end{aligned}$$

where the last inequality holds because $w_{ni} \leq C(n)$, and n_2, the number of voters who are delegated to, is at most the number of voters who delegate, which is at most $n/g(n)$ by assumption.

This means that

$$\mathrm{Var}[X_M] \leq \mathrm{Var}[X_D] + \frac{1}{4} \cdot \frac{n}{g(n)} (C(n)^2 - 1)$$

and therefore

$$\begin{aligned}
\frac{\mathrm{Var}[X_M]}{\mathrm{Var}[X_D]} &\leq \frac{\mathrm{Var}[X_D] + \frac{1}{4} \cdot \frac{n}{g(n)} (C(n)^2 - 1)}{\mathrm{Var}[X_D]} \\
&= 1 + \frac{\frac{n}{g(n)} (C(n)^2 - 1)}{4\,\mathrm{Var}[X_D]}.
\end{aligned}$$

Now, note that by Equation (1.25),

$$\mathrm{Var}[X_D] \geq n\beta(1-\beta)$$

and therefore

$$\begin{aligned}
\mathrm{Var}[X_M] &\leq \mathrm{Var}[X_D] \left(1 + \frac{\frac{n}{g(n)} (C(n)^2 - 1)}{4n\beta(1-\beta)} \right) \\
&= \mathrm{Var}[X_D] \left(1 + \frac{1}{g(n)} \cdot \frac{C(n)^2 - 1}{4\beta(1-\beta)} \right).
\end{aligned}$$

Let

$$\eta = \frac{1}{g(n)} \cdot \frac{C(n)^2 - 1}{4\beta(1-\beta)}$$

and note that as n goes to infinity, η goes to 0 because we chose $g(n)$ to grow asymptotically more quickly than $C(n)^2$.

Therefore, revisiting the original integrals, we have

$$P_D = \int_{\frac{n}{2}}^{n} \frac{1}{\sqrt{2\pi \operatorname{Var}[X_D]}} e^{\frac{-(x-\mathbb{E}[X_D])^2}{2\operatorname{Var}[X_D]}} \, dx$$

and

$$P_M \geq \int_{\frac{n}{2}}^{n} \frac{1}{\sqrt{2\pi \operatorname{Var}[X_D](1+\eta)}} e^{\frac{-(x-\mathbb{E}[X_D])^2}{2\operatorname{Var}[X_D](1+\eta)}} \, dx.$$

Simplifying the above yields

$$P_D = \frac{1}{2}\left(\operatorname{Erf}\left(\frac{n - \mathbb{E}[X_D]}{\sqrt{2\operatorname{Var}[X_D]}}\right) - \operatorname{Erf}\left(\frac{n - 2\,\mathbb{E}[X_D]}{2\sqrt{2\operatorname{Var}[X_D]}}\right)\right) \tag{1.29}$$

and

$$\begin{aligned} P_M \geq \frac{1}{2}\Bigg(&\operatorname{Erf}\left(\frac{n - \mathbb{E}[X_D]}{\sqrt{2\operatorname{Var}[X_D](1+\eta)}}\right) \\ &-\operatorname{Erf}\left(\frac{n - 2\,\mathbb{E}[X_D]}{2\sqrt{2\operatorname{Var}[X_D](1+\eta)}}\right)\Bigg) \end{aligned} \tag{1.30}$$

Furthermore, again by Equation (1.25), we know that $\operatorname{Var}[X_D] \in [\beta(1-\beta)n, n/4]$ and therefore $\sqrt{\operatorname{Var}[X_D]} = \sqrt{cn}$, where $c \in [\beta(1-\beta), 1/4]$. From this, note that as n goes to infinity, the argument to the first error function in each expression goes to infinity.

Let

$$h_1(n) = \frac{n - 2\,\mathbb{E}[X_D]}{2\sqrt{2\operatorname{Var}[X_D]}} \tag{1.31}$$

be the argument to the second error function in (1.29), and let

$$h_2(n) = \frac{n - 2\,\mathbb{E}[X_D]}{2\sqrt{2\operatorname{Var}[X_D](1+\eta)}} \tag{1.32}$$

be the argument to the second error function in (1.30). As n goes to infinity, note that the argument to (1.31) must go to one of four states: infinity, negative infinity, zero, or a constant. In the case that it goes to infinity, negative infinity, or zero, the presence of the extra $\frac{1}{\sqrt{1+\eta}}$ term in (1.32) does nothing to change the sign of the arguments, and therefore they each converge to the same state (infinity, negative infinity, or zero) as n approaches infinity. When the argument to (1.31) goes to a constant, note that as n goes to infinity, η goes to 0, and therefore the two converge once again.

We conclude that (an upper bound on) the difference between P_D and P_M converges to 0, and hence DNH is satisfied. □

1.2 Minimizing the Maximum Weight of Voters in Liquid Democracy

The foregoing section indicates that, even if delegations go only to more competent agents, a high concentration of power might still be harmful for social welfare, by neutralizing benefits corresponding to the Condorcet Jury Theorem. Concentration of power is also a concern in real-world implementations of liquid democracy: Certain individuals, the so-called super-voters, seem to amass enormous weight, whereas most agents do not receive any delegations. As Kling et al. [147] describe, super-voters in the Pirate Party were so controversial that "the democratic nature of the system was questioned, and many users became inactive." Besides the negative impact of super-voters on perceived legitimacy, super-voters might also be more exposed to bribing. Although delegators can retract their delegations as soon as they become aware of suspicious voting behavior, serious damage might be done in the meantime. Furthermore, if super-voters jointly have sufficient power, they might find it more efficient to organize majorities through deals between super-voters behind closed doors, rather than to try to win a broad majority through public discourse.

While all these concerns suggest that the weight of super-voters should be limited, the exact metric to optimize for varies between them and is often not even clearly defined. For the purposes of this chapter, we choose to minimize the weight of the heaviest voter. As is evident in the Spiegel article, the weight of individual voters plays a direct role in the perception of super-voters. But even beyond that, we are confident that minimizing this measure will lead to substantial improvements across all presented concerns.

Just how can the maximum weight be reduced? One approach might be to restrict the power of delegation by imposing caps on the weight. However, as argued by Behrens et al. [26], delegation is always possible by coordinating outside of the system and copying the desired delegate's ballot. Pushing delegations outside of the system would not alleviate the problem of super-voters, just reduce transparency. Therefore, we instead adopt a voluntary approach: If agents are considering multiple potential delegates, all of whom they trust, they are encouraged to leave the decision for one of them to a centralized mechanism. With the goal of avoiding high-weight agents in mind, our research challenge is twofold:

> *First, investigate the algorithmic problem of selecting delegations to minimize the maximum weight of any agent, and, second, show that allowing multiple delegation options does indeed provide a significant reduction in the maximum weight compared to the status quo.*

1.2.1 Our Approach and Results

We formally define our problem in Section 1.2.3. In addition to minimizing the maximum weight of any voter, we specify how to deal with delegators whose vote cannot possibly reach any voter. In general, our problem is closely related to minimizing congestion for confluent flow as studied by Chen et al. [68]. Not only does this connection suggest an optimal algorithm based on mixed integer linear programming, but we also get a polynomial-time $(1 + \ln |V|)$-approximation algorithm, where V is the set of voters. In addition, we show

that approximating our problem to within a factor of $\frac{1}{2}\log_2|V|$ is NP-hard.

In Section 1.2.4, to evaluate the benefits of allowing multiple delegations, we propose a probabilistic model for delegation behavior — inspired by the well-known *preferential attachment* model [25] — in which we add agents successively. With a certain probability d, a new agent delegates; otherwise, she votes herself. If she delegates, she chooses k many delegation options among the previously inserted agents. A third parameter γ controls the bias of this selection towards agents who already receive many delegations. Assuming $\gamma = 0$, i.e., that the choice of delegates is unbiased, we prove that allowing two choices per delegator ($k = 2$) asymptotically leads to dramatically lower maximum weight than classical liquid democracy ($k = 1$) using an argument based on the "power of choice." In the latter case, with high probability, the maximum weight is at least $\Omega(t^\beta)$ for some $\beta > 0$, whereas the maximum weight in the former case is only $\mathcal{O}(\log\log t)$ with high probability, where t denotes simultaneously the time step of the process and the number of agents. Our analysis draws on a phenomenon called the *power of choice* that can be observed in many different load balancing models. In fact, even a greedy mechanism that selects a delegation option to locally minimize the maximum weight as agents arrive exhibits this asymptotic behavior, which upper-bounds the maximum weight for optimal resolution.

In Section 1.2.5, we complement our theoretical findings with empirical results. Our simulations demonstrate that our approach continues to outperform classical preferential attachment for higher values of γ. We also show that the most substantial improvements come from increasing k from one to two, i.e., that increasing k even further only slightly reduces the maximum weight. We continue to see these improvements in terms of maximum weight even if just some fraction of delegators give two options while the others specify a single delegate. Finally, we compare the optimal maximum weight with the maximum weight produced by the approximation algorithm and greedy heuristics.

1.2.2 Related Work

Kling et al. [147] conduct an empirical investigation of the existence and influence of super-voters. The analysis is based on daily data dumps, from 2010 until 2013, of the German Pirate Party installation of LiquidFeedback. As noted above, Kling et al. find that super-voters exist, and have considerable power. The results do suggest that super-voters behave responsibly, as they "do not fully act on their power to change the outcome of votes, and they vote in favour of proposals with the majority of voters in many cases." Of course, this does not contradict the idea that a balanced distribution of power would be desirable.

In recent years, there has been an increasing number of theoretical analyses of liquid democracy. In the field of political theory, Blum and Zuber [38] give a normative justification of liquid democracy. They consider two accounts of democracy, which differ in the stated goal of a democratic system. In the *epistemic* framework, the success of a democratic system should lead to good decisions with respect to some objective notion of quality, whereas, in the *egalitarian* framework, a democratic system should allow each individual to impose her particular interests to the same degree. Blum and Zuber conclude that liquid democracy improves upon purely representative democracy with respect to both metrics. They see unequal voting weights as problematic and suggest public deliberation before a

vote to attenuate this problem.

In the spirit of the egalitarian framework, Green-Armytage [121] justifies liquid democracy in a spatial model of political preferences similar to facility placement. When an agent has incomplete information about a topic, transitive delegations can help to express her preferences more accurately by harnessing the expertise of like-minded, more qualified agents.

In this paper, we consider a single delegation network. Other works allow agents to specify different delegations for multiple interconnected issues, where the binary preferences and outcomes are restricted to satisfy a propositional formula [72] or to correspond to binary comparisons in a ranking [51]. Both papers propose ways of reconciling contradictory choices made by different delegates.

We also highlight related work that considers models of network formation and influence attenuation in the context of liquid democracy. Bloembergen et al. [37] introduce a game-theoretic model of delegation in order to study rational delegation behavior in liquid-democracy networks. In their model, delegation networks might be formed by a best-response dynamic or as Nash equilibria of a delegation game. Escoffier et al. [98] study a similar delegation game with different incentives.

Boldi et al. [40] study a variant of liquid democracy in which a voter's weight decreases by a discount factor every time her vote is transitively delegated, penalizing long delegation chains. They argue that this variant is more appropriate in online communities, where trust relationships are typically less deep than in the real world. While not intended as such, this variant of liquid democracy can also reduce the weight of super-voters, at least of those who receive most of their delegations indirectly. However, such a variant violates the principle of "one person, one vote" and incentivizes delegation outside of the system [26]. By contrast, our approach reduces the weight of super-voters while preserving each voter's individual influence.

1.2.3 Algorithmic Model and Results

Let us consider a delegative voting process where agents may specify multiple potential delegations. This gives rise to a directed graph, whose nodes represent agents and whose edges represent potential delegations. In the following, we will conflate nodes and the agents they represent. A distinguished subset of nodes corresponds to agents who have voted directly, the *voters*. Since voters forfeit the right to delegate, the voters are a subset of the sinks of the graph. We call all non-voter agents *delegators*.

Each agent has an inherent voting weight of 1. When the delegations will have been resolved, the weight of every agent will be the sum of weights of her delegators plus her inherent weight. We aim to choose a delegation for every delegator in such a way that the maximum weight of any voter is minimized.

This task closely mirrors the problem of congestion minimization for confluent flow (with infinite edge capacity): There, a flow network is also a finite directed graph with a distinguished set of graph sinks, the *flow sinks*. Every node has a non-negative *demand*. If we assume unit demand, this demand is 1 for every node. Since the flow is confluent, for every non-sink node, the algorithm must pick exactly one outgoing edge, along which

the flow is sent. Then, the *congestion* at a node n is the sum of congestions at all nodes who direct their flow to n plus the demand of n. The goal in congestion minimization is to minimize the maximum congestion at any flow sink. (We remark that the close connection between our problem and confluent flow immediately suggests a variant corresponding to splittable flow; we discuss this variant at length in Section 1.3.)

In spite of the similarity between confluent flow and resolving potential delegations, the two problems differ when a node has no path to a voter / flow sink. In confluent flow, the result would simply be that no flow exists. In our setting however, this situation can hardly be avoided. If, for example, several friends assign all of their potential delegations to each other, and if all of them rely on the others to vote, their weight cannot be delegated to any voter. Our mechanism cannot simply report failure as soon as a small group of voters behaves in an unexpected way. Thus, it must be allowed to leave these votes unused. At the same time, of course, our algorithm should not exploit this power to decrease the maximum weight, but must primarily maximize the number of utilized votes. We formalize these issues in the following section.

Problem Statement

All graphs $G = (N, E)$ mentioned in this section will be finite and directed. Furthermore, they will be equipped with a set V of distinguished sinks in the graph. For the sake of brevity, these assumptions will be implicit in the notion "graph G with V".

Some of these graphs represent situations in which all delegations have already been resolved and in which each vote reaches a voter: We call a graph (N, E) with V a *delegation graph* if it is acyclic, its sinks are exactly the set V, and every other vertex has outdegree one. In such a graph, define the *weight* $w(n)$ of a node $n \in N$ as

$$w(n) \coloneqq 1 + \sum_{(m,n)\in E} w(m).$$

This is well-defined because E is a well-founded relation on N.

Resolving the delegations of a graph G with V can now be described as the MinMaxWeight problem: Among all delegation subgraphs (N', E') of G with voting vertices V of maximum $|N'|$, find one that minimizes the maximum weight of the voting vertices.

Connections to Confluent Flow

We recall definitions from the flow literature as used by Chen et al. [68]. We slightly simplify the exposition by assuming unit demand at every node.

Given a graph (N, E) with V, a *flow* is a function $f : E \to \mathbb{R}_{\geq 0}$. For any node n, set $in(n) \coloneqq \sum_{(m,n)\in E} f(m, n)$ and $out(n) \coloneqq \sum_{(n,m)\in E} f(n, m)$. At every node $n \in N \setminus V$, a flow must satisfy *flow conservation*:

$$out(n) = 1 + in(n).$$

Note that all nodes in V are sinks in the graph, and thus have no outflow. The congestion at any node n is defined as $1 + in(n)$. A flow is *confluent* if every node has at most one

outgoing edge with positive flow. We define MinMaxCongestion as the problem of finding a confluent flow on a given graph such that the maximum congestion is minimized.

To relate the two presented problems, we need to refer to the parts of a graph (N, E) with V from which V is reachable: The *active* nodes $active_V(N, E)$ are all $n \in N$ such that there exists a path from n to a sink $v \in V$ using edges in E. The *active subgraph* is the restriction of (N, E) to $active_V(N, E)$. In particular, V is part of this subgraph.

Lemma 1.2.1. *Let $G = (N, E)$ with V be a graph. Its delegation subgraphs (N', E') that maximize $|N'|$ are exactly the delegation subgraphs with $N' = active_V(N, E)$. At least one such subgraph exists.*

Proof. First, we show that all nodes of a delegation subgraph are active. Indeed, consider any node n_1 in the subgraph. By following outgoing edges, we obtain a sequence of nodes $n_1\, n_2 \ldots$ such that n_i delegates to n_{i+1}. Since the graph is finite and acyclic, this sequence must end with a vertex n_j without outgoing edges. This must be a voter; thus, n_1 is active.

Furthermore, there exists a delegation subgraph of (N, E) with nodes exactly $active_V(N, E)$. Indeed, the shortest-paths-to-set-V forest (with edges pointed in the direction of the paths) on the active subgraph is a delegation graph.

By the first argument, all delegation subgraphs must be subgraphs of the active subgraph. By the second argument, to have the maximum number of nodes, they must include all nodes of this subgraph. □

Lemma 1.2.2. *Let (N, E) with V be a graph and let $f : E \to \mathbb{R}_{\geq 0}$ be a confluent flow (for unit demand). By eliminating all zero-flow edges from the graph, we obtain a delegation graph.*

Proof. We first claim that the resulting graph is acyclic. Indeed, for the sake of contradiction, suppose that there is a cycle including some node n. Consider the flow out of n, through the cycle and back into n. Since the flow is confluent, and thus the flow cannot split up, the demand can only increase from one node to the next. As a result, $in(n) \geq out(n)$. However, by flow conservation and unit demand, $out(n) = in(n) + 1$, which contradicts the previous statement.

Furthermore, the sinks of the graph are exactly V: By assumption, the nodes of V are sinks in the original graph, and thus in the resulting graph. For any other node, flow conservation dictates that its outflow be at least its demand 1, thus every other node must have outgoing edges.

Finally, every node not in V must have outdegree 1. As detailed above, the outdegree must be at least 1. Because the flow was confluent, the outdegree cannot be greater.

As a result of these three properties, we have a delegation graph. □

Lemma 1.2.3. *Let (N, E) with V be a graph in which all vertices are active, and let (N, E') be a delegation subgraph. Let $f : E \to \mathbb{R}_{\geq 0}$ be defined such that, for every node $n \in N \setminus V$ with (unique) outgoing edge $e \in E'$, $f(e) \coloneqq w(n)$. On all other edges $e \in E \setminus E'$, set $f(e) \coloneqq 0$. Then, f is a confluent flow.*

Proof. For every non-sink, flow conservation holds by the definition of weight and flow. By construction, the flow must be confluent. □

Algorithms

The observations made above allow us to apply algorithms — even approximation algorithms — for MinMaxCongestion to our MinMaxWeight problem; that is, we can reduce the latter problem to the former.

Theorem 1.7. *Let $\mathcal{A}$ be an algorithm for* MinMaxCongestion *with approximation ratio $c \geq 1$. Let $\mathcal{A}'$ be an algorithm that, given (N, E) with V, runs $\mathcal{A}$ on the active subgraph, and translates the result into a delegation subgraph by eliminating all zero-flow edges. Then $\mathcal{A}'$ is a c-approximation algorithm for* MinMaxWeight.

Proof. By Lemma 1.2.1, removing inactive parts of the graph does not change the solutions to MinMaxWeight, so we can assume without loss of generality that all vertices in the given graph are active.

Suppose that the optimal solution for MinMaxCongestion on the given instance has maximum congestion α. By Lemma 1.2.2, it can be translated into a solution for MinMaxWeight with maximum weight α. By Lemma 1.2.3, the latter instance has no solution with maximum weight less than α, otherwise it could be used to construct a confluent flow with the same maximum congestion. It follows that the optimal solution to the given MinMaxWeight instance has maximum weight α.

Now, $\mathcal{A}$ returns a confluent flow with maximum congestion at most $c \cdot \alpha$. Using Lemma 1.2.2, $\mathcal{A}'$ constructs a solution to MinMaxWeight with maximum weight at most $c \cdot \alpha$. Therefore, $\mathcal{A}'$ is a c-approximation algorithm. □

Note that Theorem 1.7 works for $c = 1$, i.e., even for exact algorithms. Therefore, it is possible to solve MinMaxWeight by adapting any exact algorithm for MinMaxFlow. In particular, congestion minimization for confluent flow can be expressed as a mixed integer linear program (MILP).

To stress the connection to MinMaxWeight, denote the congestion at a voter i by $w(i)$. For each potential delegation (u, v), $f(u, v)$ gives the amount of flow between u and v. This flow must be nonnegative (1.34) and satisfy flow conservation (1.35). Congestion is defined in Equation (1.36). To minimize maximum congestion, we introduce a variable z that is higher than the congestion of any voter (1.37), and minimize z (1.33).

So far, we have described a Linear Program for optimizing splittable flow. To restrict the solutions to confluent flow, we must enforce an 'all-or-nothing' constraint on outflow from any node, i.e. at most one outgoing edge per node can have positive flow. We express this using a convex-hull reformulation. We introduce a binary variable $x_{u,v}$ for each edge (1.38), and set the sum of binary variables for all outgoing edges of a node to 1 (1.39). If M is a constant larger than the maximum possible flow, we can then bound $f(u, v) \leq M\, x_{u,v}$ (1.40) to have at most one positive outflow per node.

The final MILP is thus

$$
\begin{aligned}
&\text{minimize} && z && && (1.33)\\
&\text{subject to} && f(m,n) \geq 0 && \forall (m,n) \in E, && (1.34)\\
& && \sum_{(n,m)\in E} f(n,m) = 1 + \sum_{(m,n)\in E} f(m,n) && \forall n \in N \setminus V, && (1.35)\\
& && w(v) = 1 + \sum_{(n,v)\in E} f(n,v) && \forall v \in V, && (1.36)\\
& && z \geq w(v) && \forall v \in V, && (1.37)\\
& && x_{m,n} \in \{0,1\} && \forall (m,n) \in E, && (1.38)\\
& && \sum_{(n,m)\in E} x_{n,m} = 1 && \forall n \in N \setminus V, && (1.39)\\
& && f(m,n) \leq M \cdot x_{m,n} && \forall (m,n) \in E. && (1.40)
\end{aligned}
$$

Since the foregoing algorithm is based on solving an NP-hard problem, it might be too inefficient for typical use cases of liquid democracy with many participating agents. Fortunately, it might be acceptable to settle for a slightly non-optimal maximum weight if this decreases computational cost. To our knowledge, the best polynomial approximation algorithm for MinMaxCongestion is due to Chen et al. [68] and achieves an approximation ratio of $1 + \ln |V|$. Their algorithm starts by computing the optimal solution to the splittable-flow version of the problem, by solving a linear program. The heart of their algorithm is a non-trivial, deterministic rounding mechanism. This scheme drastically outperforms the natural, randomized rounding scheme, which leads to an approximation ratio of $\Omega(|N|^{1/4})$ with arbitrarily high probability [67].

Hardness of Approximation

In this section, we demonstrate the NP-hardness of approximating the MinMaxWeight problem to within a factor of $\frac{1}{2} \log_2 |V|$. On the one hand, this justifies the absence of an exact polynomial-time algorithm. On the other hand, this shows that the approximation algorithm is optimal up to a multiplicative constant.

Theorem 1.8. *It is NP-hard to approximate the* MinMaxWeight *problem to a factor of* $\frac{1}{2} \log_2 |V|$*, even when each node has outdegree at most* 2*.*

Not surprisingly, we derive hardness via a reduction from MinMaxCongestion, i.e., a reduction in the opposite direction from the one given in Theorem 1.7. As shown by Chen et al. [68], approximating MinMaxCongestion to within a factor of $\frac{1}{2} \log_2 |V|$ is NP-hard. However, in our case, nodes have unit demands. Moreover, we are specifically interested in the case where each node has outdegree at most 2, as in practice we expect outdegrees to be very small, and this case plays a special role in Section 1.2.4.

We begin with a lemma that slightly strengthens a hardness result by Fortune et al. [106]:

Lemma 1.2.4. *Let G be a directed graph in which all vertices have an outdegree of at most 2. Given vertices s_1, s_2, t_1, t_2, it is NP-hard to decide whether there exist vertex-disjoint paths from s_1 to t_1 and from s_2 to t_2.*

Proof. Without the restriction on the outdegree, the problem is NP-hard [106]. We reduce the general case to our special case.

Let G' be an arbitrary directed graph; let s'_1, s'_2, t'_1, t'_2 be distinguished vertices. To restrict the outdegree, replace each node n with outdegree d by a binary arborescence (directed binary tree with edges facing away from the root) with d sinks. All incoming edges into n are redirected towards the root of the arborescence; outgoing edges from n instead start from the different leaves of the arborescence. Call the new graph G, and let s_1, s_2, t_1, t_2 refer to the roots of the arborescences replacing s'_1, s'_2, t'_1, t'_2, respectively.

Clearly, our modifications to G' can be carried out in polynomial time. It remains to show that there are vertex-disjoint paths from s_1 to t_1 and from s_2 to t_2 in G iff there are vertex-disjoint paths from s'_1 to t'_1 and from s'_2 to t'_2 in G'.

If there are disjoint paths in G', we can translate these paths into G by visiting the arborescences corresponding to the nodes on the original path one after another. Since both paths visit disjoint arborescences, the new paths must be disjoint.

Suppose now that there are disjoint paths in G. Translate the paths into G' by visiting the nodes corresponding to the sequence of visited arborescences. Since each arborescence can only be entered via its root, disjointness of the paths in G implies disjointness of the translated paths in G'. □

Now, we can strengthen the hardness of approximation for MinMaxCongestion by Chen et al. [68]. We believe the lemma is of independent interest, as it shows a surprising separation between the case of outdegree 1 (where the problem is moot) and outdegree 2, and that the asymptotically optimal approximation ratio is independent of degree. But it also allows us to prove Theorem 1.8 almost directly.

Lemma 1.2.5. *It is NP-hard to approximate the* MinMaxCongestion *problem to a factor of* $\frac{1}{2}\log_2 k$*, where* k *is the number of sinks, even when each node has unit demand and outdegree at most* 2*.*

Proof of Lemma 1.2.5. We adapt the proof of Theorem 1 of Chen et al. [68].

Let $G = (V, E), s_1, s_2, t_1, t_2$ be given as in Lemma 1.2.4. Without loss of generality, G only contains nodes from which t_1 or t_2 is reachable, t_1 and t_2 are sinks and all four vertices are distinct. Let $\ell = \lceil \log_2 |V| \rceil$ and $k = 2^\ell$. Build the same auxiliary network as that built by Chen et al. [68], which consists of a binary arborescence whose $k - 1$ nodes are copies of G. The construction is illustrated in Figure 1.2. For more details, refer to [68].

For ease of exposition, we describe our reduction as returning a flow network with polynomially-bounded positive integer demands. Implicitly, the described network is subsequently translated into one with unary demand; to express a demand of d at a node n in our unit-demand setting, add $d - 1$ fresh nodes with a single outgoing edge to n.

Denote the number of nodes in the network by $\phi \coloneqq (k-1)\cdot|V|+k$, and set $\Phi \coloneqq \ell\cdot\phi+1$. In [68], every copy of s_2 and t_2 has demand 1, the copy of s_1 at the root has demand 2, and all other nodes have demand 0. Instead, we give these nodes demands of Φ, 2Φ and 1, respectively. Note that the size of the generated network[4] is polynomial in the size of G and that the outdegree of each node is at most 2. From every node, one of the sinks

[4]Even after unfolding our non-unitary-demand nodes.

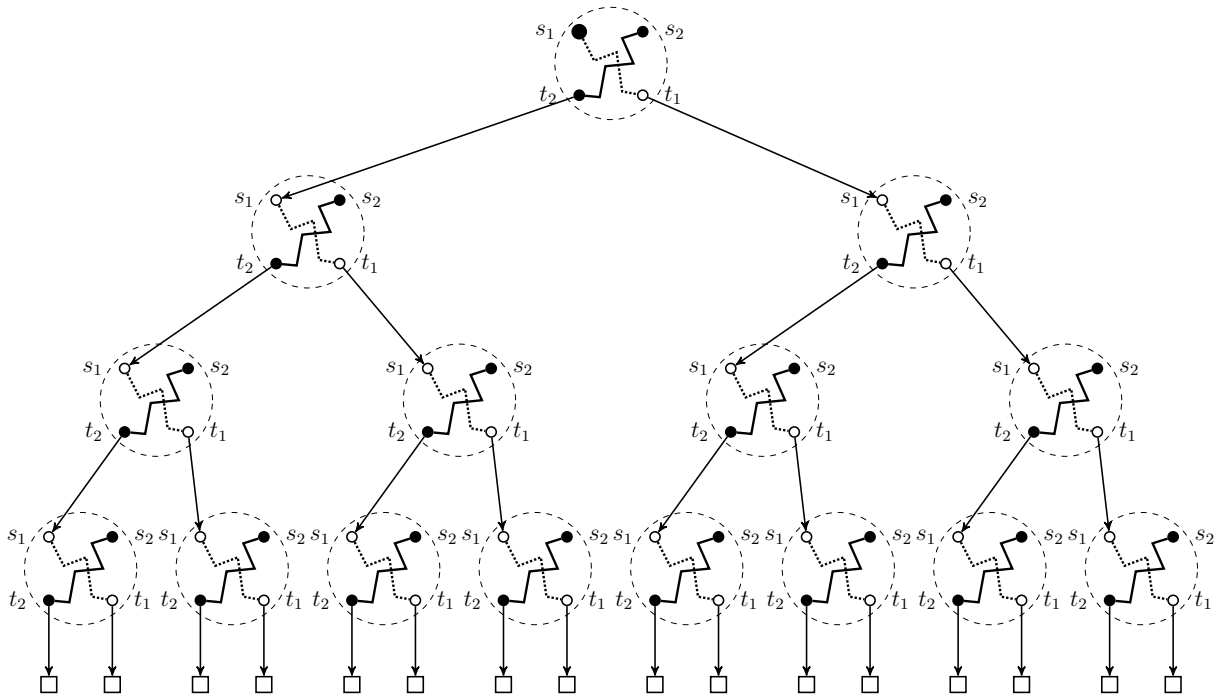


Figure 1.2: Auxiliary network generated from G, here for $k = 16$. Recreation of [68, Fig. 2].

S displayed as rectangles in Figure 1.2 is reachable. Since the minimum-distance-to-S spanning forest describes a flow, a flow in the network exists.

Suppose that G contains vertex-disjoint paths P_1 from s_1 to t_1 and P_2 from s_2 to t_2. In each copy of G in the network, route the flow along these paths. We can complete the confluent flow inside of this copy in such a way that the demand of every node is routed to t_1 or t_2: By assumption, each of the nodes can reach one of these two path endpoints. Iterate over all nodes in order of ascending distance to the closest endpoint and make sure that their flow is routed to an endpoint. For the endpoints themselves, there is nothing to do. For positive distance, a node might be part of a path and thus already connected to an endpoint. Else, look at its successor in a shortest path to an endpoint. By the induction hypothesis, all flow from this successor is routed to an endpoint, so route the node's flow to this successor. If we also use the edges between copies of G and between the copies and the sinks, we obtain a confluent flow. Each sink except for the rightmost one can only collect the demand of two nodes with demand Φ plus a number of nodes with demand 1. The rightmost sink collects the demand from the single node with demand 2Φ plus some unitary demands. Thus, the congestion of the system can be at most $2\Phi + \phi$.

Now, consider the case in which G does not have such vertex-disjoint paths. In every confluent flow and in every copy, there are three options:

- the flow from s_1 flows to t_2 and the flow from s_2 flows to t_1,
- the flow from s_1 and s_2 flows to t_1, or
- the flow from s_1 and s_2 flows to t_2.

In each case, the flow coming in through s_1 is joined by additional demand of at least

Φ. Consider the path from the copy of s_1 at the root to a sink. By a simple inductive argument, the congestion at the endpoint of the ith copy of G on this path is at least $(i+1)\cdot\Phi$. Thus, the total congestion at the sink must be at least $(\ell+1)\cdot\Phi$. The lemma now follows from the fact that

$$\frac{\log_2 k}{2}(2\Phi+\phi) = \frac{\ell}{2}(2\Phi+\phi) < (\ell+1)\cdot\Phi. \qquad \square$$

Proof of Theorem 1.8. We reduce (gap) MinMaxCongestion with unit demand and outdegree at most 2 to (gap) MinMaxWeight with outdegree at most 2. First, we claim that if there are inactive nodes, there is no confluent flow. Indeed, let n_1 be an inactive node. For the sake of contradiction, suppose that there exists a flow f. Follow the positive flow to obtain a sequence $n_1\, n_2 \ldots$. By definition, none of the nodes reachable from n_1 can be a voter. Since, by flow conservation and unit demand, each node must delegate, the sequence must be infinite. As detailed in the proof of Lemma 1.2.2, a confluent flow with unit demand cannot contain cycles. Thus, the sequence contains infinitely many different nodes, which contradicts the finiteness of G.

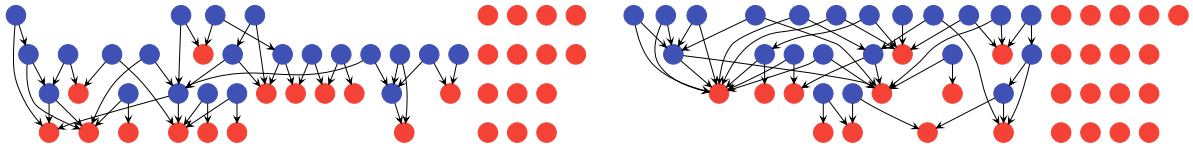
Therefore, we can assume without loss of generality that in the given instance of MinMaxCongestion, all nodes are active (as the problem is still NP-hard). The reduction creates an instance of MinMaxWeight that has the same graph as the given instance of MinMaxCongestion. Using an argument analogous to the proof of Theorem 1.7 (reversing the roles of Lemma 1.2.2 and Lemma 1.2.3 in its proof), we see that this is a strict approximation-preserving reduction. $\square$

1.2.4 Probabilistic Model and Results

Our generalization of liquid democracy to multiple potential delegations aims to decrease the concentration of weight. Accordingly, the success of our approach should be measured by its effect on the maximum weight in real elections. Since, at this time, we do not know of any available datasets,[5] we instead propose a probabilistic model for delegation behavior, which can serve as a credible proxy. Our model builds on the well-known preferential attachment model, which generates graphs possessing typical properties of social networks.

The evaluation of our approach will be twofold: In Sections 1.2.4 and 1.2.4, for a certain choice of parameters in our model, we establish a striking separation between traditional liquid democracy and our system. In the former case, the maximum weight at time t is $\Omega(t^\beta)$ for a constant β with high probability, whereas in the latter case, it is in $\mathcal{O}(\log\log t)$ with high probability, even if each delegator only suggests two options. For other parameter settings, we empirically corroborate the benefits of our approach in Section 1.2.5.

[5] There is one relevant dataset that we know of, which was analyzed by Kling et al. [147]. However, due to stringent privacy constraints, the data privacy officer of the German Pirate Party was unable to share this dataset with us.



(a) $\gamma = 0$ (b) $\gamma = 1$

Figure 1.3: Example graphs generated by the preferential delegation model for $k = 2$ and $d = 0.5$.

The Preferential Delegation Model

Many real-world social networks have degree distributions that follow a power law [153; 180]. Additionally, in their empirical study, Kling et al. [147] observed that the weight of voters in the German Pirate Party was "power law-like" and that the graph had a very unequal indegree distribution. In order to meld the previous two observations in our liquid democracy delegation graphs, we adapt a standard preferential attachment model [25] for this specific setting. At a high level, our *preferential delegation* model is characterized by three parameters: $0 < d < 1$, the probability of delegation; $k \geq 1$, the number of delegation options from each delegator; and $\gamma \geq 0$, an exponent that governs the probability of delegating to nodes based on current weight.

At time $t = 1$, we have a single node representing a single voter. In each subsequent time step, we add a node for agent i and flip a biased coin to determine her delegation behavior. With probability d, she delegates to other agents. Else, she votes independently. If i does not delegate, her node has no outgoing edges. Otherwise, add edges to k many independently selected, previously inserted nodes, where the probability of choosing node j is proportional to $(\mathit{indegree}(j)+1)^\gamma$. Note that this model might generate multiple edges between the same pair of nodes, and that all sinks are voters. Figure 1.3 shows example graphs for different settings of γ.

In the case of $\gamma = 0$, which we term *uniform delegation*, a delegator is equally likely to attach to any previously inserted node. Already in this case, a "rich-get-richer" phenomenon can be observed, i.e., voters at the end of large networks of potential delegations will likely see their network grow even more. Indeed, a larger network of delegations is more likely to attract new delegators. In traditional liquid democracy, where $k = 1$ and all potential delegations will be realized, this explains the emergence of super-voters with excessive weight observed by Kling et al. [147]. We aim to show that for $k \geq 2$, the resolution of potential delegations can strongly outweigh these effects. In this, we profit from an effect known as the "power of two choices" in load balancing described by Azar et al. [13].

For $\gamma > 0$, the "rich-get-richer" phenomenon additionally appears at the degrees of nodes. Since the number of received potential delegations is a proxy for an agent's competence and visibility, new agents are more likely to attach to agents with high indegree. In total, this is likely to further strengthen the inherent inequality between voters. For increasing γ, the graph becomes increasingly flat, as a few super-voters receive nearly all delegations. This matches observations from the LiquidFeedback dataset [147] that "the delegation network is slowly becoming less like a friendship network, and more like a bi-

partite networks of super-voters connected to normal voters." The special case of $\gamma = 1$ corresponds to preferential attachment as described by Barabási and Albert [25].

The most significant difference we expect to see between graphs generated by the preferential delegation model and real delegation graphs is the assumption that agents always delegate to more senior agents. In particular, this causes generated graphs to be acyclic, which need not be the case in practice. It does seem plausible that the majority of delegations goes to agents with more experience on the platform. Even if this assumption should not hold, there is a second interpretation of our process if we assume — as do Kahng et al. [137] — that agents can be ranked by competence and only delegate to more competent agents. Then, we can think of the agents as being inserted in decreasing order of competence. When a delegator chooses more competent agents to delegate to, her choice would still be biased towards agents with high indegree, which is a proxy for popularity.

It may be useful to note that the MinMaxWeight approach based on confluent flow does not require the underlying delegation graph to be acyclic, as the objective tries to minimize the maximum weight of any voter over all possible delegation choices that maximize the total number of utilized votes. In this sense, unavoidable cycles result in lost voting power.

In our theoretical results, we focus on the cases of $k = 1$ and $k = 2$, and assume $\gamma = 0$ to make the analysis tractable. The parameter d can be chosen freely between 0 and 1. Note that our upper bound for $k = 2$ directly translates into an upper bound for larger k, since the resolution mechanism always has the option of ignoring all outgoing edges except for the first two. Therefore, to understand the effect of multiple delegation options, we can restrict our attention to $k = 2$. This crucially relies on $\gamma = 0$, where potential delegations do not influence the probabilities of choosing future potential delegations. Based on related results by Malyshkin and Paquette [167], it seems unlikely that increasing k beyond 2 will reduce the maximum weight by more than a constant factor.

Lower Bounds for Single Delegation ($k = 1$, $\gamma = 0$)

As mentioned above, we first assume uniform delegation and a single delegation option per delegator, and derive a lower bound on the maximum weight. To state our results rigorously, we say that a sequence $(\mathcal{E}_m)_m$ of events happens *with high probability* if $\Pr[\mathcal{E}_m] \to 1$ for $m \to \infty$. Since the parameter going to infinity is clear from the context, we omit it.

Theorem 1.9. *In the preferential delegation model with $k = 1$, $\gamma = 0$, and $d \in (0, 1)$, with high probability, the maximum weight of any voter at time t is in $\Omega(t^\beta)$, where $\beta > 0$ is a constant that depends only on d.*

Proof. It suffices to show that, with high probability, there exists a voter at every time t whose weight is bounded from below by a function in $\Omega(t^\beta)$.

For ease of exposition, we pretend that $i_{max} := \log_2 \frac{t}{\ln t}$ is an integer.[6] We divide the t agents into $i_{max} + 1$ blocks $B_0, \dots, B_{i_{max}}$. The first block B_0 contains agents 1 to $\tau := \ln t$, and every subsequent block B_i contains agents $(\tau\, 2^{i-1}, \tau\, 2^i]$.

[6] The same argument works for $i_{max} := \lfloor \log_2 \frac{t}{\ln t} \rfloor$ if we appropriately bound the term.

We keep track of the total weight S_i of all voters in B_0 after the entirety of block B_i has been added. Furthermore, we define an event X_i saying that a high enough number of agents in block B_i transitively delegate into B_0. If all X_i hold, $S_{i_{max}}$ scales like a power function. Then, we show that, as t increases, the probability of any X_i failing goes to zero. Thus, our bound on $S_{i_{max}}$ holds with high probability. The total weight of B_0 and the weight of the maximum-weight voter in B_0 can differ by at most a factor of τ, which is logarithmic in t. Thus, with high probability, there is a voter in B_0 whose weight is a power function.

In more detail, let $\varepsilon \coloneqq \frac{1}{2}$ and let $d' \coloneqq (1-\varepsilon)\, d = \frac{d}{2}$. For each $i \geq 0$, let Y_i denote the number of votes from block i transitively going into B_0. Clearly, $S_i = \sum_{j=0}^{i} Y_i$. For $i > 0$, let X_i denote the event that

$$Y_i > d' \, \frac{\tau \left(1 + \frac{d'}{2}\right)^{i-1}}{2}.$$

Bounding the Expectation of Y_i We first prove by induction on i that, if X_1 through X_i hold, then

$$S_i \geq \tau \left(1 + \frac{d'}{2}\right)^{i}. \tag{1.41}$$

For $i = 0$, $S_0 = \tau$ and the claim holds. For $i > 0$, by the induction hypothesis, $S_{i-1} \geq \tau \left(1 + \frac{d'}{2}\right)^{i-1}$. By the assumption X_i,

$$Y_i > d' \, \frac{\tau \left(1 + \frac{d'}{2}\right)^{i-1}}{2}.$$

Thus,

$$S_i = S_{i-1} + Y_i \geq \tau \left(1 + \frac{d'}{2}\right)^{i-1} + d' \, \frac{\tau \left(1 + \frac{d'}{2}\right)^{i-1}}{2} = \tau \left(1 + \frac{d'}{2}\right)^{i-1} \left(1 + \frac{d'}{2}\right) = \tau \left(1 + \frac{d'}{2}\right)^{i}.$$

This concludes the induction and establishes Equation (1.41).

Now, for any agent j in B_i, the probability of transitively delegating into B_0 is

$$d \, \frac{\sum_{v \in V \cap B_0} w_{j-1}(v)}{j-1} \geq d \, \frac{S_{i-1}}{\tau \, 2^i}.$$

Conditioned on $X_1, \ldots, X_{i-1}$, we can thus lower-bound Y_i by a binomial variable $Bin\left(\tau \, 2^{i-1}, d \, \frac{S_{i-1}}{\tau \, 2^i}\right)$ to obtain

$$\mathbb{E}[Y_i \mid X_1, \ldots, X_{i-1}] \geq \tau \, 2^{i-1} \, d \, \frac{S_{i-1}}{\tau \, 2^i} = d \, \frac{S_{i-1}}{2} \geq d \, \frac{\tau \left(1 + \frac{d'}{2}\right)^{i-1}}{2}.$$

Denoting the right hand side by

$$\mu :- d \, \frac{\tau \left(1 + \frac{d'}{2}\right)^{i-1}}{2},$$

note that X_i holds if $Y_i > (1-\varepsilon)\, \mu$.

Failure Probability Goes to 0 Now, we must show that, with high probability, all X_i hold. By underapproximating the probability of delegation by a binomial random variable as before and by using a Chernoff bound, we have for all $i > 0$

$$\Pr[X_i \mid X_1, \ldots, X_{i-1}] \geq \Pr\left[Bin\left(\tau\, 2^{i-1}, d\,\frac{\tau\,(1+d'/2)^{i-1}}{\tau\, 2^i}\right) > (1-\varepsilon)\,\mu\right] \geq 1 - e^{-\frac{\varepsilon^2\mu}{2}}.$$

By the union bound,

$$\Pr[\exists i, 1 \leq i \leq i_{max} \text{ such that } X_i \text{ fails}] \leq \sum_{i=1}^{i_{max}} e^{-\frac{\varepsilon^2\, d\, \tau\, (1+d'/2)^{i-1}}{4}}.$$

We wish to show that the right hand side goes to 0 as t increases. We have

$$\begin{aligned}
\sum_{i=1}^{i_{max}} e^{-\frac{\varepsilon^2\, d\, \tau\, (1+d'/2)^{i-1}}{4}} &\leq i_{max}\left(e^{-\frac{\varepsilon^2\, d\, \tau}{4}}\right) && \text{(by monotonicity)}\\
&= \left(\log_2 \frac{t}{\ln t}\right)\left(t^{-\frac{\varepsilon^2\, d}{4}}\right), && \text{(by definitions of } i_{max},\, \tau)
\end{aligned}$$

which indeed approaches 0 as t increases.

Bounding the Maximum Weight Note that the weight of B_0 at time t is exactly $S_{i_{max}}$. Set $x \coloneqq 1 + d'/2 > 1$, which is a constant. With high probability, by Equation (1.41),

$$\frac{S_{i_{max}}}{\tau} \geq \left(1 + \frac{d'}{2}\right)^{i_{max}} = x^{\log_2 \frac{t}{\ln t}} = \left(\frac{t}{\ln t}\right)^{\log_2 x}.$$

Since $x > 1$, $\log_2 x > 0$. For any $0 < \beta < \log_2 x$, $\frac{S_{i_{max}}}{\tau} \in \Omega(t^\beta)$ with high probability. Since B_0 has weight $S_{i_{max}}$ and contains at most τ voters, with high probability there is some voter in B_0 with that much weight. □

Before proceeding to the upper bound and showing the separation, we would like to point out that — with a minor change to our model — these lower bounds also hold for $\gamma = 1$. Consider a model in which the probability of attaching to a delegator n remains proportional to $(1 + \mathit{indegree}(n))^\gamma$, but the probability for voters n is now proportional to $(2 + \mathit{indegree}(n))^\gamma$.[7] If we represent voters with a self-loop edge, both terms just equal $\mathit{degree}(n)^\gamma$, which arguably makes this implementation of preferential attachment cleaner to analyze (e.g., [41]). Thus, we can interpret preferential attachment for $\gamma = 1$ as uniformly picking an edge and then flipping a fair coin to decide whether to attach the new node to the edge's start or endpoint. Since every node has exactly one outgoing edge, this is equivalent to uniformly choosing a node and then, with probability $\frac{1}{2}$, instead picking its successor. This has the same effect on the distribution of weights as just uniformly choosing a node in uniform delegation, so Theorem 1.9 also holds for $\gamma = 1$ in our modified setting. Real-world delegation networks, which we suspect to resemble the case of $\gamma = 1$, should therefore exhibit similar behavior.

[7] Clearly, our results for $\gamma = 0$ hold for both variants.

Upper Bound for Double Delegation ($k = 2$, $\gamma = 0$)

Analyzing cases with $k > 1$ is considerably more challenging. One obstacle is that we do not expect to be able to incorporate optimal resolution of potential delegations into our analysis, because the computational problem is hard even when $k = 2$ (see Theorem 1.8). Therefore, we give a pessimistic estimate of optimal resolution via a greedy delegation mechanism, which we can reason about alongside the stochastic process. Clearly, if this stochastic process can guarantee an upper bound on the maximum weight with high probability, this bound must also hold if delegations are optimally resolved to minimize maximum weight.

In more detail, whenever a new delegator is inserted into the graph, the greedy mechanism immediately selects one of the delegation options. As a result, at any point during the construction of the graph, the algorithm can measure the weight of the voters. Suppose that a new delegator suggests two delegation options, to agents a and b. By following already resolved delegations, the mechanism obtains voters a^* and b^* such that a transitively delegates to a^* and b to b^*. The greedy mechanism then chooses the delegation whose voter currently has lower weight, resolving ties arbitrarily.

This situation is reminiscent of a phenomenon known as the "power of choice." In its most isolated form, it has been studied in the *balls-and-bins* model, for example by Azar et al. [13]. In this model, n balls are to be placed in n bins. In the classical setting, each ball is sequentially placed into a bin chosen uniformly at random. With high probability, the fullest bin will contain $\Theta(\log n / \log \log n)$ balls at the end of the process. In the choice setting, two bins are independently and uniformly selected for every ball, and the ball is placed into the emptier one. Surprisingly, this leads to an exponential improvement, where the fullest bin will contain at most $\Theta(\log \log n)$ balls with high probability.

We show that, at least for $\gamma = 0$ in our setting, this effect outweighs the "rich-get-richer" dynamic described earlier:

Theorem 1.10. *In the preferential delegation model with $k = 2$, $\gamma = 0$, and $d \in (0, 1)$, the maximum weight of any voter at time t is $\log_2 \ln t + \Theta(1)$ with high probability.*

Because the proof of Theorem 1.10 is quite intricate and technical, we only present a sketch of its structure here. In our proof we build on work by Malyshkin and Paquette [167], who study the maximum *degree* in a graph generated by preferential attachment with the power of choice. In addition, we incorporate ideas by Haslegrave and Jordan [124]. Proofs for the individual lemmas can be found in the full version of the paper [120].

For our analysis, it would be natural to keep track of the number of voters v with a specific weight $w_j(v) = k$ at a specific point j in time. In order to simplify the analysis, we instead keep track of random variables

$$F_j(k) \coloneqq \sum_{\substack{v \in V \\ w_j(v) \geq k}} w_j(v),$$

i.e., we sum up the weights of all voters with weight at least k. Since the total weight

increases by one in every step, we have

$$\forall j.\ F_j(1) = j, \text{ and} \tag{1.42}$$

$$\forall j, k.\ F_j(k) \leq j. \tag{1.43}$$

If $F_j(k) < k$ for some j and k, the maximum weight of any voter must be below k.

If we look at a specific $k > 1$ in isolation, the sequence $(F_j(k))_j$ evolves as a Markov process initialized at $F_1(k) = 0$ and then governed by the rule

$$F_{m+1}(k) - F_m(k) = \begin{cases} 1 & \text{with probability } d \left(\frac{F_m(k)}{m}\right)^2 \\ k & \text{with probability } d \left(\left(\frac{F_m(k-1)}{m}\right)^2 - \left(\frac{F_m(k)}{m}\right)^2\right) \\ 0 & \text{else} \end{cases}. \tag{1.44}$$

In the first case, both potential delegations of a new delegator lead to voters who already had weight at least k. We must thus give her vote to one of them, increasing $F_m(k)$ by one. In the second case, a new delegator offers two delegations leading to voters of weight at least $k-1$, at least one of which has exactly weight $k-1$. Our greedy algorithm will then choose a voter with weight $k-1$. Because this voter is counted in the definition of $F_j(k)$, $F_m(k)$ increases by k. Finally, if a new voter appears, or if a new delegator can transitively delegate to a voter with weight less than $k-1$, then $F_m(k)$ does not change.

In order to bound the maximum weight of a voter, we first need to get a handle on the general distribution of weights. For this, we define a sequence of real numbers $(\alpha_k)_k$ such that, for every $k \geq 1$, the sequence $\frac{F_j(k)}{j}$ converges in probability to α_k. Set $\alpha_1 \coloneqq 1$. For every $k > 1$, let α_k be the unique root $0 < x < \alpha_{k-1}$ of the polynomial

$$a_k(x, p) \coloneqq d\, x^2 + k\, d\, (p^2 - x^2) - x \tag{1.45}$$

for p set to α_{k-1}.[8] Since $a_k(0, \alpha_{k-1}) > 0$ and $a_k(\alpha_{k-1}, \alpha_{k-1}) < 0$, such a solution exists by the intermediate value theorem. Because the polynomial is quadratic, such a solution must be unique in the interval. It follows that the α_k form a strictly decreasing sequence in the interval $(0, 1]$.

The sequence $(\alpha_k)_k$ converges to zero, and eventually does so very fast. However, this is not obvious from the definition and, depending on d, the sequence can initially decrease slowly. In the full proof, we demonstrate convergence to zero, as well as show that the sequence decreases at a rate in $\mathcal{O}(k^{-2})$. Based on this, we choose an integer k_0 such that the sequence decreases very fast from there. In the same lemma, we define a more nicely behaved sequence $(f(k))_{k \geq k_0}$ that is a strict upper bound on $(\alpha_k)_{k \geq k_0}$ and that is contained between two doubly-exponentially decaying functions.

Lemma 1.2.6. *For all $k \geq 1$, $\varepsilon > 0$ and functions $\omega(m)$ such that $\omega(m) \to \infty$ and $\omega(m) < m$ (for sufficiently large m),*

$$\Pr\left[\exists j,\ \omega(m) \leq j \leq m \text{ s.t. } \frac{F_j(k)}{j} > \alpha_k + \varepsilon\right] \to 0.$$

[8] The equation $0 = a_k(x, p)$ can be obtained from Equation (1.44) by naïvely assuming that $\frac{F_j(k-1)}{j}$ converges to a value p and $\frac{F_j(k)}{j}$ converges to x, then plugging these values in the expectation of the recurrence.

Proof sketch. The proof proceeds by induction on k. For $k = 1$, the claim directly holds. For larger k, we use a suitably chosen δ in place of ε and ω_0 in place of ω for the induction hypothesis. With the induction hypothesis, we bound the $\frac{F_m(k-1)}{m}$ term in the recurrence in Equation (1.44). Furthermore, all increments $F_j(k) - F_{j-1}(k)$ where $\frac{F_{j-1}(k)}{j-1} \geq \alpha_k$ holds can be dominated by independent and *identically distributed* random variables η'_j.

Denote by π the first point $j \geq \omega_0(m)$ such that $\frac{F_j(k)}{j} \leq \alpha_k + \frac{\varepsilon}{2}$. The η'_j then dominate all increments $F_j(k) - F_{j-1}(k)$ for $\omega_0(m) < j \leq \pi$. Using Chernoff's bound and suitably chosen δ and ω_0, we show that, with high probability, $\pi \leq \omega(m)$.

Because of this, if $\frac{F_j(k)}{j} > \alpha_k + \varepsilon$ for some $j \geq \omega(m)$, the sequence $\left(\frac{F_j(k)}{j}\right)_j$ must eventually cross from below $\alpha_k + \frac{\varepsilon}{2}$ to above $\alpha_k + \varepsilon$ without in between falling below α_k. On this segment, we can overapproximate the sequence by a random walk with increments distributed as η'_j. Since the sequence might previously decrease below α_k an arbitrary number of times, we overapproximate the probability of ever crossing $\alpha_k + \varepsilon$ for $j \geq \omega(m)$ by a sum over infinitely many random walks. This sum converges to 0 for $m \to \infty$, which shows our claim. □

The above lemma gives us a good characterization of the behavior of $(F_j(k))_j$ for any fixed k (and large enough j). To prove an upper bound on the maximum weight, however, we are ultimately interested in statements about $F_j(k(m))$, where $k(m) \in \Theta(\log_2 \ln m)$ and the range of j varies with m. In order to obtain such results, we will first show in Lemma 1.2.7 that whole ranges of k simultaneously satisfy bounds with high probability.

As in the previous lemma, we can only show our bounds with high probability for j past a certain period of initial chaos. We will define a function, $\phi(m, k)$, that takes a role similar to $\omega(m)$ in Lemma 1.2.6. The function $\phi(m, k)$ gives each k a certain amount of time to satisfy the bounds, depending on m: Let $\rho(m) \coloneqq (\ln \ln m)^{\frac{1}{3}}$ and define $\phi(m, k) \coloneqq \rho(m)\, C^{2^{k+1}}$, where C is an integer that is sufficiently large to satisfy

$$\ln C > \max\left(1,\ c_1,\ \ln\left(\frac{2}{1-d}\right) + \frac{c_1}{2}\right). \tag{1.46}$$

In the above, c_1 is a postive constant defining the lower bound on $f(k)$.

Additionally, let $k_*(m)$ be the smallest integer such that

$$C^{2^{k_*(m)+1}} \geq \sqrt{m}. \tag{1.47}$$

Note that $C^{2^{k_*(m)+1}} < m$ because increasing the double exponent in increments of 1 is equivalent to squaring the term. By applying logarithms to $C^{2^{k_*(m)+1}} \geq \sqrt{m}$ and $C^{2^{k_*(m)+1}} < m$, we obtain $\log_2 \log_C m - 2 \leq k_*(m) < \log_2 \log_C m - 1$, from which it follows that $k_*(m) = \log_2 \ln m + \Theta(1)$.

Lemma 1.2.7. *With high probability, for all $k_0 \leq k \leq k_*(m)$, and for all $\phi(m, k) \leq j \leq m$,*

$$\frac{F_j(k)}{j} \leq f(k).$$

Proof sketch. Let $\mathcal{G}_k$ be the event

$$\mathcal{G}_k \coloneqq \left\{ \forall j, \phi(m,k) \le j \le m.\ \frac{F_j(k)}{j} \le f(k) \right\}.$$

Our goal is to show that $\mathcal{G}_k$ holds for all k in our range. In the spirit of an inductive argument, we begin by showing $\mathcal{G}_{k_0}$ with high probability and then give evidence for how, under the assumption $\mathcal{G}_k$, $\mathcal{G}_{k+1}$ is likely to happen. Instead of an explicit induction, we piece together these parts in a union bound.

The base case $\mathcal{G}_{k_0}$ follows from Lemma 1.2.6 with $\omega(m) \coloneqq \phi(m, k_0)$ and $\varepsilon \coloneqq f(k_0) - \alpha_{k_0}$.

For the step, fix some $k \ge k_0$, and assume $\mathcal{G}_k$. We want to give an upper bound on the probability that $\mathcal{G}_{k+1}$ happens. We split this into multiple substeps: First, we prove that, given $\mathcal{G}_k$, some auxiliary event $\mathcal{E}(k+1)$ happens only with probability converging to 0. Then, we show that $\overline{\mathcal{E}(k+1)} \subseteq \mathcal{G}_{k+1}$ where $\overline{\mathcal{E}}$ denotes the complement of an event $\mathcal{E}$. This means that, whenever the unlikely event does not take place, $\mathcal{G}_{k+1}$ holds. This allows the step to be repeated.

If $\mathcal{G}_k$ does not hold for any $k_0 \le k \le k_*(m)$, then $\overline{\mathcal{G}_{k_0}}$ or one of the $\mathcal{E}(k)$ must have happened. The union bound converges to zero for $m \to \infty$, proving our claim. □

As promised, the last lemma enables us to speak about the behavior of $F_j(k(m))$. We will use a sequence of such statements to show that, with high probability, $F_j(k(m))$ for some $k(m)$ does not change over a whole range of j:

Lemma 1.2.8. *There exists $M > 0$ and an integer $r > 0$ such that, for $j_0(m) \coloneqq (\ln \ln m)^M$, $F_m(k_*(m) + r) = F_{j_0(m)}(k_*(m) + r)$ holds with high probability. In addition, there is $\beta > \frac{1}{2}$ such that, with high probability,*

$$F_{j_0(m)}(k_*(m) + r - 1) \le j_0(m)^{1-\beta}. \tag{1.48}$$

Proof sketch. We finally get a statement about $F_j(k_*(m))$: By choosing different k for different j in Lemma 1.2.7, we obtain a constant $\beta_0 > 0$ such that, with high probability,

$$\forall j, \ln \ln m \le j \le m.\ \frac{F_j(k_*(m))}{j} \le j^{-\beta_0}.$$

We now increase β_0 until it is larger than $\frac{1}{2}$. Set $r_0' \coloneqq 0$ and $M_0 \coloneqq 1$. In fact, we obtain a stronger proposition of the form

$$\forall j, (\ln \ln m)^{M_i} \le j \le m.\ \frac{F_j(k_*(m) + r_i')}{j} \le j^{-\beta_i}$$

holding with high probability to obtain, for some $M_{i+1} > 0$ and with high probability,

$$\forall j, (\ln \ln m)^{M_{i+1}} \le j \le m.\ \frac{F_j(k_*(m) + r_i' + 1)}{j} \le j^{-\frac{3}{2}\beta_i}.$$

If we set $r_{i+1}' \coloneqq r_i' + 1$ and $\beta_{i+1} \coloneqq \frac{3}{2}\beta_i$, we can repeatedly apply this argument until some $\beta_i > \frac{1}{2}$. Let M, r' and β denote M_i, r_i' and β_i, respectively, for this i. If, furthermore, $r \coloneqq r' + 1$, Equation (1.48) follows as a special case.

We then simply union-bound the probability of $F_j(k_*(m)+r)$ increasing for any j between $j_0(m)$ and m. Using the above over-approximation in Equation (1.44) gives us an over-harmonic series, whose value goes to zero with $m \to \infty$. □

We can now prove Theorem 1.10. Let Q_i denote the maximum weight after i time steps.

Proof of Theorem 1.10. By Lemma 1.2.8, with high probability, $F_m(k_*(m)+r) = F_{j_0(m)}(k_*(m)+r)$. Therefore, we have that with high probability

$$\begin{aligned} F_m(k_*(m)+r) &= F_{j_0(m)}(k_*(m)+r) \\ &\le F_{j_0(m)}(k_*(m)+r-1) && \text{(by monotonicity)} \\ &\le j_0(m)^{1-\beta} && \text{(by Equation (1.48))} \\ &= \left((\ln\ln m)^M\right)^{1-\beta} \\ &\le (\ln\ln m)^{M+1}. \end{aligned}$$

For any j and k, $Q_j \le \max\{k, F_j(k)\}$. Since, for large enough m, $k_*(m)+r < (\ln\ln m)^{M+1}$, the maximum weight Q_m is at most $(\ln\ln m)^{M+1}$ with high probability. This result holds for general m, so we are allowed to plug in $j_0(m)$ for m. Then, $Q_{j_0(m)} \le (\ln\ln j_0(m))^{M+1}$. Moreover, $(\ln\ln j_0(m))^{(M+1)^2} < j_0(m)$ for sufficiently large m because M is a constant and polylogarithmic terms grow asymptotically slower than polynomial terms. Rewriting this yields

$$Q_{j_0(m)} \le (\ln\ln j_0(m))^{M+1} < j_0(m)^{1/(M+1)}. \tag{1.49}$$

Now, note that $k_*(m)+r \ge \left(j_0(m)^{1/(M+1)}\right)$ for large enough m. Therefore, Equation (1.49) implies that, with high probability, a graph generated in $j_0(m)$ time steps has no voters of weight $k_*(m)+r$ or higher. In other words, with high probability, $F_{j_0(m)}(k_*(m)+r) = 0$, so with high probability $F_m(k_*(m)+r) = 0$ (again by Lemma 1.2.8). This means that the maximum weight after m time steps is also upper-bounded by $k_*(m)+r = \log_2 \ln m + \Theta(1)$. □

1.2.5 Simulations

In this section, we present our simulation results, which support the two main messages of this paper: that allowing multiple delegation options significantly reduces the maximum weight, and that it is computationally feasible to resolve delegations in a way that is close to optimal. x Our simulations were performed on a MacBook Pro (2017) on MacOS 10.12.6 with a 3.1 GHz Intel Core i5 and 16 GB of RAM. All running times were measured with at most one process per processor core. Our simulation software is written in Python 3.6 using Gurobi 8.0.1 to solve MILPs. All of our simulation code is open-source and available at https://github.com/pgoelz/fluid.

Multiple vs. Single Delegations

For the special case of $\gamma = 0$, we have established a doubly exponential, asymptotic separation between single delegation ($k = 1$) and two delegation options per delegator ($k = 2$). While the strength of the separation suggests that some of this improvement will carry over to the real world, we still have to examine via simulation whether improvements are visible for realistic numbers of agents and other values of γ.

To this end, we empirically evaluate two different mechanisms for resolving delegations. First, we optimally resolve delegations by solving the MILP for confluent flow with the Gurobi optimizer. Our second mechanism is the greedy "power of choice" algorithm used in the theoretical analysis and introduced in Section 1.2.4.

In Figure 1.4, we compare the maximum weight produced by a single-delegation process to the optimal maximum weight in a double-delegation process, for different values of γ and d. Since our theoretical analysis used a greedy over-approximation of the optimum, we also run the greedy mechanism on the double-delegation process.

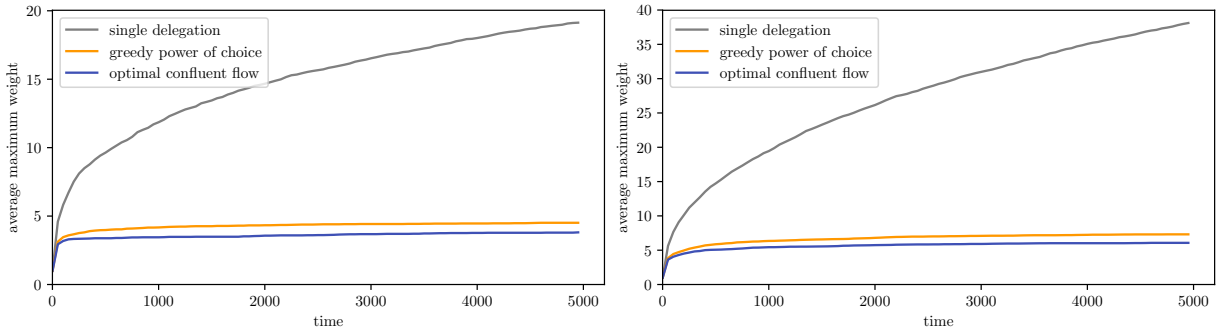
These simulations show that our asymptotic findings translate into considerable differences even for small numbers of agents, across different values of d. Moreover, these differences remain nearly as pronounced for values of γ up to 1, which corresponds to classical preferential attachment. This suggests that our mechanism can outweigh the social tendency towards concentration of votes; however, evidence from real-world elections is needed to settle this question. Lastly, we would like to point out the similarity between the graphs for the optimal maximum weight and the result of the greedy algorithm, which indicates that a large part of the separation can be attributed to the power of choice.

If we increase γ to large values, the separation between single and double delegation disappears. In Figure 1.5a, for $\gamma = 2$, all three curves are hardly distinguishable from the linear function $d \cdot \mathit{time}$, meaning that one voter receives nearly all the weight. The reason is simple: In the simulations used for that figure, 99% of all delegators give two identical delegation options, and 99.8% of these delegators (98.8% of all delegators) give both potential delegations to the heaviest voter in the graph. There are even values of $\gamma > 1$ and d such that the curve for single delegation falls below the ones for double delegation. This can be seen in Figure 1.5b, where 87.7% of voters give two identical delegation options. Since adding two delegation options per step makes the indegrees grow faster, the delegations concentrate toward a single voter more quickly, and again lead to a wildly unrealistic concentration of weight. Thus, it seems that large values of γ do not actually describe our scenario of multiple delegations.

As we have seen, switching from single delegation to double delegation greatly improves the maximum weight in plausible scenarios. It is natural to wonder whether increasing k beyond 2 will yield similar improvements. As Figure 1.6 shows, however, the returns to increasing k quickly diminish, which is common to many incarnations of the power of choice [13].

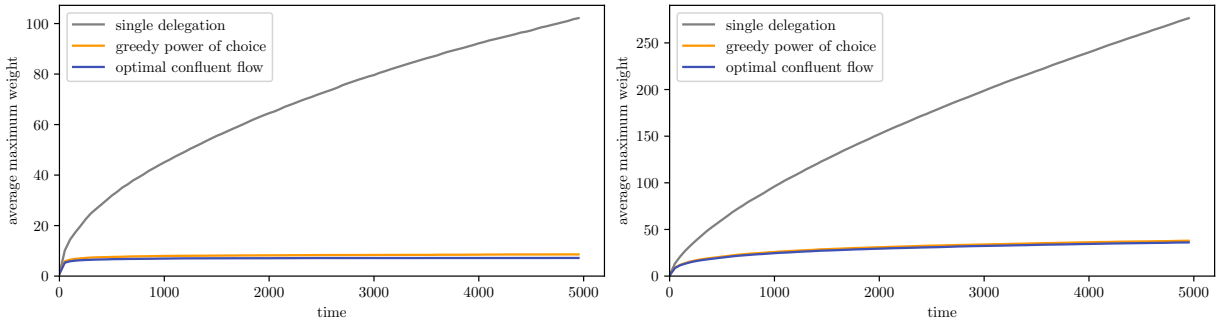
Evaluating Mechanisms

Already the case of $k = 2$ appears to have great potential; but how easily can we tap it?



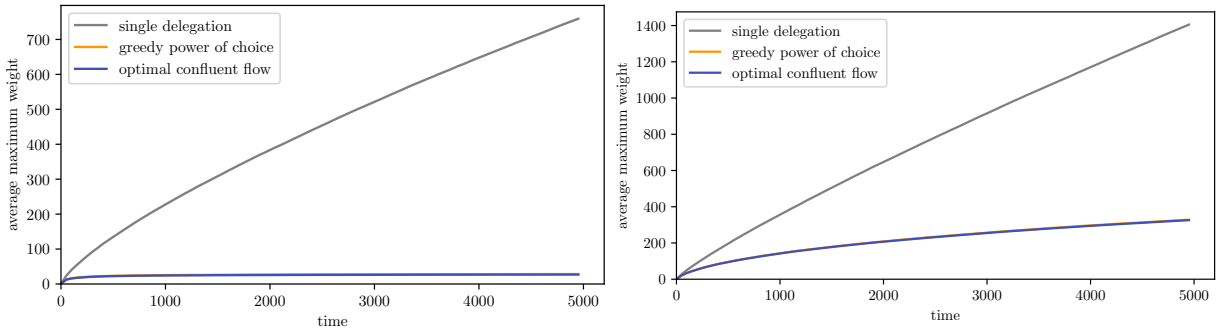
(a) $\gamma = 0$, $d = 0.25$

(b) $\gamma = 1$, $d = 0.25$



(c) $\gamma = 0$, $d = 0.5$

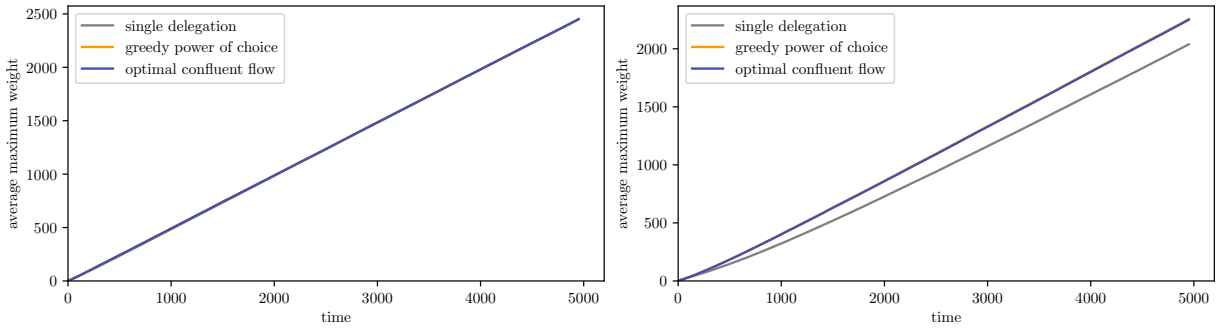
(d) $\gamma = 1$, $d = 0.5$



(e) $\gamma = 0$, $d = 0.75$

(f) $\gamma = 1$, $d = 0.75$

Figure 1.4: Maximum weight averaged over 100 simulations of length 5 000 time steps each. Maximum weight has been computed every 50 time steps.



(a) $\gamma = 2$, $d = 0.5$ (b) $\gamma = 1.5$, $d = 0.5$

Figure 1.5: Maximum weight averaged over 100 simulations, computed every 50 time steps.

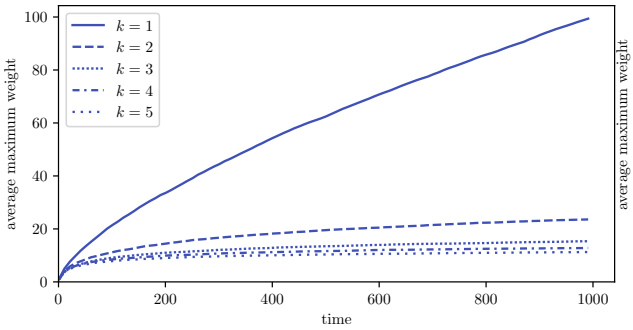


Figure 1.6: Optimal maximum weight for different k averaged over 100 simulations, computed every 10 steps. $\gamma = 1$, $d = 0.5$.

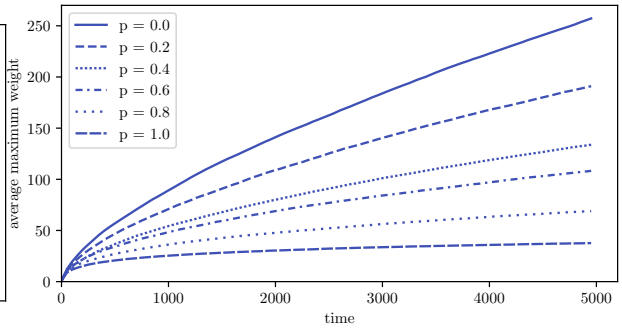
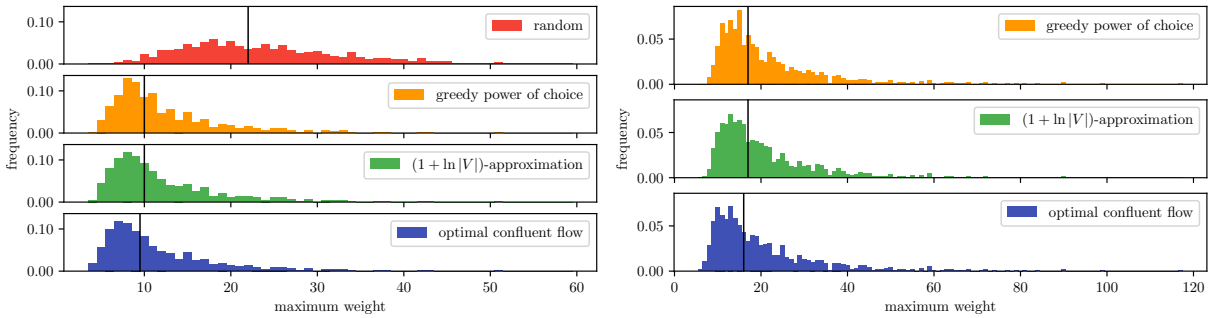


Figure 1.7: Optimal maximum weight averaged over 100 simulations. Voters give two delegations with probability p; else one. $\gamma = 1$, $d = 0.5$.

We have observed that, on average, the greedy "power of choice" mechanism comes surprisingly close to the optimal solution. However, this greedy mechanism depends on seeing the order in which our random process inserts agents and on the fact that all generated graphs are acyclic, which need not be true in practice. If the graphs were acyclic, we could simply first sort the agents topologically and then present the agents to the greedy mechanism in reverse order. On arbitrary active graphs, we instead proceed through the strongly connected components in reverse topological order, breaking cycles and performing the greedy step over the agents in the component. To avoid giving the greedy algorithm an unfair advantage, we use this generalized greedy mechanism throughout this section. Thus, we compare the generalized greedy mechanism, the optimal solution, the $(1+\ln|V|)$-approximation algorithm[9] and a random mechanism that chooses a uniformly chosen option per delegator.

At a high level, we find that both the generalized greedy algorithm and the approximation algorithm perform comparably to the optimal confluent flow solution, as shown in

[9]For one of their subprocedures, instead of directly optimizing a convex program, Chen et al. [68] reduce this problem to finding a lexicographically optimal maximum flow in $\mathcal{O}(n^5)$. We choose to directly optimize the convex problem in Gurobi, hoping that this will increase efficiency in practice.



(a) $t = 100$ (b) $t = 500$

Figure 1.8: Frequency of maximum weights at time t over 1 000 runs. $\gamma = 1$, $d = 0.5$, $k = 2$. The black lines mark the medians.

Figure 1.8 for $d = 0.5$ and $\gamma = 1$. As Figure 1.9 suggests, all three mechanisms seem to exploit the advantages of double delegation, at least on our synthetic benchmarks. These trends persist for other values of d and γ.

The similar success of these three mechanisms might indicate that our probabilistic model for $k = 2$ generates delegation networks that have low maximum weights for arbitrary resolutions. However, this is not the case: The random mechanism does quite poorly on instances with as few as $t = 100$ agents, as shown in Figure 1.8a. With increasing t, the gap between random and the other mechanisms only grows further, as indicated by Figure 1.9. In general, the graph for random delegations looks more similar to single delegation than to the other mechanisms on double delegation. Indeed, for $\gamma = 0$, random delegation is equivalent to the process with $k = 1$, and, for higher values of γ, it performs even slightly worse since the unused delegation options make the graph more centralized. Because of the poor performance of random delegation, if simplicity is a primary desideratum, we recommend using the generalized greedy algorithm instead.

As Figure 1.10 demonstrates, all three other mechanisms, including the optimal solution, easily scale to input sizes as large as the largest implementations of liquid democracy to date. Whereas the three mechanisms were close with respect to maximum weight, our implementation of the approximation algorithm is typically slower than the optimal solution (which requires a single call to Gurobi), and the generalized greedy algorithm is blazing fast. These results suggest that it would be possible to resolve delegations almost optimally even at a national scale.

1.3 Conclusions

1.3.1 An Algorithmic Perspective on Liquid Democracy

How realistic is the model? We revisit an important point, which has already come up several times. Our assumption that voters only delegate their votes to more competent voters is clearly restrictive. But we feel it allows us, in a sense, to distill the essence of liquid democracy (e.g., by avoiding complications that have to do with delegation cycles)

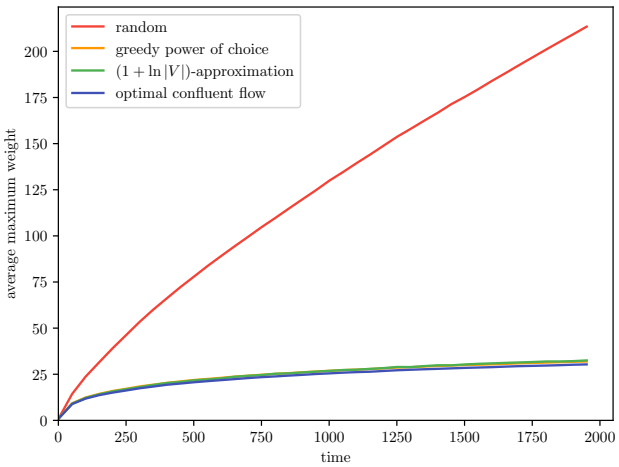


Figure 1.9: Maximum weight per algorithm for $d = 0.5$, $\gamma = 1$, $k = 2$, averaged over 100 simulations.

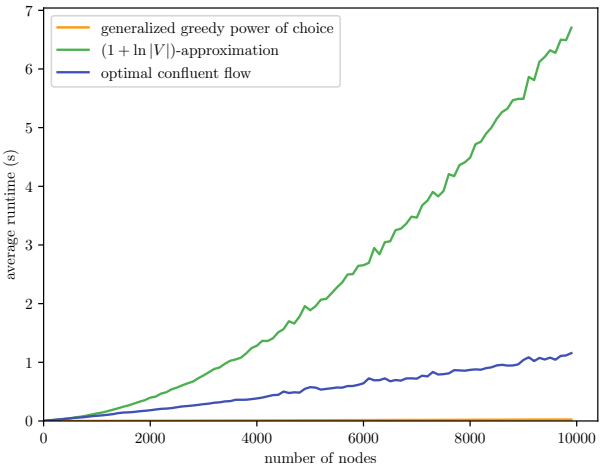


Figure 1.10: Running time of mechanisms on graphs for $d = 0.5$, $\gamma = 1$, averaged over 20 simulations.

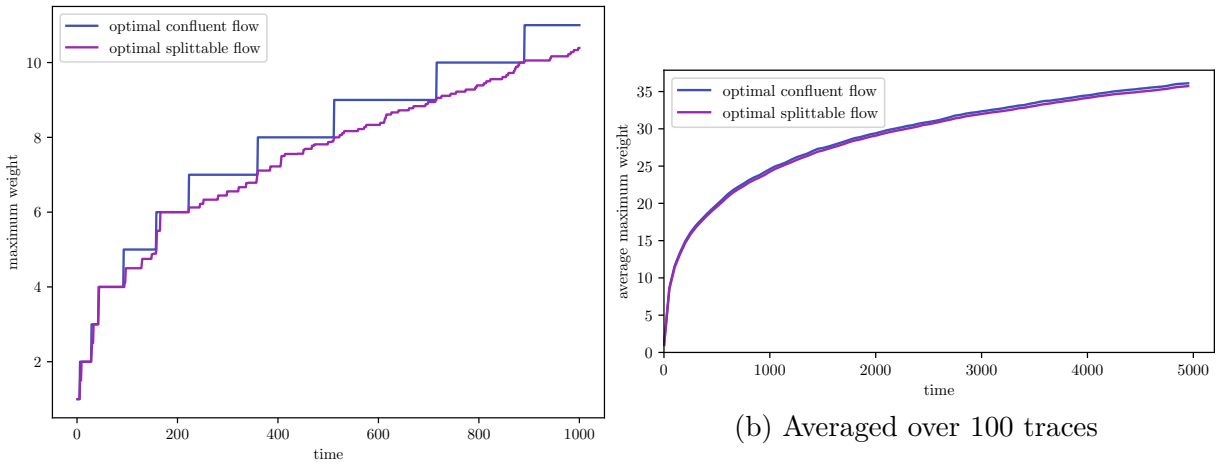
and focus on central issues such as vote correlation. Moreover, as noted earlier, our negative result—Theorem 1.1—is especially powerful in this model, that is, it holds *despite* the foregoing assumption. And the positive result—Theorem 1.3—should (informally speaking) still hold in a relaxed model where voters may delegate their votes to less competent voters, as long as the average competence level increases by a constant due to delegation. We view this as a realistic assumption.

Beyond binary issues. In our model, there are only two alternatives, one correct and one incorrect. While this setting is of practical importance, it is natural to ask whether our results extend to the case of three or more alternatives. However, there are several obstacles.

First, a representation of the ground truth, and of voters' perceptions thereof, is required. A popular option is the *Mallows* [166] *Model*, where the ground truth is a ranking of the alternatives, and the probability that a voter cast a given ranking as his vote decreases exponentially with its "distance" from the ground truth, in a way that depends on a (competence) parameter ϕ_i. This model coincides with ours (using a suitable transformation between ϕ_i and p_i) when the number of alternatives is 2.

Second, we have assumed that votes are aggregated using the majority rule, which is the only reasonable voting rule when there are two alternatives. By contrast, when choosing among three or more alternatives, there are many voting rules one can use.

Additionally, we assumed that the probability of each voter choosing the correct alternative is an independent Bernoulli parameterized by that voter's competence. However, our negative result would likely go through if we relaxed the independence assumption and allowed for bounded covariance between voters by using a similar construction.



(a) Single trace

(b) Averaged over 100 traces

Figure 1.11: Confluent vs. splittable flow: $\gamma = 1$, $d = 0.5$, $k = 2$.

1.3.2 Minimizing the Maximum Weight of Voters

The approach we have presented and analyzed revolves around the idea of allowing agents to specify multiple delegation options, and selecting one such option per delegator. As mentioned in Section 1.2.3, a natural variant of this approach corresponds to splittable — instead of confluent — flow. In this variant, the mechanism would not have to commit to a single outgoing edge per delegator. Instead, a delegator's weight could be split into arbitrary fractions between her potential delegates. Indeed, such a variant would be computationally less expensive, and the maximum voting weight can be no higher than in our setting. However, we view our concept of delegation as more intuitive and transparent: Whereas, in the splittable setting, a delegator's vote can disperse among a large number of agents, our mechanism assigns just one representative to each delegator. As suggested in the introduction, this is needed to preserve the high level of accountability guaranteed by classical liquid democracy.

We find that this fundamental shortcoming of splittable delegations is not counterbalanced by a marked decrease in maximum weight. Indeed, representative empirical results given in Figure 1.11 show that the maximum weight trace is almost identical under splittable and confluent delegations. Figure 1.11a plots a single run of the two solutions over time and suggests that the confluent solution is very close to the ceiling of the fractional LP solution. Figure 1.11b averages the optimal confluent and splittable solutions over 100 traces to demonstrate that, in our setting, the solution for confluent flow closely approximates the less constrained solution to splittable flow on average.

Furthermore, note that in the preferential delegation model with $k = 1$, splittable delegations do not make a difference, so the lower bound given in Theorem 1.9 goes through. And, when $k \geq 2$, the upper bound of Theorem 1.10 directly applies to the splittable setting. Therefore, our main technical results in Section 1.2.4 are just as relevant to splittable delegations.

To demonstrate the benefits of multiple delegations as clearly as possible, we assumed

that every agent provides two possible delegations. In practice, of course, we expect to see agents who want to delegate but only trust a single person to a sufficient degree. This does not mean that delegators should be required to specify multiple delegations. For instance, if this was the case, delegators might be incentivized to pad their delegations with very popular agents who are unlikely to receive their votes. Instead, we encourage voters to specify multiple delegations on a voluntary basis, and we hope that enough voters participate to make a significant impact. Fortunately, as demonstrated in Figure 1.7, most of the benefits of multiple delegation options persist even if only a fraction of delegators specify two delegations.

The question remains whether sufficiently many agents will indeed be sufficiently close to indifferent between multiple delegates for these benefits to be relevant in practice. Leaving individual incentives aside, how should one trade off the limitation of super-voters against the level of trust in realized delegations? Such questions can be posed in models like the one by Kahng et al. [137], which we leave for future work.

Without doubt, a centralized mechanism for resolving delegations wields considerable power. Even though we only use this power for our specific goal of minimizing the maximum weight, agents unfamiliar with the employed algorithm might suspect it of favoring specific outcomes. To mitigate these concerns, we propose to divide the voting process into two stages. In the first, agents either specify their delegation options or register their intent to vote. Since the votes themselves have not yet been collected, the algorithm can resolve delegations without seeming partial. In the second stage, voters vote using the generated delegation graph, just as in classic liquid democracy, which allows for transparent decisions on an arbitrary number of issues. Additionally, we also allow delegators to change their mind and vote themselves if they are dissatisfied with how delegations were resolved. This gives each agent the final say on their share of votes, and can only further reduce the maximum weight achieved by our mechanism. We believe that this process, along with education about the mechanism's goals and design, can win enough trust for real-world deployment.

Beyond our specific extension, one can consider a variety of different approaches that push the current boundaries of liquid democracy. For example, in a recent position paper, Brill [50] raises the idea of allowing delegators to specify a ranked list of potential representatives. His proposal is made in the context of alleviating delegation cycles, whereas our focus is on avoiding excessive concentration of weight. But, at a high level, both proposals envision centralized mechanisms that have access to richer inputs from agents. Making and evaluating such proposals now is important, because, at this early stage in the evolution of liquid democracy, scientists can still play a key role in shaping this exciting paradigm.

Being good is easy, what is difficult is being just.

Victor Hugo

2

District-Fair Participatory Budgeting

In this chapter, we study participatory budgeting, which is a method used by city governments to select public projects to fund based on residents' votes. Typically, cities that use participatory budgeting divide their budget among districts proportionally to their population. Each district then holds an election over local projects and uses its budget to fund the projects most preferred by its voters. However, district-level participatory budgeting can yield poor social welfare because it does not necessarily fund projects supported across multiple districts. On the other hand, decision making that only takes global social welfare into account can be unfair to districts: A social-welfare-maximizing solution might not fund any of the projects preferred by a district, despite the fact that its constituents pay taxes to the city. Thus, we study how to fairly maximize social welfare in a participatory budgeting setting with a single city-wide election. We propose a notion of fairness that guarantees each district at least as much welfare as it would have received in a district-level election. We show that, although optimizing social welfare subject to this notion of fairness is NP-hard, we can efficiently construct a lottery over welfare-optimal outcomes that is fair in expectation. Moreover, we show that, when we are allowed to slightly relax fairness, we can efficiently compute a fair solution that is welfare-maximizing, but which may overspend the budget.

2.1 Introduction

Participatory budgeting is a democratic approach to the allocation of public funds. In the participatory budgeting paradigm, city governments fund public projects based on constituents' votes. In contrast to budget committees, which operate behind closed doors,

participatory budgeting promises to directly take the voices of the community into account. Since 2014, Paris has allocated more than €100 million per year using constituents' votes. Many other cities around the globe—including Porto Alegre, New York City, Boston, Chicago, San Francisco, Lisbon, Madrid, Seoul, Chengdu, and Toronto—employ participatory budgeting [60; 62; 20].

Typically, participatory budgeting is used at a district-level. Each district of the city is allotted a budget proportional to its size. Constituents living in a given district vote on projects such as park, road or school improvements local to the district, using some version of approval voting. Then, the district's budget is spent according to these votes. For instance, in Paris a participatory budget is split between 20 districts (a.k.a. arrondissements), constituents vote and then each district runs a greedy algorithm to maximize the total social welfare—i.e., the total number of votes—of the funded projects. In this algorithm, projects are selected in descending order of vote count until the budget runs out.

Having separate elections for each district leads to several problems. Foremost, projects that are not local to a single district cannot be accommodated. For this reason, Paris must run an additional election for city-wide projects. However, this splits the available budget for participatory budgeting between district-level and city-wide elections in an ad hoc manner, which is not informed by votes.[1] Further, people may have interests in multiple districts, such as those who live and work in different districts. For this reason, Paris has to allow residents to choose the district in which they vote. Lastly, a project that only benefits voters at the edge of a district may receive a number of votes that is not proportional to the number of potential beneficiaries.

A simple solution to these problems is a single city-wide election. However, such a voting scheme may result in unfair outcomes. For instance, if votes are aggregated to maximize social welfare (i.e., as is presently done in Paris on the district level) then it is possible that some districts might have none of their preferred projects funded despite deserving a large proportion of the budget. Such outcomes are likely when some districts are much more populous than others, in which case projects local to small districts cannot gather sufficiently many votes. Ideally, we would like a system that balances the tradeoff between social welfare and fairness without an arbitrary, pre-determined split between district-specific and city-wide funding. This motivates our central research question:

How can we maximize social welfare in a way that is fair to all districts?

Intuitively, a solution that is fair to all districts should somehow represent each districts' constituents. One way to formalize this intuition is to stipulate that no district should be able to obtain higher utility by purchasing projects with its proportional share of the budget. In particular, each district should receive at least as much utility as it would have received had it held a district-level election with its proportional share of the budget. We call this guarantee *district fairness*.[2] A district-fair allocation of funds always exists, since an outcome obtained by holding separate district elections is district fair. We aim to find

[1] In 2016, this split in Paris was €64.3 million for district elections and €30 million for city-wide elections [61].

[2] Our notion of district fairness can be thought of as a form of *individual rationality* where every district is seen as an "individual."

district-fair outcomes that maximize social welfare. Such an outcome will be a Pareto-improvement on the status quo of district-level participatory budgeting, in the sense that each district's welfare has increased.

Our Results. In our model we think of (utilitarian) social welfare as induced by a given value assigned by each district to each project; our goal is to maximize the sum of these values over districts and selected projects. Note that this model captures the setting of approval votes, where each voter decides on a collection of projects to vote for; the social welfare of a district for a project would then be interpreted as the project's overall number of approvals from voters in that district. This observation is important because some variant of approval voting is used in most real-world participatory budgeting elections, including in Paris.

We also assume that each district is endowed with an arbitrary fraction of the total budget. Clearly this captures, as a special case, the common setting where the endowment of each district is proportional to its size. Moreover, the reasoning behind the existence of district-fair outcomes immediately applies to the more general setting.

We first show that it is NP-complete to compute an allocation that is welfare-maximizing subject to district fairness. This result holds even for the case of approval votes and proportional budgets, and therefore the generality of our model only strengthens our positive (algorithmic) results without weakening the main negative (hardness) result. We also show that the natural linear program (LP) formulation of the problem has an unbounded integrality gap. Since participatory budgeting elections can be large—hundreds of projects are proposed and hundreds of thousands of votes are cast in Paris—computational complexity can become a problem in practice. Thus, we seek polynomial-time solutions with reasonable approximation guarantees.

There are several ways one might relax our problem or trade-off between parameters in our problem. In this work, we design polynomial-time algorithms that work when we relax or approximate some of the following: (1) the achieved social welfare; (2) the spent budget; (3) the fairness of the solution; and (4) the absence of randomization.

We first relax (4) by considering distributions over outcomes, a.k.a. "lotteries". We show that using a multiplicative-weights-type algorithm, one can efficiently find a lottery that guarantees budget feasibility (ex post), optimum social welfare (ex post), and district-fairness in expectation up to an ϵ (ex ante). Since the fairness guarantee only holds in expectation, some districts may be underserved once the lottery is realized. However, since participatory budgeting typically happens repeatedly (e.g., annually), such districts could be compensated in the next election, for example by increasing their share of the budget in the next year. Additionally, conducting the lottery in a completely transparent way would help in practice.

We next consider what sort of deterministic guarantees are achievable. To this end, we show how to use techniques from submodular optimization to find an outcome that is district fair "up to one project" and which achieves optimum social welfare with the caveat that the outcome may need to spend 64.7% more money than was originally budgeted. We also give a randomized algorithm with the same guarantees but which overshoots the

budget by only a $1/e$ ($\approx 37\%$) fraction with high probability. Additionally, as a corollary of these results, we give both deterministic and randomized algorithms that achieve weaker utility and fairness guarantees but do not overspend the available budget.

2.2 Related Work

The social choice literature on participatory budgeting has both studied the voting rules used in practice, and designed original voting schemes. Goel et al. [118] study knapsack voting, used for example in Madrid [62], where voters cannot approve more projects than fit into the budget constraint. Talmon and Faliszewski [213] axiomatically study a variety of approval-based rules that maximize social welfare, both greedy and optimal ones.

The unit cost case (where all projects have the same cost) is best-studied, as *multi-winner* or *committee* elections [101]. For example, this setting models the election of a parliament. A main focus of that literature is the computational complexity of the winner determination of various voting rules. More relevant for our purposes are fairness axioms used in this setting. The most prominent such axioms are variants of *justified representation* [18]. These axioms are formulated for approval votes, and require that arbitrary subgroups of the electorate need to be represented in the outcome if they are *cohesive*, in the sense that there are a sufficient number of projects that are approved by every member of the subgroup. Several voting rules are known to satisfy these conditions, including Phragmén's rule and Thiele's Proportional Approval Voting [132; 201; 52; 19]. By contrast, district-fairness gives guarantees to a specific selection of subgroups (i.e., disjoint districts) but does not require these groups to be cohesive.

A very strong fairness axiom that is sometimes discussed in the context of committee elections and participatory budgeting is the *core* [99; 18; 100]. It insists that every subgroup (or *coalition*) must be represented (in the sense that it should not be possible for the subgroup to propose an alternative use of their proportional share of the budget that each group member prefers to the chosen outcome), without a cohesiveness requirement. For approval-based elections, it is a major open question whether there always exists a core outcome. For general additive utilities, there are instance where no core outcome exists [100], but several researchers have proved the existence of approximations to the core [135; 100; 69; 188]. A district-fair outcome is, in a sense, in the core: no subgroup which coincides with a district can block the outcome. Thus, our work shows that for general utilities, a core-like outcome exists if we only allow a specific collection of (disjoint) coalitions to block.

The problem of *knapsack sharing* [56] has a similar motivation to our problem. The knapsack sharing problem supposes that the *projects* are separated into districts (instead of, in our case, the voters), and each project comes with a cost and a value. The aim is to find a budget-feasible set of projects that maximize the minimum total value of the projects in a district. Note that in this formulation all districts are treated equally (there is no weighting by district population) and that there is no notion of the value of a project to a specific district. The literature contains a variety of algorithms for solving this NP-hard problem [e.g., 237; 238; 126; 110].

2.3 Formal Problem, Notation and Definitions

Formally, the setting we consider is as follows. We are given a budget $b \in \mathbb{Z}_{\geq 1}$. There are m possible projects $\mathcal{P} = \{x_1, \ldots, x_m\}$ with associated nonnegative costs $c : \mathcal{P} \to \mathbb{Z}_{\geq 0}$. We refer to a subset $W \subseteq \mathcal{P}$ as an *outcome*. The cost of an outcome W is $c(W) \coloneqq \sum_{x_j \in W} c(x_j)$. We say that a subset W is *budget-feasible* if $c(W) \leq b$.

There are k districts $d_1, \ldots, d_k$. The *social welfare* (or *utility*) that project x_j provides to district d_i is $\mathrm{sw}_i(x_j) \in \mathbb{Z}_{\geq 0}$. We assume that utilities are additive; i.e., the utility that an outcome $W \subseteq \mathcal{P}$ provides to district d_i is $\mathrm{sw}_i(W) := \sum_{x_j \in W} \mathrm{sw}_i(x_j)$. Furthermore, the total social welfare of $W \subseteq \mathcal{P}$ is $\mathrm{sw}(W) \coloneqq \sum_{i \in [k]} \mathrm{sw}_i(W)$.

Throughout this work we assume that $\mathrm{sw}(x_j)$ and $c(x_j)$ are both $\mathrm{poly}(k, m)$ for each j. (A function f is $\mathrm{poly}(x, y)$ if there exists a $k \geq 0$ such that $f = O((xy)^k)$.) We can relax this assumption using well-known bucketing techniques at the cost of an arbitrarily small ϵ in the guarantees of our algorithms. See the fully polynomial time approximation scheme for the knapsack problem [66] for an example of this technique.

To model the participatory budgeting setting, we assume that each district deserves some portion of the budget and, in turn, deserves at least the utility it could achieve if it spent its budget on its most preferred projects. Specifically, each district d_i deserves some budget $b_i \geq 0$ where $\sum_i b_i = b$. District d_i deserves utility $f_i := \mathrm{sw}_i(W_i)$, where $W_i := \arg\max_{W : c(W) \leq b_i} \mathrm{sw}_i(W)$ is d_i's favorite outcome costing at most b_i.

Definition 2.1 (District-Fair Outcome)**.** *We say that an outcome W is district-fair (DF) if $\mathrm{sw}_i(W) \geq f_i$ for all i.*

Computing f_i is precisely an instance of the knapsack problem; by our assumption that utilities and costs are polynomially bounded, this knapsack instance is solvable in polynomial time [66]. Thus, we will assume f_i is known.

Note that the outcome $\bigcup_i W_i$ is both budget-feasible and district-fair, so an outcome with both properties always exists. Our goal is to find a budget-feasible and district-fair outcome W which maximizes social welfare $\mathrm{sw}(W)$. We call our problem *district-fair welfare maximization*. Throughout this paper, we let $W^* := \arg\max_W \mathrm{sw}(W)$ be some optimal solution, where the argmax is taken over budget-feasible and district-fair solutions. Similarly, we let $\mathrm{OPT} := \mathrm{sw}(W^*)$.

We consider two relaxations of district fairness. The first relaxation extends the concept to *lotteries* over outcomes. We require that each district only needs to be approximately satisfied in expectation. We give an efficient algorithm to compute optimal district-fair lotteries in Section 2.4.

Definition 2.2 (ϵ-District-Fair Lottery)**.** *Given $\epsilon > 0$, we say that a probability distribution $\mathcal{W}$ over outcomes of cost at most b is an ϵ-district-fair (ϵ-DF) lottery if $E_{W \sim \mathcal{W}}[\mathrm{sw}_i(W)] \geq f_i - \epsilon$ for every district d_i.*

The second relaxation is *district-fairness up to one good* (DF1). Intuitively, an allocation is DF1 if each district would be satisfied if one additional project was funded.

Definition 2.3 (DF1)**.** *An outcome W is* DF1 *if for every d_i,*

$$\mathrm{sw}_i(W) + \max_{x_j \in (\mathcal{P} \setminus W)} \mathrm{sw}_i(x_j) \geq f_i.$$

DF1 is inspired by the well-studied notion of EF1 (envy-freeness up to one good) from the private goods setting [58]. This relaxation is mild, and unlike relaxations that require district-fairness to hold on average over districts, it is a *uniform* relaxation which provides guarantees for all districts. We study DF1 outcomes in Section 2.5.

2.3.1 NP-Hardness

Our first result shows that the problem of optimizing social welfare subject to district-fairness is NP-hard even in the restricted setting of approval votes (i.e., voters provide binary yes/no opinions over projects) and budgets proportional to district sizes. In fact, our problem remains NP-hard in this restricted setting even when each district contains only one voter and projects have unit costs.

We reduce from exact 3-cover (X3C), which is known to be NP-hard [115]. The idea of our reduction is as follows. Given an instance of X3C, we define a district for each of the elements in the universe, and then add a large amount of dummy districts. We then define a project for each set in our problem instance which gives one utility to the districts corresponding to the elements which it covers. We also define a large set of dummy projects that are approved by all dummy districts. We then ask whether there exists a district-fair outcome that attains high social welfare. An optimal solution for our district-fair welfare maximization problem, then, will first try to solve the X3C instance as efficiently as possible so that it can spend as much of its budget as possible on high-utility dummy projects. We formalize this idea in the following proof.

Theorem 2.4. *It is NP-complete to decide, given an instance of district-fair welfare maximization and an integer M, whether there exists a budget-feasible and district-fair outcome W such that $\mathrm{sw}(W) \geq M$. NP-hardness holds even in the restricted setting of approval votes and budgets proportional to district sizes, and when each district contains one voter and all projects have unit cost.*

Proof. The stated problem is trivially in NP. For NP-hardness we reduce from X3C. In an instance of X3C, we are given a universe $U = \{e_1, \ldots, e_{3n}\}$ and a collection $\{S_1, \ldots, S_m\}$ of 3-element subsets of U. It is a "yes"-instance if there exists a selection $S_{j_1}, \ldots, S_{j_n}$ such that $S_{j_1} \cup \cdots \cup S_{j_n} = U$.

Given an instance of X3C, we construct an instance of our problem as follows. Let $M = 3mn + 1$. We have $3n + M$ districts, $D \cup D'$. Let $D = \{d_1, \ldots, d_{3n}\}$, where each d_i in D corresponds to element e_i. Additionally, let $D' = \{d_{3n+1}, \ldots d_{3n+M}\}$, where each $d_i \in D'$ is a dummy district. We have $m + 2n + M$ projects, $X \cup X'$. Let $X = \{x_1, \ldots, x_m\}$, where $x_j \in X$ corresponds to set S_j, and let $X' = \{x_{m+1}, \ldots, x_{m+2n+M}\}$, where each $x_j \in M'$ is a dummy project. Utilities are as follows: every dummy district approves every dummy project, so $\mathrm{sw}_i(x_j) = 1$ for each $i \geq 3n + 1$ and $x_j \in X'$. Also, each non-dummy district approves of non-dummy sets to reflect the structure of the X3C instance: that is, for each $i \leq 3n$ we have $\mathrm{sw}_i(x_j) = 1$ if $x_j \in X$ and $e_i \in S_j$. All other utilities are 0: that is, $\mathrm{sw}_i(x_j) = 0$ for all other i and j. Each project has cost 1, and our budget is $b = 3n + M$. We assume all districts contain 1 voter, so $b_i = 1$ for every district d_i. Clearly, $f_i = 1$ for

each i. We ask whether there exists a district fair committee with social welfare at least $3n + (2n + M)M$.

If there exists a solution $S_{j_1}, \ldots, S_{j_n}$ to the X3C instance, then $W = \{x_{j_1}, \ldots, x_{j_n}\} \cup X'$ is an outcome with cost $n + (2n + M) = 3n + M = b$. Clearly, W is district-fair, and its social welfare is $3n + (2n + M)M$, so this is a "yes"-instance for the district-fair welfare-maximization problem.

Conversely suppose that there exists a district-fair budget-feasible outcome W with social welfare at least $3n + (2n + M)M$. Note that all projects in X together give overall welfare at most $3mn < M$. Thus, we must have $X' \subseteq W$ since otherwise the total welfare of W is less than $(2n + M)M$. Hence $|X \cap W| \leq n$. By district-fairness, for each $i = 1, \ldots, 3n$, there must be some $x_j \in W$ such that $e_i \in S_j$. These two facts together imply that $\{S_j : x_j \in W\}$ is a solution to the X3C instance. □

This NP-hardness result holds even if each district consists of a single voter and all projects have unit cost. As we show in the full paper, this special case admits a polynomial-time $\frac{1}{2}$-approximation. Our algorithm is based on a greedy algorithm and a combinatorial argument which "matches away" high utility goods of the optimal solution. One might hope to achieve an approximation result for the general case. A natural approach would be to round the optimal solution to the LP relaxation of the natural ILP formulation of our problem. However, a simple example in the full paper shows that the integrality gap of that formulation is unboundedly large, so this approach will not work.

2.4 Optimal District-Fair Lottery

In this section, we allow randomness and consider lotteries over outcomes. Our main result for the lottery setting is an ϵ-DF lottery which always achieves the optimal social welfare subject to district fairness. The welfare guarantee is ex post, so that every outcome in the lottery's support achieves optimal welfare. For the remainder of this section we let $\epsilon > 0$ refer to the ϵ in the ϵ-DF definition.

Theorem 2.5. *There is an algorithm which, in* $\text{poly}\left(m, k, \frac{1}{\epsilon}\right)$ *time, returns an ϵ-DF lottery $\mathcal{W}$ such that for all outcomes W in the support of $\mathcal{W}$, we have* $\text{sw}(W) \geq \text{OPT}$.

The intuition for our algorithm is as follows. We begin by showing that our problem is polynomial-time solvable if the number of districts k is constant. Such an algorithm is useful because we can artificially make the number of districts constant by convexly combining all districts into a single district $\tilde{d}$. We can, then, compute as our solution a utility-optimal outcome W which is fair for $\tilde{d}$ but not necessarily fair for each d_i individually. However, we can bias our solution to try and satisfy fairness for certain districts by increasing the weights of these districts in our convex combination. Thus, if W is not fair for d_i, we might naturally increase the proportional share of d_i in the convex combination and recompute W in the hopes that the new outcome we compute will be fair for d_i. We obtain our lottery by repeatedly increasing the weight of districts that do not have their fairness constraint satisfied, and then take a uniform distribution over the resulting outcomes.

Turning to the proof, we begin by describing how to solve our problem in polynomial time when k is a constant. Our algorithm will solve the natural dynamic program (DP). Specifically, consider the true/false value $R(\mathrm{sw}^{(1)}, \ldots, \mathrm{sw}^{(k)}, j, b)$ which is the answer to the question, "Does there exists an outcome of cost at most b using projects $x_1, x_2, \ldots, x_j$ wherein district d_i achieves social welfare at least $\mathrm{sw}^{(i)}$?" If the answer to this question is yes, then either the desired utilities are possible with the stated budget without using x_j or there is an outcome which uses at most $b - c(x_j)$ budget that doesn't use x_j in which every district gets at least its specified utility minus how much it values x_j. Thus, $R(\mathrm{sw}^{(1)}, \ldots, \mathrm{sw}^{(k)}, j, b)$ is true if and only if either $R(\mathrm{sw}^{(1)}, \ldots, \mathrm{sw}^{(k)}, j-1, b)$ is true or $R(\mathrm{sw}^{(1)} - \mathrm{sw}_1(x_j), \ldots, \mathrm{sw}^{(k)} - \mathrm{sw}_k(x_j), j-1, b - c(x_j))$ is true, giving us a definition by recurrence.

By our assumption that all costs and utilities are polynomially bounded, we can easily solve the dynamic program (DP) for the above recurrence, giving the following result.

Lemma 2.6. *There is an algorithm that finds a budget-feasible district-fair outcome W with $\mathrm{sw}(W) = \mathrm{OPT}$ in $m^{O(k)}$ time.*

Proof. Our algorithm simply fills in the DP table and returns the outcome corresponding to the entry in our DP table which is true, satisfies $\mathrm{sw}^{(i)} \geq f_i$ for all i and which maximizes $\sum_i \mathrm{sw}^{(i)}$. The recurrence is correct by the above reasoning.

To see why we can fill in the DP table in the stated time, note that we can trivially solve our base case, $R(\mathrm{sw}^{(1)}, \ldots, \mathrm{sw}^{(k)}, j, 1)$, for each j and possible value for each $\mathrm{sw}^{(i)}$ in polynomial time. Since $\max_{i,j} \mathrm{sw}_i(x_j)$ is polynomially bounded in m, we need only check polynomially-many in m values for each $\mathrm{sw}^{(i)}$. Lastly, since j and b are bounded by a polynomial in m, we conclude that our DP table has $m^{O(k)}$ entries, giving the desired runtime. □

We now describe our multiplicative-weights-type algorithm to produce our lottery using the above algorithm.[3] We let $w_i^{(t)} \geq 0$ be the "weight" of district i in iteration t and let $w^{(t)} := \sum_i w_i^{(t)}$ be the total weight in iteration t. Initially, our weights are uniform: $w_i^{(1)} = 1$ for all i.

For any iteration t and district d_i we let $p_i^{(t)} := \frac{w_i^{(t)}}{w^{(t)}}$ be the proportion of the weight that district i has in iteration t. These $p_i^{(t)}$ will induce our convex combination over districts; in particular we let $\tilde{d}^{(t)}$ be a district which values project x_j to extent $\tilde{\mathrm{sw}}^{(t)}(x_j) := \sum_i p_i^{(t)} \cdot \mathrm{sw}_i(x_j)$ and which deserves $\tilde{f}^{(t)} := \sum_i p_i^{(t)} \cdot f_i$ utility. Also, let $\mathrm{sw}_{\max}$ be the maximum welfare of an outcome.

With the above notation in hand, we can give our instantiation of multiplicative weights where $T := \frac{4 \ln k}{\epsilon^2} \cdot \mathrm{sw}_{\max}^2$ is the number of iterations of our algorithm.

1. For all iterations $t \in [T]$:
   (a) Let W_t be an outcome that maximizes $\mathrm{sw}(W_t)$ subject to $\tilde{\mathrm{sw}}^{(t)}(W_t) \geq \tilde{f}^{(t)}$ and $c(W_t) \leq b$. We can compute W_t using Lemma 2.6.

[3] We will only need to invoke the above algorithm for the case $k = 1$. This amounts to solving the knapsack problem with a single covering constraint, which to our knowledge is not one of the standard variants of the knapsack problem.

(b) Let $m_i^{(t)} := \mathrm{sw}_i(W_t) - f_i$ be our "mistakes", indicating how far off a district was from getting what it deserved.

(c) Update weights: $w_i^{(t+1)} \leftarrow w_i^{(t)} \cdot \exp(-\epsilon m_i^{(t)})$.

2. Return lottery $\mathcal{W}$, the uniform distribution over $\{W_t\}_t$.

We now restate the usual multiplicative weights guarantee in terms of our algorithm. This lemma guarantees that, on average, the multiplicative weights strategy is competitive with the best "expert." In the following $\langle p^{(t)}, m^{(t)} \rangle := \sum_i p_i^{(t)} \cdot m_i^{(t)}$ is the usual inner product.

Lemma 2.7 (9)**.** *For all i we have*

$$\frac{1}{T} \sum_{t \le T} \langle p^{(t)}, m^{(t)} \rangle \le \epsilon + \frac{1}{T} \sum_{t \le T} m_i^{(t)}.$$

We can use this lemma to show the desired guarantees.

Proof of Theorem 2.5. We use the algorithm described above.

Our algorithm is polynomial time since it runs for polynomially-many iterations and in each iteration we compute a solution for a problem on only one district which is solvable in polynomial time by Lemma 2.6. Also, note that by Lemma 2.6 we know that $c(W_t) \le b$ for all t, so all outcomes in the lottery are budget-feasible.

We now argue that the above lottery is utility-optimal. Fix an iteration t. Notice that since W^* is fair for all districts then it is fair for $\tilde{d}^{(t)}$. In particular,

$$\tilde{\mathrm{sw}}^{(t)}(W^*) = \sum_i p_i \cdot \mathrm{sw}_i(W^*) \ge \sum_i p_i f_i = \tilde{f}^{(t)}$$

Thus, W^* is a budget-feasible solution for the problem of finding a max-utility outcome which is fair for $\tilde{d}^{(t)}$. Thus, $\mathrm{sw}(W_t)$ can only be larger than $\mathrm{sw}(W^*)$, meaning that $\mathrm{sw}(W_t) \ge \mathrm{OPT}$.

We now argue that the above lottery is ϵ-DF in expectation. Fix a district d_i. By Lemma 2.7 we know that

$$\frac{1}{T} \sum_{t \le T} \langle p^{(t)}, m^{(t)} \rangle \le \epsilon + \frac{1}{T} \sum_{t \le T} m_i^{(t)}. \tag{2.1}$$

Now notice that by definition of $m_i^{(t)}$ and since our lottery is uniform over all W_t we know that the right-hand-side of Equation (2.1) is

$$\begin{aligned} \epsilon + \frac{1}{T} \sum_{t \le T} m_i^{(t)} &= \epsilon + \frac{1}{T} \sum_t (\mathrm{sw}_i(W_t) - f_i) \\ &= \epsilon - f_i + \frac{1}{T} \sum_t \mathrm{sw}_i(W_t) \\ &= \epsilon - f_i + \mathop{\mathbb{E}}_{W \sim \mathcal{W}}[\mathrm{sw}_i(W)] \end{aligned}$$

Thus, to show that $f_i - \epsilon \le \mathbb{E}_{W \sim \mathcal{W}}[\mathrm{sw}_i(W)]$, it suffices to show that the left-hand side of Equation (2.1) is at least 0. That is, we must show $0 \le \frac{1}{T} \sum_{t \le T} \langle p^{(t)}, m^{(t)} \rangle$. However,

this amounts to simply showing that W_t is fair for $\tilde{d}^{(t)}$; in particular, we have that the left-hand-side is

$$\begin{aligned}\frac{1}{T}\sum_{t\le T}\langle p^{(t)}, m^{(t)}\rangle &= \frac{1}{T}\sum_{t\le T}\sum_i p_i^{(t)}\cdot(\mathrm{sw}_i(W_t)-f_i)\\ &= \frac{1}{T}\sum_{t\le T}\tilde{\mathrm{sw}}^{(t)}(W_t)-\tilde{f}^{(t)}.\end{aligned}$$

It holds that $\tilde{sw}^{(t)}(W_t)-\tilde{f}^{(t)}\ge 0$ since we always choose a solution which is fair for $\tilde{d}^{(t)}$, and so we conclude that the left-hand-side of Equation (2.1) is at least 0. □

2.5 Optimal DF1 Outcome with Extra Budget

We now study how well we can do if we allow ourselves to overspend the available budget. Certainly it is possible to achieve district fairness and optimal fairness-constrained utility OPT if the algorithm can spend *double* the available budget: we can compute an outcome W_1 with $c(W_1)\le b$ that is welfare-maximizing without attempting to satisfy district-fairness, and we can compute some outcome W_2 with $c(W_2)\le b$ that is district-fair (see Section 2.3); then $W_1\cup W_2$ satisfies district fairness and we clearly have $c(W_1\cup W_2)\le 2b$ and $\mathrm{sw}(W_1\cup W_2)\ge \mathrm{OPT}$. In this section, we show that we can find a solution that requires less than twice the budget, if we slightly relax the district fairness requirement to DF1. Our main result for the DF1 setting shows that, under DF1 fairness, there is a deterministic algorithm which achieves DF1 and optimal social welfare if one overspends a 0.647 fraction of the budget.

Theorem 2.8. *For any constant $\epsilon>0$, there is a $\mathrm{poly}(m,k)$-time algorithm which, given an instance of district-fair welfare maximization, returns an outcome W such that W is DF1, $c(w)\le(1.647+\epsilon)\,b$, and $\mathrm{sw}(W)\ge(1-\epsilon)\mathrm{OPT}$.*

Overspending by 64.7% is a worst-case result, and the algorithm may often overspend less. If the context does not permit any overspending, one can run the same algorithm with a reduced budget; then the output will be feasible for the true budget, yet will satisfy weaker fairness and social welfare guarantees. More precisely, given an instance $\mathcal{I}$ and a multiplier $\beta<1$, we define an instance $\mathcal{I}'(\beta)$, which is identical to $\mathcal{I}$ but in which each district d_i contributes only $\beta\cdot b_i$ and thus deserves utility $f_i' := \mathrm{sw}_i(W_i')$, where W_i' is d_i's favorite outcome which costs at most $\beta\cdot b_i$. Additionally, let $\mathrm{OPT}'(\beta)$ represent the maximum achievable social welfare over all district-fair solutions in $\mathcal{I}'$ using a budget of at most $b':=\beta\cdot b$. Then, applying Theorem 2.8 to $\mathcal{I}'(\beta)$ results in an outcome which is DF1 and utility-optimal on this reduced instance and does not overspend the original budget b.

Corollary 2.9. *For any constant $\epsilon>0$, there is a $\mathrm{poly}(m,k)$-time algorithm which, given an instance $\mathcal{I}$ of district-fair welfare maximization, returns an outcome W such that W is DF1 for $\mathcal{I}'(\frac{1}{1.647})$, $c(W)\le(1+\epsilon)\,b$, and $\mathrm{sw}(W)\ge(1-\epsilon)\mathrm{OPT}'(\frac{1}{1.647})$.*

Our result uses a submodular optimization as a subroutine. If one allows randomization in this subroutine, algorithms with better approximation ratios are known. Thus, we can prove a similar theorem (and corollary) with a *randomized* algorithm which achieves DF1

and optimal social welfare while overspending its budget by only a $\frac{1}{e} \approx .37$ fraction of the budget, with high probability (i.e., with probability $1 - \frac{1}{p(m,k)}$ where $p(m,k)$ is some polynomial in m and k). We defer details of our randomized algorithm to the full paper.

In the remainder of this section, we will prove Theorem 2.8. Our main tool is a notion of the "coverage" of a partial outcome. An outcome has high coverage if we do not need to spend much more money to make it district-fair. On a high level, our proof consists of two main steps. First, we show how to complete an outcome with good coverage into a DF1 outcome. Second, we will show how to frame the problem of finding a solution with good coverage and social welfare as a submodular maximization problem subject to linear constraints, allowing us to use a result by Mizrachi et al. [174].

We begin by formalizing the coverage of a solution. Roughly, if we imagine that initially every district requires its portion of the budget for fairness, then fractional coverage captures how much less districts must spend to satisfy their own fairness constraints. Thus, if we imagine that our algorithm first spends its budget to satisfy fairness as efficiently as possible, and then spends the remainder of its budget on the highest utility projects, then the coverage of a collection of projects is roughly how much budget this collection "frees up" for the algorithm to spend on the highest utility projects. More formally, we define coverage by way of the notions of fractional outcomes and residual budget requirements.

Definition 2.10 (fractional outcomes)**.** *A fractional outcome is a vector $p \in \mathbb{R}^m$ where $0 \leq p_j \leq 1$. We overload notation and let the social welfare of p for district d_i be $\mathrm{sw}_i(p) := \sum_j \mathrm{sw}_i(x_j) \cdot p_j$. Similarly the social welfare of p is $\sum_i \mathrm{sw}_i(p)$. Lastly, we define the cost of p as $\sum_j c(x_j) \cdot p_j$.*

We now define the residual budget requirement of a district, given an outcome, which can be understood as the minimum amount of additional money that must be spent to satisfy the district, if fractional outcomes are allowed.

Definition 2.11 ($\mathrm{resid}_i(W)$)**.** *The residual budget requirement of district d_i given (integral) outcome W is the minimum cost of a fractional outcome p such that $\mathrm{sw}_i(W) + \mathrm{sw}_i(p) \geq f_i$ and $p_j = 0$ for all $x_j \in W$.*

We can now define the coverage of an outcome for a particular district i in terms of the total amount of budget they deserve and their residual budget requirement.

Definition 2.12 ($\mathrm{cover}_i(W)$)**.** *The coverage of an outcome W for district d_i is the difference between the amount of budget they deserve, b_i, and their residual budget requirement:* $\mathrm{cover}_i(W) := b_i - \mathit{resid}_i(W)$.

Lastly, we define the coverage of an outcome.

Definition 2.13 ($\mathrm{cover}(W)$)**.** *The overall coverage of an outcome W is the sum over all districts d_i of the coverage W affords d_i:* $\mathrm{cover}_i(W) := \sum_i \mathrm{cover}_i(W)$.

Next, we establish a useful property of DF1 solutions. In particular, given a set of projects that achieves relatively good fairness on *average*, we can then buy a small subset of projects that results in fairness up to one good for all districts. In particular, given a collection of projects that covers a $1 - \beta$ fraction of all fairness constraints, we can use at most an extra β fraction of our budget in order to complete this to a DF1 solution. Moreover, this completion is quite intuitive: purchase all projects whose total coverage exceed their cost, until there are no such projects remaining.

Formally, we state the following *DF1 completion* lemma.

Lemma 2.14 (DF1 Completion)**.** *Given an outcome W with $\mathrm{cover}(W) = b - r$, one can compute in polynomial time a set $W' \supseteq W$ such that W' is DF1 and $c(W') \leq c(W) + r$.*

Proof. We first prove that for every non-DF1 outcome W, there exists a project that we can add to W which increases its coverage by at least $c(x_j)$. Suppose that W is an outcome that fails DF1, and let d_i be a district such that $\mathrm{sw}_i(W) + \mathrm{sw}_i(x_j) < f_i$ for all $x_j \notin W$. Let p be the fractional outcome witnessing $\mathrm{resid}_i(W)$; thus $\mathrm{sw}_i(W) + \mathrm{sw}_i(p) \geq f_i$. We may assume without loss of generality that all but at most one project is integral in p_j (because there is always some optimal p with this property by additivity of sw_i). Since W fails DF1 for d_i, there is some $x_j \notin W$ such that $p(x_j) = 1$. Then $\mathrm{resid}_i(W \cup \{x_j\}) = \mathrm{resid}_i(W) - c(x_j)$ (witnessed by the fractional outcome obtained from p by removing x_j from it). Thus, from definitions, $\mathrm{cover}_i(W \cup \{x_j\}) = \mathrm{cover}_i(W) + c(x_j)$, and hence $\mathrm{cover}(W \cup \{x_j\}) \geq \mathrm{cover}(W) + c(x_j)$.

Now suppose we are given an outcome W with $\mathrm{cover}(W) = b - r$, which fails DF1. We can identify a project x_j as above, add it to W, and increase the coverage by at least $c(x_j)$. We repeat this until the outcome is DF1. This process must stop, since at each step the coverage increases by $c(x_j)$ but by definition the coverage can never exceed b. For the same reason, the cost of the projects we have added to W cannot exceed r, and thus $c(W') \leq c(W) + r$. □

With this lemma in hand, we now turn to the problem of finding high-coverage outcomes with good welfare. Let $B \geq 0$ be a lower bound on the social welfare we desire. We rephrase our problem as an optimization problem in which we maximize the coverage of an outcome subject to a linear knapsack constraint and a linear covering constraint. The knapsack constraint enforces budget feasibility, and the covering constraint encodes the requirement that the total utility of the outcome is at least B.

$$\begin{aligned} &\max_{W \subseteq \mathcal{P}} \mathrm{cover}(W) \\ &\text{s.t. } \mathrm{sw}(W) \geq B, \\ &\qquad c(W) \leq b. \end{aligned} \tag{DF1P}$$

The main tool we apply is a theorem on the maximization of nondecreasing submodular functions of Mizrachi et al. [174]. Recall that a set function is nondecreasing if its value never decreases as elements are added to its input, and submodular if it exhibits diminishing returns.

Definition 2.15. *Given a finite set Ω, a set function $f : \Omega \to \mathbb{R}_{\geq 0}$ is* nondecreasing *and* submodular *if for every $A, B \subseteq \Omega$ such that $A \subseteq B$ we have $f(A) \leq f(B)$ and $f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B)$ for all $x \in \Omega \setminus B$.*

The theorem we apply is as follows.

Theorem 2.16 (174, Theorem 5)**.** *For each constant $\epsilon > 0$, there exists a deterministic algorithm for maximizing a nondecreasing submodular function subject to one packing constraint and one covering constraint that runs in time $O(n^{O(1)})$, where $n = |\Omega|$ is the size*

of the support of the set function, satisfies the covering constraint up to a factor of $1-\epsilon$ *and the packing constraint up to a factor of* $1+\epsilon$*, and achieves an approximation ratio of* 0.353.

We apply this theorem to find a solution that satisfies a 0.353 fraction of coverage and achieves optimal fairness-constrained utility. Then, we apply Lemma 2.14 to augment our solution using an additional $1-0.353+\epsilon$ fraction of our budget in order to obtain a final solution which satisfies full DF1. However, in order to apply Theorem 2.16, we must first establish that $\text{cover}(W)$ is a nondecreasing submodular function. In particular, note that the coverage functions $\text{cover}_i(W)$ for each district are clearly nondecreasing and submodular. It follows that their sum, $\text{cover}(W)$ is also nondecreasing and submodular, yielding the following lemma.

Lemma 2.17. *The function* $\text{cover}(W)$ *is nondecreasing and submodular.*

We are now ready to prove Theorem 2.8, which applies the DF1 completion lemma to an approximately optimal solution for the problem DF1P.

Proof of Theorem 2.8. Recall that we have assumed that the maximum utility of an outcome is polynomially bounded in m and k and that the maximum utility is integral. Thus, the value of OPT falls in a polynomial range. For each value B in this range, solve the problem DF1P using the algorithm from Theorem 2.16. Now consider all values of B for which the algorithm returned a solution with $\text{cover}(W) \geq 0.353b$; such a value must exist since we are guaranteed this condition when $B = \text{OPT}$ (since for this value, the optimum of problem (DF1P) is b). Among all solutions we found that satisfy $\text{cover}(W) \geq 0.353b$, take the one that maximizes $\text{sw}(W)$. This solution provides social welfare *at least* $(1-\epsilon)\text{OPT}$.

We have obtained an outcome W with

$$\text{cover}(W) \geq 0.353b = b - 0.647b,$$

and $\text{sw}(W) \geq (1-\epsilon)\text{OPT}$ and $c(W) \leq (1+\epsilon)b$. Now apply Lemma 2.14 to W to obtain a DF1 outcome $W' \supseteq W$ with

$$c(W') \leq c(W) + 0.647b \leq (1 + 0.647 + \epsilon)b.$$

This outcome W' satisfies the requirements of Theorem 2.8. □

2.6 Conclusions

Our results extend to the special case of unit costs, also known as committee selection. In committee selection, we elect a committee to represent voters in a larger governmental body such as a parliament. Often, to ensure local representation, the electorate is split into voting districts, which elect their representatives separately. The districts may be apportioned different numbers of representatives, for example based on district size. While this scheme guarantees each district representation, it may well be possible to increase the welfare of the voters in a district, for example by electing a diverse array of candidates with expertise in various areas who can gather votes from across the electorate. Thus, it is natural for

all districts to elect the committee together if we impose district-fairness constraints. This way, we can maximize social welfare of the final committee while guaranteeing each district fair representation. This gives a more holistic view of committee selection in exactly the same way we addressed participatory budgeting, only instead of pooling the budget between districts, we now pool seats on a committee.

Our model implicitly treats districts as atoms, and so district fairness is a kind of *individual rationality* property. In turn, individual rationality is a type of strategyproofness: it incentivizes districts not to leave the central election and instead hold a separate one. Is it possible to design a voting scheme that is fully strategyproof for districts, so that districts do not have incentives to misreport the utilities of their residents? Unfortunately not: Peters [186] proves an impossibility theorem about committee elections which implies that there does not exist a voting rule that is efficient, district-fair, and also strategyproof. This result holds even for approval votes.

Several open questions remain. Most obvious is the question of whether can we achieve welfare maximization and DF1 in polynomial time while guaranteeing to overspend the budget by less than $1/e$. More broadly, it would be interesting to study our problem with more general utility functions such as submodular or even general monotone valuation functions. Additionally, it would be exciting to study approximation algorithms which promise full district fairness. Inthe full version of the paper, we present an algorithm which satisfies district fairness and provides a 1/2-approximation to optimal district-fair social welfare in the special case of unanimous districts; it would be interesting to extend this result to the general case.

One thing to note is that the algorithm proposed in Corollary 2.9, which satisfies the original budget constraints but only guarantees scaled-down optimal social welfare, is not guaranteed to result in higher social welfare than the independent-district setting. It would be very interesting to run further empirical experiments to evaluate the performance of the scaled-down DF1 algorithm against the standard district-only setting.

The greatest care should be employed in constituting this representative assembly ... it should be an equal representation, or, in other words, equal interests among the people should have equal interests in it.

John Adams.

3 Representation in Multiwinner Elections

In this chapter, we study proportionality in approval-based multiwinner elections with a variable number of winners, where both the size and identity of the winning committee are informed by voters' opinions, which are expressed as binary approval votes over candidates. While proportionality has been studied in multiwinner elections with a fixed number of winners, it has not been considered in the variable number of winners setting. The measure of proportionality we consider is *average satisfaction (AS)*, which intuitively measures the number of agreements on average between sufficiently large and cohesive groups of voters and the output of the voting rule. First, we show an upper bound on AS that any deterministic rule can provide, and that straightforward adaptations of deterministic rules from the fixed number of winners setting do not achieve better than a 1/2 approximation to AS even for large numbers of candidates. We then prove that a natural randomized and strategyproof rule achieves a 29/32 approximation to AS.

3.1 Introduction

We study *multiwinner approval-based elections*, where a group of agents, or voters, selects a committee from a set of candidates based on the agents' preferences. Each agent expresses her preferences through an approval vote, where she designates a subset of candidates she approves for the committee, and all votes are then aggregated to select a winning committee from the pool of candidates.

Some multiwinner elections include a fixed committee size: the outcome must fill exactly k seats on a committee. This is known as the fixed number of winners (FNW) setting, and there is a large body of work on the complexity and proportionality of various voting rules

in the FNW setting [21; 202; 22; 53; 187; 210]. In contrast, we are interested in the setting in which there is no a priori fixed committee size, also known as the variable number of winners (VNW) setting. In this case, both the size of the committee and the candidates chosen to sit on the committee are informed by agents' votes.

We present a setting where VNW elections are a natural fit; Faliszewski et al. [102] discuss others.

Consider an election that consists of a series of ballot measures, where each ballot question can easily be reversed such that "Yes" becomes "No" and "No" becomes "Yes". This is a practical concern, as ballots are often deliberately constructed such that a "Yes" on one question represents a vote in favor of *upholding* a current statute, while a "Yes" on another question down the ballot represents a vote in favor of *repealing* a current statute [178]. In this case, voters derive utility from every decision they agree with, whether it is an approval vote or a disapproval vote. Note that, because there is no set number of measures that must be "elected" (i.e., passed), this constitutes a VNW election.

It can be important to ensure that the selected alternatives are chosen in a proportional manner. For instance, in the case of ballot measures, we may want to ensure that all groups in the electorate are satisfied with at least some of the outcomes. In other words, a small majority of the electorate should not be able to overrule a sizable minority on *every* ballot measure.

In order to study proportionality in FNW elections, researchers have proposed the axioms of justified representation (JR), proportional justified representation (PJR), extended justified representation (EJR), and average satisfaction (AS) [21; 202], which capture the intuition that all sufficiently large groups that agree on sufficiently many candidates should achieve some measure of satisfaction. However, to our knowledge, we are the first to study representation in VNW elections.

Our Contributions. Our main research goal is to study proportionality in multiwinner elections with a variable number of winners. In particular, we study the proportionality measure of average satisfaction (AS) and show that there is a separation between the performance of deterministic and randomized voting rules.

As our first contribution, we develop a framework for thinking about proportionality in VNW elections. Previous work on proportionality in FNW elections is largely based on the concept of justified representation (and extensions thereof). However, as we discuss in Section 3.4, JR-based notions of proportionality are less compelling in VNW elections than in FNW elections. Therefore, we instead base our approach on the concept of average satisfaction, which is arguably a more robust version of justified representation.

Second, in Section 3.5, we consider the proportionality guarantees of deterministic rules in the VNW setting. We extend three existing deterministic rules for the FNW setting to the VNW setting, and show that these rules do not guarantee good approximations to average satisfaction. We also prove upper bounds on the level of average satisfaction that any deterministic rule can provide.

Finally, in Section 3.6, motivated by the shortcomings of deterministic rules, we turn our attention to randomized rules and show that a natural randomized rule provides a

good approximation to average satisfaction.

3.2 Related Work

There is a significant body of work studying proportionality in FNW elections. As mentioned above, [21] put forward the compelling axiom of justified representation (JR), as well as a stronger version of this axiom, extended justified representation (EJR) to capture the notion that any sufficiently large and cohesive group of voters deserves some measure of representation in the elected committee. Sánchez-Fernández et al. [202] build on this idea by introducing the intermediate axiom of proportional justified representation (PJR), a relaxation of EJR that is more stringent than JR.

Average satisfaction (AS) was first defined by Sánchez-Fernández et al. [202], who study the average satisfaction guaranteed by extended justified representation (EJR). Further work by Aziz et al. [22] shows that Proportional Approval Voting (PAV) guarantees a level of average satisfaction that implies EJR. Additionally, Skowron et al. [208] extend the notion of average satisfaction to the context of complete rankings as opposed to committee selection. Further work by Skowron [209] studies the proportionality degree of various multiwinner rules by considering the average satisfaction of all groups of a certain size.

There is also a significant body of work studying VNW elections; however, to the best of our knowledge, none of the proposed rules satisfy proportionality (and, in general, that is not their goal). Kilgour [141] proposes a multitude of rules for VNW elections, including satisfaction approval voting and variants thereof. In a related vein, Kilgour et al. [143] and Brams et al. [45] study the minimax and minisum rules for selecting a committee in the VNW setting. Fishburn and Pekeč [105] study threshold approaches to committee selection, which are VNW rules in the sense that the size of the selected committee depends on the approval votes. Additionally, the Mean Rule [92] and Borda Mean Rule [46] can be seen as VNW rules when given approval votes.

Finally, Faliszewski et al. [102] study the computational complexity of various VNW rules, but do not consider proportionality in their analysis.

3.3 Preliminaries

Let $N = \{v_1, \ldots, v_n\}$ be a set of n voters and $C = \{c_1, \ldots, c_m\}$ be a set of m candidates. For every voter v_i, denote by $A_i \subseteq C$ the set of candidates that are *approved* by v_i. A preference profile $\mathbf{A} = \{A_1, \ldots, A_n\}$ is the set of all voter preferences A_i.

A variable number of winners (VNW) voting rule f takes as input a preference profile $\mathbf{A}$ and outputs some set of candidates $f(\mathbf{A}) \subseteq C$. Note that we allow $f(\mathbf{A}) = \emptyset$ or $f(\mathbf{A}) = C$. We will also consider randomized VNW voting rules that output a distribution over sets of candidates.

Throughout this paper, we will denote by W the set of candidates included in the committee, and we will denote by $C \setminus W$ the set of candidates excluded from the committee.

We say that a group of voters $V \subseteq N$ is ℓ-large if $|V| \geq \ell \cdot \frac{n}{m}$, and ℓ-cohesive if $|\cap_{i \in V} A_i| + |\cap_{i \in V} C \setminus A_i| \geq \ell$. We will also say that a group of voters V *agrees* on a

candidate c_j if $c_j \in A_i$ for all $i \in V$ or $c_j \notin A_i$ for all $i \in V$. Otherwise, we say that V *disagrees* on c_j. Intuitively, a group of voters is ℓ-large and ℓ-cohesive if they constitute an ℓ/m fraction of all voters who agree on ℓ out of m candidates.

In our work, we consider a different measure of representation than in the FNW setting. In the FNW setting, voters derive utility from the number of their approved candidates elected to the committee. However, this definition cannot be easily adapted to the VNW setting because then a rule could maximally satisfy all voters by including all candidates on the committee. Therefore, we assume that voters derive utility from agreeing with the placement of candidates either on the committee or not on the committee. For instance, in an election with two candidates, c_1 and c_2, if a voter i has approval set $A_i = \{c_1\}$ (i.e., she approves c_1 and disapproves c_2), then she receives one unit of utility for the output committee $\{c_1, c_2\}$ because she agrees with the inclusion of c_1 but disagrees with the inclusion of c_2.

With this in mind, the following definition of average satisfaction is adapted from the definition of Sánchez-Fernández et al. [2017] in the FNW setting.

Definition 3.1. *Given a set of candidates $W \subseteq C$, the* average satisfaction *of a group of voters $V \subseteq N$ is*

$$avs_W(V) = \frac{1}{|V|} \sum_{i \in V} (|A_i \cap W| + |(C \setminus A_i) \cap (C \setminus W)|).$$

We can now define AS in the VNW setting.[1] The intuition behind the following definition is that any sufficiently large and cohesive group of voters deserves to be adequately represented *on average*, which is a departure from justified representation-based axioms that have been studied in the FNW setting. Intuitively, JR-like notions of proportionality only require that some member of each cohesive group is represented to some extent, whereas average satisfaction requires all members of each cohesive group to be represented (at least on average).

Definition 3.2. *A set of candidates $W \subseteq C$ satisfies* α-AS *if, for all ℓ-large and ℓ-cohesive groups of voters $V \subseteq N$, $avs_W(V) \geq \alpha \cdot \ell$ for all $\ell \in [m]$. For brevity, we refer to the special case of* 1*-AS as AS.*

The following example demonstrates cohesiveness and average satisfaction.

Example 3. *Consider the following profile with $n = 8$ voters, $v_1, \ldots, v_8$, and $m = 4$ candidates, $c_1, \ldots, c_4$, with preferences*

$$A_1 = A_2 = \{c_1, c_2, c_3, c_4\} \qquad A_6 = \{c_2, c_3\}$$
$$A_3 = A_4 = \{c_1, c_2\} \qquad A_7 = \{c_3\}$$
$$A_5 = \{c_1, c_3\} \qquad A_8 = \{c_4\}.$$

Now, consider the output $W = \{c_4\}$. Note that each voter agrees with the output on the placement of at least one candidate, so for any 1*-large and* 1*-cohesive group $V_{(1)}$ (i.e., a group of $1 \cdot \frac{n}{m} = 2$ voters who agrees on the placement of* 1 *candidate), $avs_W(V_{(1)}) \geq 1$.*

[1] Note that we overload the use of the term "average satisfaction" to refer to both the numerical quantity from Definition 3.1 (average satisfaction) as well as the axiomatic property in Definition 3.2 (AS).

Furthermore, note that there is only one 2-large and 2-cohesive group of voters: v_1, v_2, v_3, and v_4 agree on the placement of c_1 and c_2, but disagree on the placement of c_3 and c_4, so they constitute a 2-large group of voters who agree on 2 candidates. Let $V_{(2)} = \{v_1, v_2, v_3, v_4\}$. Note that $avs_W(V_{(2)}) = 1$ because each $v \in V$ agrees with W on exactly one placement, but because this group of voters is 2-large and 2-cohesive, we see that W only satisfies 1/2-AS in this scenario.

Given our definition of voter satisfaction, we can straightforwardly extend the following deterministic multiwinner rules from the FNW setting to the VNW setting.

Proportional Approval Voting (PAV). Under the PAV rule [218], voter i derives utility $H_k = 1+1/2+\cdots+1/k$ from a committee W, where $k = |A_i \cap W| + |(C \setminus A_i) \cap (C \setminus W)|$ is the number of candidate placements that i agrees with. The goal of PAV is to maximize the sum of all voters' utilities, and thus PAV outputs the subset $W \subseteq C$ with highest PAV-score.

Sequential Phragmén (seq-Phragmén). The seq-Phragmén rule [189; 133; 53] is defined as follows. Each candidate carries a load of one unit, and this load is distributed among voters who agree with the placement of this candidate in either the included set or excluded set. The seq-Phragmén rule proceeds iteratively by, in each round, placing the candidate that results in the smallest increase in the maximal load of any voter.

Let $x_i^{(t)}$ denote the load of voter i, and $s^{(t)}$ the maximal load, after t candidates have been placed. All voters start out with no load, $x_i^{(0)} = 0$. Furthermore, let $N_j = \{i \in N : c_j \in A_i\}$ represent the set of voters that approve of candidate c_j. The maximal voter load if, on the t^{th} placement, candidate c_j is included in the committee is

$$s^{(t)}(c_j) = \frac{1 + \sum_{i \in N_j} x_i^{(t-1)}}{|N_j|},$$

and the maximal voter load if candidate c_j is excluded from the committee is

$$s^{(t)}(\overline{c_j}) = \frac{1 + \sum_{i \in N \setminus N_j} x_i^{(t-1)}}{|N \setminus N_j|}$$

because the load is distributed so as to equalize the loads of all voters who agree with the placement of c_j. At each step t, seq-Phragmén places the candidate c_j that minimizes $\min(s^{(t)}(c_j), s^{(t)}(\overline{c_j}))$ and updates voter loads accordingly: in the case that c_j is included in the committee,

$$x_i^{(t)} = \begin{cases} s^{(t)}(c_j) & \text{if } i \in N_j \\ x_i^{(t-1)} & \text{otherwise,} \end{cases}$$

and in the case that c_j is excluded from the committee,

$$x_i^{(t)} = \begin{cases} s^{(t)}(\overline{c_j}) & \text{if } i \in N \setminus N_j \\ x_i^{(t-1)} & \text{otherwise.} \end{cases}$$

This rule proceeds until all candidates have been placed, and then returns the included and excluded candidates.

Rule X. Rule X [187] allocates each voter a budget of one dollar, which they then spend on placing candidates either in the included set or excluded set. Placing a candidate costs n/m dollars, and the set of voters who agree on the placement of this candidate must be able to collectively afford the placement. The rule starts with an empty included set W and an empty excluded set $\overline{W}$, and it iteratively places candidates in the committee or its complement as follows.

Let $b_i(t)$ be the amount of money that voter i has remaining after the t^{th} candidate is placed; i.e., $b_i(0) = 1$ for all voters $v_i \in N$. At the t^{th} step, we say that a candidate $c \notin W \cup \overline{W}$ is q-affordable for some $q \geq 0$ if

$$\max\left(\sum_{i:c\in A_i} \min(q, b_i(t-1)), \sum_{i:c\in C\setminus A_i} \min(q, b_i(t-1))\right) \geq n/m.$$

In other words, candidate c is q-affordable if it can be placed in either the included or excluded set while voters who approve or disapprove of c each pay a maximum of q dollars. If no candidate is q-affordable for any $q \geq 0$, then the rule stops, placing the current set of included candidates into W, the current set of excluded candidates into $C \setminus W$, and placing arbitrarily any candidates not already put into W or into $C \setminus W$. Else, the rule places the candidate which is q-affordable for the minimum value q in the approved or disapproved committee, according to voter preferences. Each voter who agrees with this placement has their budget updated to $b_i(t) = b_i(t-1) - \min(q, b_i(t-1))$, and the process continues.

3.4 Justified Representation in VNW Elections

In order to build intuition about why we focus on AS instead of (E/P)JR, we begin by defining JR, PJR, and EJR for VNW elections. In each case, the definition is a straightforward adaptation of the corresponding definition for the FNW setting, where we intuitively replace "agreement with members on the committee" with "agreement on the placement of each candidate." We slightly overload notation—namely, JR, PJR, and E JR—from the FNW setting in the following definitions.

Definition 3.3 (JR)**.** *Consider a ballot profile* $\mathbf{A}$*. A set of candidates* $W \subseteq C$ *satisfies* justified representation (JR) *with respect to* $\mathbf{A}$ *if, for all sets of* 1*-large and* 1*-cohesive voters* N^**, there exists an* $i \in N^*$ *such that* $|A_i \cap W| + |(C \setminus A_i) \cap (C \setminus W)| \geq 1$.

Definition 3.4 (PJR)**.** *Consider a ballot profile* $\mathbf{A}$*. A set of candidates* $W \subseteq C$ *satisfies* proportional justified representation (PJR) *with respect to* $\mathbf{A}$ *if, for all* ℓ*-large and* ℓ*-cohesive groups of voters* N^**,* $|\bigcup_{i\in N^*} A_i \cap W| + |(\bigcup_{i\in N^*}(C \setminus A_i)) \cap (C \setminus W)| \geq \ell$ *for all* $\ell \in [m]$.

Definition 3.5 (EJR)**.** *Consider a ballot profile* $\mathbf{A}$*. A set of candidates* $W \subseteq C$ *satisfies* extended justified representation (EJR) *with respect to* $\mathbf{A}$ *if, for all* ℓ*-large and* ℓ*-cohesive groups of voters* N^**, there exists an* $i \in N^*$ *such that* $|A_i \cap W| + |(C \setminus A_i) \cap (C \setminus W)| \geq \ell$ *for all* $\ell \in [m]$.

The following example illustrates these definitions.

Example 4. *Consider the same profile as in Example 3 with $n = 8$ voters, $v_1, \ldots, v_8$, and $m = 4$ candidates, $c_1, \ldots, c_4$.*

Again, consider the output $W = \{c_4\}$. W satisfies JR because each voter agrees with the output on the placement of at least one candidate. Furthermore, W satisfies PJR because, on the only 2-large and 2-cohesive group of voters, $\{v_1, v_2, v_3, v_4\}$, two of them agree with the placement of c_3 and two of them agree with the placement of c_4. However, W does not satisfy EJR because no voter in the coalition agrees with two placements of W—they all agree with exactly one placement.

We also study the relationship between the extensions of PAV, seq-Phragmén, and Rule X, and different notions of justified representation in the VNW setting. The proofs of the following propositions are omitted due to space constraints.[2]

Proposition 1. *PAV satisfies PJR.*

Proposition 2. *Seq-Phragmén satisfies PJR.*

Proposition 3. *Rule X satisfies PJR but not EJR.*

Notably, in the VNW setting, JR and PJR are less compelling notions of representation than in the FNW setting. In particular, whenever an ℓ-cohesive group of voters does not agree on the placement of a particular candidate, PJR automatically counts that candidate toward the group's representation quota, since at least one member of the group agrees with the candidate's placement. In other words, any disagreement within an ℓ-cohesive group results in partial representation, no matter the outcome of the election. This is particularly problematic for JR: any 1-large, 1-cohesive group of voters that disagrees on even a single candidate will never be witness to a violation of JR.

Proposition 3 is also notable because Rule X satisfies EJR for FNW elections, but the straightforward extension of this rule does not satisfy EJR for VNW elections, demonstrating a qualitative difference between proportionality properties in the FNW and VNW settings. It is still an open question whether or not PAV and seq-Phragmén satisfy EJR for VNW elections.

3.5 Deterministic Rules

We begin by showing an upper bound on the level of average satisfaction that deterministic rules can provide.

Theorem 3.6. *No deterministic rule satisfies $(\frac{m-1}{m} + \epsilon)$-AS for any m and any $\epsilon > 0$.*

Proof. First, suppose that m is odd. Then set $n = 2$, with $A_1 = \{c_1, \ldots, c_m\}$ and $A_2 = \emptyset$. Without loss of generality, suppose that the output W is such that $|W| > \frac{m}{2}$. But then voter v_2 is an $\frac{m}{2}$-large, $\frac{m}{2}$-cohesive group with average satisfaction at most $\frac{m-1}{2}$, which yields an $(\frac{m-1}{m})$-AS approximation.

Next, suppose m is even, and set $n = 4m$. Consider the profile

[2] All omitted proofs can be found in the full version of the paper on the authors' websites.

$$A_1 = \{c_1, \ldots, c_m\} \qquad A_3 = \{c_m\}$$
$$A_2 = \{c_1, \ldots, c_{m-1}\} \quad A_4 = \emptyset$$

Again, without loss of generality, suppose that the output W is such that $|W| \geq \frac{m}{2}$. We consider two cases. In the first case, suppose that the output W has $|W| \geq \frac{m}{2} + 1$. Consider the $\frac{m}{2}$-large, $\frac{m}{2}$-cohesive group of voters $V = \{v_3, v_4\}$. We have

$$avs_W(V) \leq \frac{1}{2}(m - |W| + m - |W| + 1) \leq \frac{m-1}{2}$$

which yields at most an $(\frac{m-1}{m})$-AS approximation.

In the second case, suppose that the output W has $|W| = \frac{m}{2}$. Suppose that $c_m \notin W$ (the case of $c_m \in W$ follows symmetrically). Then again consider $V = \{v_3, v_4\}$. We have

$$avs_W(V) \leq \frac{1}{2}\left(\frac{m}{2} - 1 + \frac{m}{2}\right) = \frac{m-1}{2}$$

again yielding an $(\frac{m-1}{m})$-AS approximation. This completes the proof. □

Theorem 3.6 leaves open the possibility that there exists a deterministic rule that provides quite good average satisfaction guarantees when the number of candidates is large. Finding such a rule or lowering the upper bound is an interesting open question. However, we show that none of the natural adaptations of FNW rules that we consider is able to guarantee better than a 0.5 approximation to AS even when m is large.

Theorem 3.7. *PAV does not satisfy* $(0.5 + \epsilon)$*-AS, for any* $\epsilon > 0$ *for* $m \geq 2$.

Proof. Consider a profile with $n = 2m$ voters with preferences

$$A_1 = \ldots = A_{m-1} = \{c_1, \ldots, c_m\}$$
$$A_m = \ldots = A_{2m-2} = \{c_1, \ldots, c_{m-1}\}$$
$$A_{2m-1} = \{c_m\}$$
$$A_{2m} = \emptyset.$$

This profile is symmetric in c_m, so without loss of generality suppose that c_m is included. Suppose that some $k - 1 < m - 1$ of the candidates $c_1, \ldots, c_{m-1}$ are included. Then, the *change* in PAV score that would result from including an additional candidate is

$$\frac{m-1}{k+1} + \frac{m-1}{k} - \frac{1}{m-k} - \frac{1}{m-k+1}$$
$$\geq \frac{m-1}{m} + 1 - 1 - \frac{1}{2} \geq 0,$$

where the first inequality holds because $k < m$.

Therefore, the maximum PAV score is achieved when all candidates $c_1, \ldots, c_{m-1}$ are included. But then the group $N^* = \{v_{2m-1}, v_{2m}\}$ is 1-large and 1-cohesive but is only satisfied 0.5 times on average. □

Theorem 3.8. *seq-Phragmén does not satisfy* $(0.5 + \epsilon)$*-AS, for any* $\epsilon > 0$ *for* $m \geq 2$.

Proof. Consider the same profile as in the proof of Theorem 3.7. It is easy to check that seq-Phragmén begins by including candidates $c_1, \ldots, c_{m-2}$, after which each voter of the first and second type has load $\frac{m-2}{2(m-1)}$. In the $(m-1)$-th round, the algorithm has four choices: to include or exclude c_{m-1}, or to include or exclude c_m.

Including c_{m-1} results in a load of $\frac{m-1}{2(m-1)} = \frac{1}{2}$ on voters $v_1, \ldots, v_{2m-2}$. Excluding c_{m-1} results in a load of $\frac{1}{2}$ to voters v_{2m-1} and v_{2m}. Including c_m (which is symmetric to excluding c_m) results in a load x to voters $v_1, \ldots, v_{m-1}, v_{2m-1}$, where x is the solution to $mx - (m-1)\frac{m-2}{2(m-1)} = 1$, which yields a solution of $x = \frac{1}{2}$.

The algorithm is therefore indifferent between all possible actions; breaking ties adversarially yields the inclusion of c_{m-1}. Regardless of the inclusion or exclusion of candidate c_m, the group $N^* = \{v_{2m-1}, v_{2m}\}$ is 1-large and 1-cohesive but is only satisfied 0.5 times on average. □

We note that the dependence on tiebreaking in the proof of Theorem 3.8 can be removed by taking multiple copies of the profile used in the proof and changing the preference of a single voter.

Theorem 3.9. *Rule X does not satisfy* $(0.5+\epsilon)$*-AS, for any* $\epsilon > 0$ *for* $m \geq 3$.[3]

Proof. Consider the same profile used in the proof of Theorem 3.7. Rule X begins by including each of candidates $c_1, \ldots, c_{m-1}$. Each of these candidates costs $\frac{n}{m(2m-2)} = \frac{1}{m-1}$ for each voter $v_1, \ldots, v_{2m-2}$. In comparison, placing the last candidate at any point costs $n/2$ voters $\frac{n/m}{n/2} = \frac{2}{m}$, which is a greater cost than $\frac{1}{m-1}$ when $m \geq 3$. Including each of $c_1, \ldots, c_{m-1}$ therefore costs $v_1, \ldots, v_{2m-2}$ one dollar each. Regardless of the placement of c_m, the 1-large and 1-cohesive group of voters $N^* = \{v_{2m-1}, v_{2m}\}$ is satisfied only 0.5 times on average. □

3.6 Randomized Rules

We now turn our attention to randomized rules in order to achieve better average satisfaction guarantees. A randomized rule is one that outputs a distribution over committees rather than a single committee, and our approximation guarantee will hold in expectation over the possible committees.[4] We consider a simple and natural randomized rule that, for each candidate c_j, includes c_j in the set of winners W with probability equal to the fraction[5] of the voters who approve c_j.

[3] When $m = 2$, we know from Theorem 3.6 that no deterministic rule, including Rule X, can achieve better than a 0.5 approximation.

[4] Recent work by Cheng et al. [69] has applied randomization to proportionality in the FNW setting as well.

[5] The marginal probabilities for each candidate being included in the committee are the same under this rule as the random dictatorship rule. The distribution over committees induced by the two rules is different, however.

Definition 3.10. *Given a preference profile* $\mathbf{A}$*, the* Proportional Random Rule (PRR) *independently includes each* $c_j \in C$ *in the winning committee* W *with probability*

$$p_j = \frac{|\{v_i \in N \text{ s.t. } c_j \in A_i\}|}{n}.$$

Theorem 3.11. *PRR satisfies* 29/32*-AS in expectation for any* m*.*

In the proof of Theorem 3.11, it will be helpful to think about the effect that an individual candidate has on the satisfaction of a group G. For an outcome W, a group of voters G, and a candidate c_j, we say that the *contribution from* c_j *to the average satisfaction of* G is $avs_{c_j}(G) = |\{i : c_j \in A_i\}|/|G|$ if $c_j \in W$ or $avs_{c_j}(G) = |\{i : c_j \notin A_i\}|/|G|$ if $c_j \notin W$. Note that $avs_W(G) = \sum_{j=1}^{m} avs_{c_j}(G)$.

Proof. We prove the result in two steps. First, we show that when $\ell \leq m/3$, PRR achieves an average satisfaction of ℓ; second, we show that when $\ell > m/3$, PRR achieves an average satisfaction of $(29/32)\ell$.

Case 1: $\ell \leq m/3$**.** Consider an ℓ-cohesive group, G, of size $\ell n/m$, and a candidate c_j. Note that it is sufficient to consider groups of size exactly $\ell n/m$ because if there exists an ℓ-cohesive larger group that violates the desired guarantee, there must exist a subset of size $\ell n/m$ that also violates the guarantee. Let $k_A = |\{v_i \in G : c_j \in A_i\}|$ denote the number of voters in G who approve c_j, and $k_D = \ell n/m - k_A$ denote the number of voters in G who disapprove c_j. Without loss of generality, let $k_A \leq k_D$. Further, suppose that x of the voters in $N \setminus G$ approve c_j and $y = n - \ell n/m - x$ voters in $N \setminus G$ disapprove c_j.

The expected contribution from c_j to the average satisfaction of G is

$$\mathbb{E}[avs_{c_j}(G)] = \frac{k_A}{|G|}\left(\frac{k_A + x}{n}\right) + \frac{k_D}{|G|}\left(\frac{k_D + y}{n}\right).$$

Because $k_A \leq k_D$ and $x + y$ is fixed, this expression is minimized when $y = 0$. We therefore have

$$\begin{aligned}
\mathbb{E}[avs_{c_j}(G)] &\geq \frac{k_A}{|G|}\left(\frac{k_A + n - \ell n/m}{n}\right) + \frac{k_D}{|G|}\left(\frac{k_D}{n}\right) \\
&= \frac{1}{n|G|}\left(|G|^2 + k_A(n - \ell n/m - 2k_D)\right) \\
&\geq \frac{|G|}{n} = \frac{\ell}{m},
\end{aligned}$$

where the inequality holds because $k_D \leq \ell n/m$ by definition, and we can assume $m \geq 3$ because ℓ must be at least 1.

Case 2: $\ell > m/3$**.** Consider an ℓ-cohesive group, G, of size $\ell n/m$, and a candidate c_j. Let $k_A = |\{v_i \in G : c_j \in A_i\}|$ denote the number of voters in G who approve c_j, and $k_D = \ell n/m - k_A$ denote the number of voters in G who disapprove c_j. Without loss of generality, let $k_A \leq k_D$. As in the previous case, it is easy to show that the expected

contribution from c_j to G's average satisfaction is minimized when all voters in $N \setminus G$ approve c_j.

We therefore have that

$$\mathbb{E}[avs_{c_j}(G)] = \frac{k_A}{|G|}\left(\frac{k_A + n - \ell n/m}{n}\right) + \frac{k_D}{|G|}\left(\frac{k_D}{n}\right).$$

Substituting $k_D = \ell n/m - k_A$, taking the derivative with respect to k_A, and setting to 0 yields

$$\frac{1}{n}\left(4k_A - 3(\ell n/m) + n\right) = 0 \implies k_A = \frac{3\ell n/m - n}{4} > 0,$$

where the inequality follows from the assumption that $\ell > m/3$. Furthermore, the second derivative with respect to k_A is $4/n > 0$, and therefore $k_A = (3\ell n/m - n)/4$ is a local minimum.

The expected contribution from c_j to G's average satisfaction can therefore be as low as

$$\begin{aligned}\mathbb{E}[avs_{c_j}(G)] &= \frac{k_A}{|G|}\left(\frac{k_A + n - \ell n/m}{n}\right) + \frac{k_D}{|G|}\left(\frac{k_D}{n}\right) \\ &= \frac{-\ell}{8m} + \frac{3}{4} - \frac{m}{8\ell}.\end{aligned}$$

We also note that, because G is ℓ-cohesive, there exist at least ℓ candidates that G agrees on. Each of these candidates has

$$avs_{c_j}(G) \geq |G|/n \geq \ell/m,$$

where the first inequality follows from G being ℓ-cohesive and the second from G being ℓ-large.

Summing over the contributions of all candidates, the average satisfaction of G is at least

$$\begin{aligned}&\ell\frac{\ell}{m} + (m-\ell)\left(\frac{3}{4} - \frac{\ell}{8m} - \frac{m}{8\ell}\right) \\ &= \left(\frac{9\ell}{8m} - \frac{m^2}{8\ell^2} - \frac{7}{8} + \frac{7m}{8\ell}\right)\ell. \qquad (3.1)\end{aligned}$$

Our goal is to lower bound the term in parentheses by $\frac{29}{32}$, thus providing the desired approximation guarantee. Setting $\ell = \alpha m$, where $\alpha \in (\frac{1}{3}, 1)$, and differentiating with respect to α yields

$$\frac{d}{d\alpha}\left(\frac{9\alpha}{8} - \frac{1}{8\alpha^2} - \frac{7}{8} + \frac{7}{8\alpha}\right) = \frac{9}{8} + \frac{2}{8\alpha^3} - \frac{7}{8\alpha^2}.$$

Setting equal to 0 yields

$$9\alpha^3 - 7\alpha + 2 = (1+\alpha)(3\alpha - 2)(3\alpha - 1) = 0,$$

so the only critical point in the interval $\alpha \in (1/3, 1]$ is $\alpha = 2/3$. It is easy to check that the second derivative is positive at $\alpha = 2/3$, so average satisfaction is minimized at this point. Plugging $\ell = 2m/3$ into Equation 3.1 yields a 29/32 approximation to AS, as desired. □

Guided by Theorem 3.11, we show that the bound is tight.

Theorem 3.12. *PRR does not satisfy* $(29/32 + \epsilon)$*-AS for any* $\epsilon > 0$.

Proof. Let $m = 3$ and $n = 12$. Consider the profile

$$\begin{aligned} A_1 = A_2 = A_3 = A_4 = A_5 &= \{c_1, c_2, c_3\} \\ A_6 = A_7 = A_8 &= \{c_1, c_2\} \\ A_9 = A_{10} = A_{11} = A_{12} &= \emptyset. \end{aligned}$$

In particular, note that the first 8 voters form a 2-large and 2-cohesive group. Then the expected satisfaction of the first five voters is $\frac{2}{3} + \frac{2}{3} + \frac{5}{12} = \frac{21}{12}$ and the expected satisfaction of the next three voters is $\frac{2}{3} + \frac{2}{3} + \frac{7}{12} = \frac{23}{12}$. Taking the average yields $\frac{1}{8}(5\frac{21}{12} + 3\frac{23}{12}) = \frac{29}{16} = \frac{29}{32}\ell$ for $\ell = 2$. □

Whether there exists a randomized rule that achieves better than a 29/32-AS approximation remains an open problem.

Before concluding this section, we note a final interesting and desirable property of PRR: strategyproofness. Since decisions are made on each candidate independently, voters maximize their expected satisfaction by reporting their true approval preferences.

3.7 Conclusions

We have initiated the study of representation in approval elections with a variable number of winners. We believe that this topic, and the study of VNW elections more generally, deserves further research.

Many open problems remain. In particular, we do not have matching upper and lower bounds for the average satisfaction guarantees that can be provided by deterministic and randomized rules. Determining the existence of rules that satisfy EJR is also an interesting question; while we have argued that natural extensions of JR and PJR make less sense for VNW elections than for FNW, EJR remains a compelling property.

More broadly, we have assumed that voters gain utility whenever they agree with the placement of a candidate, either included or excluded. This is a natural model when the notions of inclusion and exclusion are symmetric, as in the ballot measure example. In other settings it makes sense to consider other utility models. For instance, a natural extension of our model would consider voters who derive different levels of utility for an approved candidate being selected and a disapproved candidate being excluded, or even negative utility from an approved candidate not being selected or a disapproved candidate being included. The latter utility model is reminiscent of rules such as net satisfaction approval voting (NSAV) [142], and precision and recall metrics in information retrieval. Extending our results to this setting appears nontrivial.

Doing what's right isn't the problem. It's knowing what's right.

Lyndon B. Johnson

4

Virtual Democracy

Virtual democracy is an approach to automating decisions, by learning models of the preferences of individual people, and, at runtime, aggregating the *predicted* preferences of those people on the dilemma at hand. One of the key questions is which aggregation method — or *voting rule* — to use; we offer a novel statistical viewpoint that provides guidance. Specifically, we seek voting rules that are *robust* to prediction errors, in that their output on people's true preferences is likely to coincide with their output on noisy estimates thereof. In this chapter, we prove that the classic Borda count rule is robust in this sense, whereas any voting rule belonging to the wide family of pairwise-majority consistent rules is not. Our empirical results further support, and more precisely measure, the robustness of Borda count.

Furthermore, we present WeBuildAI, a collective participatory framework based on the theoretical findings above. We applied this framework to a matching algorithm that operates an on-demand food donation transportation service in order to adjudicate equity and efficiency trade-offs. The service's stakeholders—donors, volunteers, recipient organizations, and nonprofit employees—used the framework to design the algorithm through a series of studies in which we researched their experiences. Our findings suggest that the framework successfully enabled participants to build models that they felt confident represented their own beliefs. Participatory algorithm design also improved both procedural fairness and the distributive outcomes of the algorithm, raised participants' algorithmic awareness, and helped identify inconsistencies in human decision-making in the governing organization. Our work demonstrates the feasibility, potential and challenges of community involvement in algorithm design.

4.1 Virtual Democracy in Theory

One of the most basic ideas underlying democracy is that complicated decisions can be made by asking a group of people to vote on the alternatives at hand. As a decision-making framework, this paradigm is versatile, because people can express a sensible opinion about a wide range of issues. One of its seemingly inherent shortcomings, though, is that voters must take the time to cast a vote—hopefully an informed one—every time a new dilemma arises.

But what if we could *predict* the preferences of voters—instead of explicitly asking them each time—and then aggregate those predicted preferences to arrive at a decision? This is exactly the idea behind the work of Noothigattu et al. [182], who are motivated by the challenge of automating *ethical* decisions. Specifically, their approach consists of three steps: first, collect preferences from voters on example dilemmas; second, learn models of their preferences, which generalize to any (previously unseen) dilemma; and third, at runtime, use those models to predict the voters' preferences on the current dilemma, and aggregate the predicted preferences to reach a decision. The idea is that we would ideally like to consult the voters on each decision, but in order to automate those decisions we instead use the models that we have learned as a proxy for the actual voters. In other words, the models serve as virtual voters, which is why we refer to this paradigm as *virtual democracy*.

Since 2017, we have been building on this approach in a collaboration with a Pittsburgh-based non-profit, 412 Food Rescue, that provides on-demand food donation distribution services. The goal is to design and deploy an algorithm that would automatically make the decisions they most frequently face: given an incoming food donation, which recipient organization (such as a housing authority or food pantry) should receive it? The voters in our implementation are stakeholders: donors, recipients, volunteers (who pick up the food from the donor and deliver it to the recipient), and employees. We have collected roughly 100 pairwise comparisons from each voter, where in each comparison, the voter is provided information about the type of donation, as well as seven relevant features of the two alternatives that are being compared; for example, the distance between donor and recipient, and when the recipient last received a donation. Using this data, we have learned a model of the preferences of each voter, which allows us to predict the voter's preference *ranking* over hundreds of recipients. And given a predicted ranking for each voter, we map them into a ranking over the alternatives by applying a *voting rule*.

While this implementation sounds simple enough, the choice of voting rule can have a major impact on the efficacy of the system. In fact, the question of which voting rule to employ is one of the central questions in computational social choice [47], and in social choice theory more broadly. A long tradition of impossibility results establishes that there are no perfect voting rules [10], so the answer, such as it is, is often context-dependent.

The central premise of this theoretical body of work is that, in the context of virtual democracy, certain statistical considerations should guide the choice of voting rule. Indeed, the voting rule inherently operates on noisy predictions of the voters' true preferences, yet one might hope that it would still output the same ranking as it would in the 'real' election based on the voters' true preferences (after all, this is the ideal that virtual democracy is

trying to approximate). Our theoretical research question, therefore, is

> *... which voting rules have the property that their output on the true preferences is likely to coincide with their output on noisy estimates thereof?*

In addition to answering this theoretical question, we work with 412 Food Rescue in order to study the effects of virtual democracy as a tool for algorithmic governance. Our work makes three contributions. First, we offer a framework and tools that enable participatory algorithm design, contributing to emerging research on human-centered algorithms and participatory design for technology. Second, through a case study with stakeholders of a real-world nonprofit, we demonstrate the feasibility, potential, and challenges of community involvement in algorithm design. Finally, our work provides insights on the effects of procedurally-fair algorithms and tools that can further understanding of algorithmic fairness and moral expectations.

Our Approach and Results. Our technical approach relies on the observation that the classic Mallows [166] model is an unusually good fit with our problem. Typically the Mallows model describes situations where there is a true ranking of the alternatives σ^*. The probability that voter i would be associated with a given ranking σ_i decreases exponentially with the number of pairs of alternatives on which σ_i and σ^* disagree (formally known as the *Kendall tau* distance). The model is parameterized by a parameter $\phi \in (0, 1]$, which is directly related to the probability that σ_i agrees with σ^* on any particular pair of alternatives. This model is very well studied (see Section 4.1.1), but, even in situations where there is a ground-truth ranking, the Mallows model may not be an accurate representation of reality [169]. This observation has motivated a body of work on generalized [65; 64] and adversarial [193; 27] noise models.

In our setting each voter has a (possibly different) true ranking σ_i^*, and the voter's predicted ranking σ_i is drawn from a Mallows distribution around σ_i^*. Crucially, since the learning algorithm is, in fact, trying to predict pairwise comparisons (which make up the training set), the accuracy of the predictor can be directly mapped to the Mallows parameter ϕ. In other words, instead of making the classic assumption that voters may fail to identify the ordering of some pairs of alternatives with some probability, we are essentially observing that the machine learning algorithm fails to accurately predict some of the pairwise comparisons, and mapping that to a separate Mallows model for each voter. To drive the point home, although the Mallows model is widely believed to be a tenuous fit with previously studied applications (as discussed earlier), it is intuitively the correct way of reasoning about the errors that arise when machine learning algorithms predict rankings based on pairwise comparisons. This insight is a key part of our conceptual contribution.

Our main positive result (Theorem 4.3) is that the classic Borda count rule is *robust* to random noise, that is, it satisfies the property stated earlier, in a precise sense. Specifically, we establish an upper bound on the probability that two alternatives are ranked differently when Borda count is applied to the true preferences and to their noisy estimates. The bound depends on the parameters of the model, as well as on the difference between the scores of the two alternatives in the true profile. On a high level, the theorem implies that if one alternative is stronger than another by a moderate margin under the true profile,

Borda count is highly unlikely to swap the two when given noisy preferences.

By contrast, we show that voting rules belonging to the wide family of *pairwise-majority consistent* rules are *not* robust (Theorem 4.6). We do this by constructing an instance where there are significant margins between alternatives, yet any voting rule belonging to this family is likely to flip a pair of alternatives.

Finally, we provide empirical results that further strengthen our case for the robustness of Borda count. Specifically, these results suggest that the probability of making a mistake on a pair of alternatives decreases very quickly with their average Borda score difference, independently of the distribution used to generate the underlying true preferences.

4.1.1 Related Work

A number of recent papers have explored the idea of automating ethical decisions via machine learning and social choice [79; 107; 182]. As mentioned above, our work builds on the framework proposed by Noothigattu et al. [182]. However, it is important to clarify why the questions we explore here do not arise in their work. Since they deal with 1.3 million voters, and split-second decisions (what should a self-driving car do in an emergency?), they cannot afford to consult the individual voter models at runtime. Hence, they have added an additional *summarization* step, whereby the individual voter models are summarized as a single, concise model of societal preferences (with possibly significant loss to accuracy). The structure of the summary model is such that, for any given set of alternatives, almost all reasonable voting rules agree on the outcome (this is their main theoretical result), hence the choice of voting rule is a nonissue under that particular implementation. By contrast, our work is motivated by the food bank application of the virtual democracy framework, where the number of voters is small and speed is not of the essence, hence we predict the preferences of individual voters at runtime.

It is worth mentioning that another prominent approach to the allocation of food donations is based on (online) fair division [4]. That said, it is important to emphasize that we study a general question about the foundations of the virtual democracy paradigm, that is, our work is not technically tied to any particular application.

Furthermore, the Mallows model underlies a large body of work in computational social choice [76; 78; 96; 95; 236; 235; 163; 192; 134; 14; 15; 16; 169; 64; 65; 234]. Our model is loosely related to that of Jiang et al. [134], where individual rankings are derived from a single ground truth ranking via a Mallows model, and then a second Mallows model is applied to obtain a noisy version of each voter's ranking. Our technical question is completely different from theirs.

Finally, there is a large body of work in social choice on finding aggregation rules that satisfy axiomatic properties that formally capture notions of fairness or efficiency [10; 215]. However, many common axiomatic properties in social choice do not apply to standard applications of virtual democracy, including the autonomous vehicle domain of Noothigattu et al. [182] and our setting of food rescue, although they may be relevant in other differently-constrained domains.

4.1.2 Preliminaries

We deal with a set of alternatives A such that $|A| = m$. Preferences over A are represented via a ranking $\sigma \in \mathcal{L}$, where $\mathcal{L} = \mathcal{L}(A)$ is the set of rankings (or permutations) over A. We denote by $\sigma(j)$ the alternative ranked in position j in σ, where position 1 is the highest, and m the lowest. We denote by $\sigma^{-1}(x)$ the position in which $x \in A$ is ranked. We use $x \succ_\sigma y$ to denote that x is preferred to y according to σ, i.e., that $\sigma^{-1}(x) < \sigma^{-1}(y)$.

The setting also includes a set of voters $N = \{1, \ldots, n\}$. Each voter $i \in N$ is associated with a ranking $\sigma_i \in \mathcal{L}$. The preferences of N are represented as a *preference profile* $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathcal{L}^n$.

Given a preference profile $\boldsymbol{\sigma} \in \mathcal{L}^n$, we say that $x \in A$ *beats* $y \in A$ *in a pairwise comparison* if a majority of voters prefer x to y, that is,

$$|\{i \in N : \ x \succ_{\sigma_i} y\}| > n/2.$$

The profile $\boldsymbol{\sigma}$ induces a *weighted pairwise majority graph* $\Gamma(\boldsymbol{\sigma})$, where we have a vertex for each alternative in A. For each $x \in A$ and $y \in A \setminus \{x\}$, there is an edge from x to y if x beats y in a pairwise comparison; the weight on this edge is

$$w_{(x,y)}(\boldsymbol{\sigma}) \triangleq |\{i \in N : \ x \succ_{\sigma_i} y\}| - |\{i \in N : \ y \succ_{\sigma_i} x\}|.$$

Voting Rules A *voting rule* (formally known as a *social welfare function*) is a function $f : \mathcal{L}^n \to \mathcal{L}$, which receives a preference profile as input, and returns a 'consensus' ranking of the alternatives. We are especially interested in two families of voting rules.

- *Positional scoring rules.* Each such rule is defined by a score vector $(\alpha_1, \ldots, \alpha_m)$. Given a preference profile $\boldsymbol{\sigma}$, the score of alternative x is

  $$\sum_{i=1}^{n} \alpha_{\sigma_i^{-1}(x)}.$$

  In words, each voter who ranks x in position p gives α_p points to x. The positional scoring rule returns a ranking of the alternatives by non-increasing score, with ties broken arbitrarily.

  Our main positive result pertains to the classic *Borda count* voting rule, which is the positional scoring rule defined by the score vector $(m-1, m-2, \ldots, 0)$. Denote the Borda count score of $x \in A$ in $\boldsymbol{\sigma} \in \mathcal{L}^n$ by

  $$B(x, \boldsymbol{\sigma}) \triangleq \sum_{i=1}^{n} \left(m - \sigma_i^{-1}(x)\right).$$

- *Pairwise-majority consistent (PMC) rules* [65]: These rules satisfy a fairly weak requirement that extends the classic notion of Condorcet consistent social *choice* functions: Given a profile $\boldsymbol{\sigma}$, if the pairwise majority graph $\Gamma(\boldsymbol{\sigma}) = (A, E)$ is such that for all $x \in A$, $y \in A \setminus \{x\}$, either $(x, y) \in E$ or $(y, x) \in E$ (i.e., it is a *tournament*), and, moreover, Γ is acyclic, then $f(\boldsymbol{\sigma}) = \tau$ for the unique ranking τ induced by $\Gamma(\boldsymbol{\sigma})$. Caragiannis et al. [65] give many examples of prominent voting rules that are PMC, including the Kemeny rule, the Slater rule, the ranked pairs method, Copeland's method, and Schulze's method.

The Mallows Model Let the *Kendall tau* distance between two rankings $\sigma, \sigma' \in \mathcal{L}$ be

$$d_{\mathrm{KT}}(\sigma, \sigma') \triangleq |\{(x, y) \in A^2 : \ x \succ_\sigma y \ \wedge \ y \succ_{\sigma'} x\}|.$$

In words, it is the number of pairs of alternatives on which σ and σ' disagree. For example, if $\sigma = (a, b, c, d)$, and $\sigma' = (a, c, d, b)$, then $d_{\mathrm{KT}}(\sigma, \sigma') = 2$.

In the Mallows [166] model, there is a *ground truth ranking* $\sigma^\star$, which induces a probability distribution over perceived rankings. Specifically, the probability of a ranking σ, given the ground truth ranking $\sigma^\star$, is given by

$$\Pr[\sigma \mid \sigma^\star] \triangleq \frac{\phi^{d_{\mathrm{KT}}(\sigma, \sigma^\star)}}{Z},$$

where $\phi \in (0, 1]$ is a parameter, and

$$Z \triangleq \sum_{\sigma' \in \mathcal{L}} \phi^{d_{\mathrm{KT}}(\sigma', \sigma^\star)}$$

is a normalization constant. Note that for $\phi = 1$ this is a uniform distribution, whereas the probability of $\sigma^\star$ goes to 1 as ϕ goes to 0. For ease of exposition, we assume that $\phi < 1$.

4.1.3 From Predictions to Mallows

In the virtual democracy framework, we are faced at runtime with a dilemma that induces a set of alternatives A. For example, when a food bank receives a donation, the set of alternatives is the current set of recipient organizations, each associated with information specific to the current donation, such as the distance between the donor and the recipient. Each voter $i \in N$ has a ranking $\sigma_i^\star \in \mathcal{L}$ over the given set of alternatives; together these rankings comprise the true preference profile $\boldsymbol{\sigma}^\star$.

One of the novel components of this paper is the assumption that, for each voter $i \in N$, we obtain a *predicted* ranking σ_i drawn from a Mallows distribution with parameter ϕ and true ranking $\sigma_i^\star$. We emphasize that, in contrast to almost all work on the Mallows Model, in our setting each voter has her own true ranking.

Why is the Mallows Model a good choice here? Recall that we are building preference models using pairwise comparisons as training data. When validating a model, we therefore test its accuracy on pairwise comparisons. And the Mallows model itself, because it is defined via the Kendall tau distance, is essentially determined by pairwise comparisons. In fact, the Mallows model (with parameter ϕ and true ranking $\sigma^\star$) is equivalent to the following generative process: for each pair of alternatives x and y such that $x \succ_{\sigma^\star} y$, x is preferred to y with probability $1/(1 + \phi)$, and y is preferred to x with probability $\phi/(1 + \phi)$; if this preference relation corresponds to a ranking (i.e., it is transitive), return that ranking, otherwise restart.

In more detail, let β be the average probability that we predict a pairwise comparison correctly; in our food bank implementation, $\beta \approx 0.9$. Based on the preceding discussion, one might be tempted to set $\beta = 1/(1 + \phi)$, i.e., set β to be the probability of getting the relative ordering of two *adjacent* alternatives correctly. While this is not unreasonable

(and would have been very convenient for us), for $\beta \approx 0.9$ it would lead to extremely high probability of correctly ranking alternatives that are, say, 30 positions apart in the ground truth ranking. In order to moderate this effect, we define another parameter $\kappa \in \{2, \ldots, m\}$, and assume that our observed pairwise comparisons are between $\sigma_i^\star(1)$ (the top-ranked alternative in the true ranking of i) and $\sigma_i^\star(\kappa)$ (the alternative ranked in position κ). Formally, the parameters β and κ are such that, for the ranking σ_i sampled from a Mallows Model with ϕ and $\sigma_i^\star$,

$$\Pr\left[\sigma_i^\star(1) \succ_{\sigma_i} \sigma_i^\star(\kappa)\right] = \beta. \tag{4.1}$$

It is worth noting that the implicit assumption that we are observing comparisons between $\sigma_i^\star(1)$ and $\sigma_i^\star(\kappa)$ specifically is not meant to be realistic. Rather, the idea is that there is *some* appropriate value of κ such that the observed accuracy β can be related to the underlying Mallows model through Equation (4.1), and, if we can establish results that are *general* with respect to the choice of κ, they would carry over to the real world.

Moving from conceptual issues to novel technical results, we start with the following lemma, which expresses the probability on the right hand side of Equation (4.1) in terms of the Mallows parameter ϕ.

Lemma 4.1. *Let σ_i be sampled from a Mallows Model with parameter ϕ and true ranking $\sigma_i^\star$. Then*

$$\Pr\left[\sigma_i^\star(1) \succ_{\sigma_i} \sigma_i^\star(\kappa)\right] = \frac{\kappa}{1-\phi^\kappa} - \frac{\kappa-1}{1-\phi^{\kappa-1}}.$$

Equation (4.1) and Lemma 4.1 imply that

$$\beta = \frac{\kappa}{1-\phi^\kappa} - \frac{\kappa-1}{1-\phi^{\kappa-1}},$$

but for subsequent results we need to express ϕ in terms of β and κ, and it is unclear whether this can be done in closed form. Nevertheless, we are are able to derive a bound that suffices for our purposes.

Lemma 4.2. *For β and κ defined as in Equation (4.1), it holds that*

$$\phi \leq \left(\frac{1-\beta}{\beta}\right)^{\frac{1}{2\kappa-1}}.$$

We relegate the proofs of both lemmas to the full version of the paper. Note that Lemma 4.1 can be proved via a theorem of Désir et al. [89]. Their theorem gives a closed form for the probability that an alternative x is ranked first out of a subset of alternatives S. This closed form is complex, and requires quite a bit of additional notation, so we instead derive the probability we are interested in, i.e., the probability that $\sigma_i^\star(\kappa)$ is ranked above $\sigma_i^\star(1)$, from scratch.

4.1.4 Robustness of Borda Count

In this section, we rigorously establish the robustness of Borda count to prediction error by showing that it satisfies a formal version of the desired property stated in Section 4.1.

We do this by building on the machinery developed in Section 4.1.3, as well as additional lemmas that we will state and prove momentarily.

As we have already discussed, we do not have access to the Mallows parameter ϕ. Instead, we can measure β, the probability that we correctly predict a pairwise comparison of alternatives that are κ positions apart. On a very high level, the theorem bounds the probability that the noisy Borda ranking (based on the sampled profile) would disagree with the true Borda ranking (based on the true profile) on a given pair of alternatives.

Theorem 4.3. *For any $\beta > 1/2$ and $\epsilon > 0$ there exists a universal constant $T = T(\beta, \epsilon)$ such that for all $n, m, \kappa \in \mathbb{N}$ such that $n, m \geq 2$, for all $s \geq T\kappa \log \kappa$, for all $\boldsymbol{\sigma}^\star \in \mathcal{L}^n$, and for all $x, x' \in A$ such that $\frac{1}{n}B(x, \boldsymbol{\sigma}^\star) \geq \frac{1}{n}B(x', \boldsymbol{\sigma}^\star) + 2s$, it holds that*

$$\Pr\left[\frac{1}{n}B(x, \boldsymbol{\sigma}) > \frac{1}{n}B(x', \boldsymbol{\sigma})\right] \geq 1 - \epsilon^n,$$

where the probability is taken over the sampling of $\boldsymbol{\sigma}$.

Let us discuss the statement of the theorem. First, note that the probability of mistake, ϵ^n, converges to 0 exponentially fast as n grows, so the theorem immediately implies a "with high probability" statement. Moreover, one can easily derive such a statement with respect to all pairs of alternatives (whose Borda scores are sufficiently separated) *simultaneously*, using a direct application of the union bound. Second, it is intuitive that the separation in Borda scores has to depend on κ, but it is encouraging (and, to us, surprising) that this dependence is almost linear. In particular, *even* if κ is almost linear in m, i.e., $\kappa \in o(m/\log m)$, the theorem implies that our noisy Borda ranking is highly unlikely to make mistakes on pairs of alternatives whose average score difference is linear in m.

Turning to the proof, we start by bounding the probability that the Borda count score $B(x, \boldsymbol{\sigma})$ of an alternative $x \in A$ in the observed profile $\boldsymbol{\sigma}$ is far from the Borda count score $B(x, \boldsymbol{\sigma}^\star)$ in the true profile $\boldsymbol{\sigma}^\star$. The proof of the following lemma adapts that of a lemma of [49], which deals with average rank (instead of average Borda count score), but in the case of a single true ranking, i.e., $\sigma_i^\star = \sigma_j^\star$, for all i, j.

Lemma 4.4. *For all alternatives $x \in A$, and all $s \geq 0$*

$$\begin{aligned}
\Pr\left[\frac{1}{n}B(x, \boldsymbol{\sigma}) \leq \frac{1}{n}B(x, \boldsymbol{\sigma}^\star) - s\right] \\
\leq \left(\frac{2e(n + ns - 1)}{n - 1} \cdot \frac{\phi^s}{1 - \phi}\right)^n, \\
\Pr\left[\frac{1}{n}B(x, \boldsymbol{\sigma}) \geq \frac{1}{n}B(x, \boldsymbol{\sigma}^\star) + s\right] \\
\leq \left(\frac{2e(n + ns - 1)}{n - 1} \cdot \frac{\phi^s}{1 - \phi}\right)^n.
\end{aligned}$$

Proof. We prove the first inequality; the proof of the second is analogous. Given a subset of voters $S \subseteq N$ and a non-negative vector $\mathbf{b} = (b_i)_{i \in S} \in \mathbb{N}^{|S|}$, let $\mathcal{E}_{S,\mathbf{b}}$ be the event that

$$B(x, \sigma_i) \leq B(x, \sigma_i^\star) - b_i$$

for all voters $i \in S$, where we abuse notation by using

$$B(x, \sigma_i) \triangleq m - \sigma_i^{-1}(x)$$

to denote the Borda count score of alternative x in the *ranking* σ_i. A lemma in the full version of the paper implies that for all $s \geq 0$,

$$\Pr[B(x, \sigma_i) \leq B(x, \sigma_i^\star) - s] \leq \frac{\phi^s}{1-\phi}. \tag{4.2}$$

Therefore,

$$\begin{aligned}\Pr[\mathcal{E}_{S,\mathbf{b}}] &= \prod_{i \in S} \Pr[B(x, \sigma_i) \leq B(x, \sigma_i^\star) - b_i] \\ &\leq \prod_{i \in S} \frac{\phi^{b_i}}{1-\phi} = \frac{\phi^{\sum_{i \in S} b_i}}{(1-\phi)^{|S|}},\end{aligned}$$

where the inequality follows from Equation (4.2).

Let $\mathcal{E}$ be the event that $\frac{1}{n}B(x, \boldsymbol{\sigma}) \leq \frac{1}{n}B(x, \boldsymbol{\sigma}^\star) - s$. Notice that

$$\mathcal{E} \subset \bigcup_{S \subseteq N, \mathbf{b} \in \mathbb{N}^{|S|} : \sum_{i \in S} b_i = ns} \mathcal{E}_{S,\mathbf{b}},$$

as there must exist a subset of voters who contribute sufficiently to the difference in Borda scores. Moreover, for a fixed S, the number of vectors $\mathbf{b} \in \mathbb{N}^{|S|}$ such that $\sum_{i \in S} b_i = ns$ is exactly $\binom{|S|+ns-1}{|S|-1}$. Therefore,

$$\begin{aligned}\Pr[\mathcal{E}] &\leq \sum_{S \subseteq N} \left| \left\{ \mathbf{b} \in \mathbb{N}^{|S|} : \sum_{i=1}^{n} b_i = ns \right\} \right| \cdot \frac{\phi^{ns}}{(1-\phi)^{|S|}} \\ &\leq 2^n \cdot \binom{n+ns-1}{n-1} \cdot \frac{\phi^{ns}}{(1-\phi)^n} \\ &\leq 2^n \cdot \left(\frac{e(n+ns-1)}{n-1} \right)^{n-1} \cdot \left(\frac{\phi^s}{1-\phi} \right)^n \\ &\leq \left(\frac{2e(n+ns-1)}{n-1} \cdot \frac{\phi^s}{1-\phi} \right)^n,\end{aligned}$$

where we used the fact that $\binom{n}{t} \leq (\frac{en}{t})^t$. □

Using Lemma 4.4 we can bound, given the Mallows parameter ϕ, the probability that two alternatives, whose Borda count scores in the true profile $\boldsymbol{\sigma}^\star$ are sufficiently far apart, are ranked by the Borda count voting rule in the correct order (in the sampled profile $\boldsymbol{\sigma}$).

Lemma 4.5. *Let $x, x' \in A$ such that $\frac{1}{n}B(x, \boldsymbol{\sigma}^\star) \geq \frac{1}{n}B(x', \boldsymbol{\sigma}^\star) + 2s$. Then*

$$\Pr\left[\frac{1}{n}B(x, \boldsymbol{\sigma}) > \frac{1}{n}B(x', \boldsymbol{\sigma})\right] \geq 1 - 2\left(\frac{2e(n+ns-1)}{n-1} \cdot \frac{\phi^s}{1-\phi} \right)^n.$$

Proof. Let $\mathcal{E}_1$ be the event that

$$\frac{1}{n}B(x,\boldsymbol{\sigma}) \leq \frac{1}{n}B(x,\boldsymbol{\sigma}^\star) - s,$$

and $\mathcal{E}_2$ be the event that

$$\frac{1}{n}B(x',\boldsymbol{\sigma}) \geq \frac{1}{n}B(x',\boldsymbol{\sigma}^\star) + s.$$

By Lemma 4.4 and a union bound we have that

$$\Pr\left[\mathcal{E}_1 \cup \mathcal{E}_2\right] \leq 2\left(\frac{2e(n+ns-1)}{n-1}\cdot\frac{\phi^s}{1-\phi}\right)^n.$$

Next, notice that every time the Borda count scores of x and x' in the sampled preference profile are in the wrong order (or tied), then at least one of $\mathcal{E}_1$, $\mathcal{E}_2$ occurred, i.e.,

$$\Pr\left[\frac{1}{n}B(x,\boldsymbol{\sigma}) \leq \frac{1}{n}B(x',\boldsymbol{\sigma})\right] \leq \Pr[\mathcal{E}_1 \cup \mathcal{E}_2].$$

The lemma directly follows. □

Recall that Lemma 4.2 gives an upper bound on ϕ as a function of β and κ. Combining with Lemma 4.5, we can bound the probability of getting the correct ranking as a function of β and κ, and prove our main result.

Proof of Theorem 4.3. By Lemma 4.5,

$$\begin{aligned}
\Pr\left[\frac{1}{n}B(x,\boldsymbol{\sigma}) > \frac{1}{n}B(x',\boldsymbol{\sigma})\right] &\geq 1 - 2\left(\frac{2e(n+ns-1)}{n-1}\cdot\frac{\phi^s}{1-\phi}\right)^n \\
&\geq 1 - 2\left(\frac{4en}{n-1}\cdot\frac{s\phi^s}{1-\phi}\right)^n \\
&\geq 1 - 2\left(8e\cdot\frac{s\phi^s}{1-\phi}\right)^n.
\end{aligned}$$

It suffices to give a bound on s such that

$$\frac{s\phi^s}{1-\phi} \leq \frac{\epsilon}{16e}. \tag{4.3}$$

By Lemma 4.2,

$$\phi \leq \left(\frac{1-\beta}{\beta}\right)^{\frac{1}{2\kappa-1}}.$$

Since $\beta > 1/2$, there is a universal constant $c > 1$ such that $\frac{1-\beta}{\beta} = \frac{1}{c}$. Therefore,

$$\begin{aligned}\frac{s\phi^s}{1-\phi} &\leq s \cdot \frac{\left(\frac{1-\beta}{\beta}\right)^{\frac{s}{2\kappa-1}}}{1-\left(\frac{1-\beta}{\beta}\right)^{\frac{1}{2\kappa-1}}} = s \cdot \frac{c^{-\frac{s}{2\kappa-1}}}{1-c^{-\frac{1}{2\kappa-1}}} \\ &= \frac{s}{c^{\frac{s}{2\kappa-1}} - c^{\frac{s-1}{2\kappa-1}}} = \frac{s}{c^{\frac{s-1}{2\kappa-1}}\left(c^{\frac{1}{2\kappa-1}}-1\right)} \\ &\leq \frac{s}{c^{\frac{s-1}{2\kappa-1}} \cdot \frac{c^{\frac{1}{2\kappa-1}}(c-1)}{c(2\kappa-1)}} \leq \frac{c}{c-1} \cdot \frac{s(2\kappa-1)}{c^{\frac{s}{2\kappa-1}}},\end{aligned}$$

where for the penultimate inequality we use the inequality

$$rz(z^{1/r}-1) > z^{1/r}(z-1),$$

which holds for all $z, r \geq 1$,[1] with $z = c$ and $r = 2\kappa - 1$. It is now easy to verify that there is a universal constant $T > 0$ such that if $s \geq T\kappa \log \kappa$ then Equation (4.3) holds. □

It is important to note that it should be possible to extend Theorem 4.3 to other positional scoring rules defined by a score vector $(\alpha_1, \ldots, \alpha_m)$ where $\alpha_j > \alpha_{j+1}$ for all $j = 1, \ldots, m-1$. However, Borda count is especially practical and easy to explain, which is why we focus on it for our positive result.

4.1.5 Non-Robustness of PMC Rules

Theorem 4.3 shows that Borda count is robust against noisy perturbations of the preference profile. It is natural to ask whether 'many' voting rules satisfy a similar property. In this section we answer this question in the negative, by proving that any voting rule that belongs to the important family of PMC rules is *not* robust in a similar sense.

Specifically, recall that under a PMC rule, when the weighted pairwise majority graph is acyclic, the output ranking is the topological ordering of the pairwise majority graph. We show that there exist profiles in which the pairwise majority graph is acyclic and all

[1] To see this, let

$$\begin{aligned}f(z,r) &\triangleq \frac{rz(z^{1/r}-1) - z^{1/r}(z-1)}{z} \\ &= (r-1)z^{1/r} + z^{1/r-1} - r.\end{aligned}$$

Taking the partial derivative with respect to z, we have

$$\frac{\partial}{\partial z} f(z,r) = \frac{(r-1)(z-1)z^{1/r-2}}{r},$$

which is clearly non-negative for $z, r \geq 1$. Also, $f(1,r) = 0$. So, we have shown that $f(z,r) \geq 0$ for all $z, r \geq 1$, which implies the claim.

edge weights are large, but, with high probability, the noisy profile also has an acyclic pairwise majority graph which induces a *different* ranking. This means that *any* PMC rule would return different rankings when applied to the true profile and the noisy profile.

Theorem 4.6. *For all $\delta > 0$, $\phi \in (0,1)$, and $m \in \mathbb{N}$ such that $m \geq 3$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, there exists a profile $\boldsymbol{\sigma}^\star \in \mathcal{L}^n$ such that $\Gamma(\boldsymbol{\sigma}^\star)$ is acyclic and all edges have weight $\Omega(n)$, but with probability at least $1-\delta$ $\Gamma(\boldsymbol{\sigma})$ is acyclic and there is a pair of alternatives on which the unique rankings induced by $\Gamma(\boldsymbol{\sigma}^\star)$ and $\Gamma(\boldsymbol{\sigma})$ disagree, where the probability is taken over the sampling of $\boldsymbol{\sigma}$.*

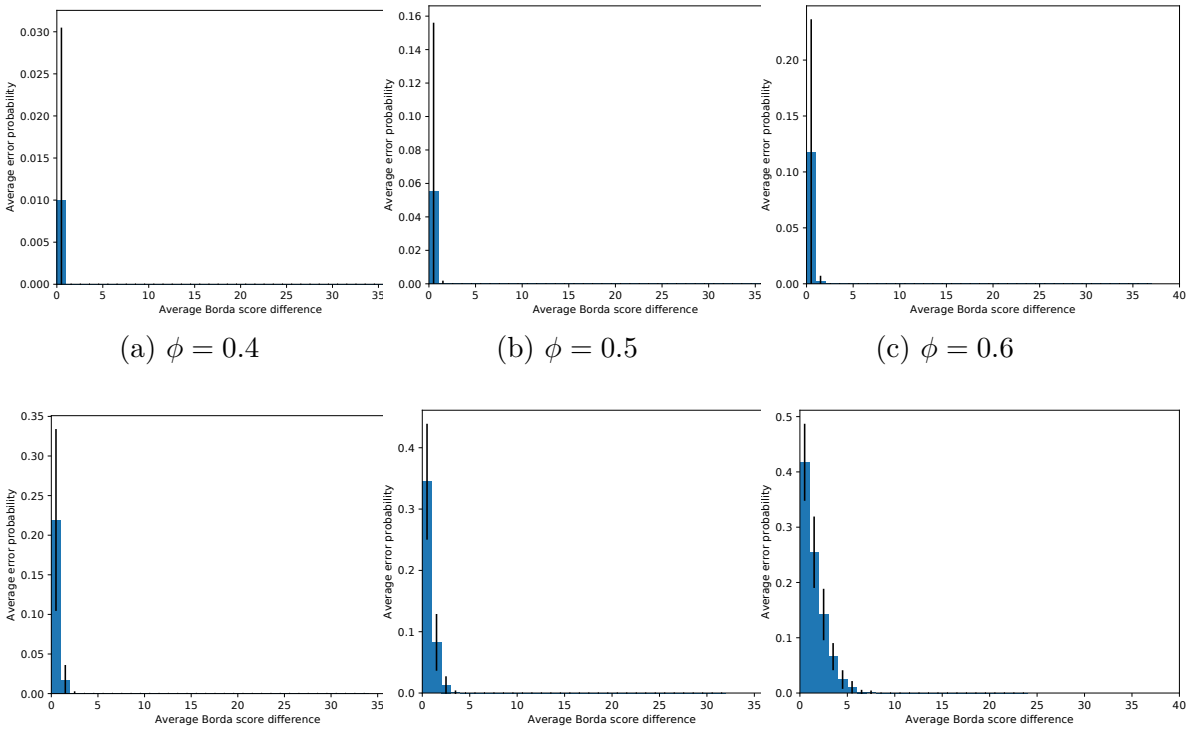
Proof Sketch. The proof of Theorem 4.6 is rather technical, and appears in the full version of the paper. In a nutshell, we construct a preference profile $\boldsymbol{\sigma}^\star$ with αn voters whose preferences are $x^\star \succ x_1 \succ \cdots$, and $(1-\alpha)n$ voters whose preferences are $x_1 \succ \cdots \succ x^\star$, for $\alpha > 1/2$. This profile induces a ranking where $x^\star$ is first and x_1 is second. However, it can be seen that, in the sampled profile $\boldsymbol{\sigma}$, many voters from the first group would flip $x^\star$ and x_1, leading to a majority who prefer x_1 to $x^\star$. Furthermore, we prove the nontrivial claim that $\Gamma(\boldsymbol{\sigma})$ is likely to be acyclic ('nontrivial' because it is unclear there would not be a cycle involving $x^\star$), which completes the argument. □

It is instructive to contrast our positive result, Theorem 4.3, with this negative result. On a very high level, the former result asserts that "if Borda count says that the gaps between alternatives are significant, then the alternatives will not flip under Borda count," whereas the latter says "even if a PMC rule says that the gaps between alternatives are very significant, some alternatives are likely to flip under that rule." On a technical level, a subtle difference is that Theorem 4.3 is stated for β and κ, whereas Theorem 4.6 is stated directly for ϕ. This actually *strengthens* the negative result, because a constant β and $\kappa \in \omega(1)$ lead to $\phi = 1 - o(1)$, i.e., very noisy distributions — and still the positive result of Theorem 4.3 holds. By contrast, the negative result of Theorem 4.6 is true *even* when ϕ is constant, i.e., for settings that are not nearly as noisy. That said, the two results are not directly comparable, as Borda count and PMC rules deal with very different notions of score or weight. Nevertheless, the take-home message is that the notion of score that defines Borda count is inherently more robust to random perturbations of the preference profile.

4.1.6 Empirical Results

In Section 4.1.4 we have established that Borda count is robust to prediction error. However, our positive theoretical result, Theorem 4.3, only provides asymptotic guarantees. In this section, we evaluate the performance of Borda count on profiles of size that is more representative of real-world instances. For our evaluation metric, we consider the probability of the rule flipping alternatives when aggregating noisy rankings against their difference in Borda score in the underlying true profile.

All of our code is open-source and can be found at https://github.com/akahng/VirtualDemocracy-ICML2019.



(a) $\phi = 0.4$ (b) $\phi = 0.5$ (c) $\phi = 0.6$

(d) $\phi = 0.7$ (e) $\phi = 0.8$ (f) $\phi = 0.9$

Figure 4.1: $p = 1$ mixture of Mallows, $n = 100$ voters, $m = 40$ alternatives

Methodology

Given n voters, m alternatives, a Mallows parameter $\phi \in (0, 1)$, and a probability $p \in [0, 1]$, we generate a true profile $\boldsymbol{\sigma}^\star = (\sigma_1^\star, \ldots, \sigma_n^\star)$ from a mixture of Mallows models. Specifically, each ranking is drawn with probability p from a Mallows model with base ranking $x_1 \succ x_2 \succ \cdots \succ x_m$ and parameter ϕ, and with probability $1 - p$ from a Mallows model with base ranking $x_m \succ x_{m-1} \succ \cdots \succ x_1$ and parameter ϕ.

We then repeatedly generate noisy profiles $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ where each σ_i is generated by a Mallows model centered at $\sigma_i^\star$ with parameter ϕ. For every pair of alternatives (x_i, x_j) such that $B(x_i, \boldsymbol{\sigma}^\star) > B(x_j, \boldsymbol{\sigma}^\star)$ — that is, x_i beat x_j when Borda count was applied to the true profile — we calculate the percentage of noisy profiles that flipped the order of x_i and x_j, i.e., those where $B(x_j, \boldsymbol{\sigma}) > B(x_i, \boldsymbol{\sigma})$. Based on the true difference in Borda scores $B(x_i, \boldsymbol{\sigma}^\star) - B(x_j, \boldsymbol{\sigma}^\star)$, we place this data point in the appropriate bucket, where the width of each bucket corresponds to an average Borda score difference of 1. This way we can relate the Borda score difference to the probability of making a pairwise prediction error. Note that starting from a mixture of 'opposite' ranking models allows us to vary the distribution over score differences in $\boldsymbol{\sigma}^\star$ by varying p.

Results

Throughout our experiments, we let $n = 100$, $m = 40$, $\phi \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$, and $p \in \{1, 0.7, 0.5\}$. Our results for $p = 1$, shown in Figure 4.1, plot the average probability of flipping the order of alternatives as a function of the difference in average Borda scores of the alternatives, where comparisons are bucketed by the difference in average Borda score. For $\phi \in \{0.1, 0.2, 0.3\}$, the observed probability of flipping any two alternatives, regardless of average Borda score difference, is 0; i.e., there are no mistakes.

At a high level, error rate decreases with true average Borda score distance in all experiments. Note that the maximum observed error rate increases with the Mallows parameter ϕ, which is intuitive because higher values of ϕ imply noisier (more uniformly random) rankings, so the probability of swapping alternatives should increase. However, for all values of ϕ and under all methods of generating profiles, the probability of making errors quickly decreases with average Borda score difference in the true profile.

Similar plots for $p = 0.7$ and $p = 0.5$ are included in the full version of the paper; these plots support the observation that the probability of making a mistake depends on the average Borda score difference, and not on the particular methods used to sample the underlying true profile.

4.2 Virtual Democracy in Practice: 412 Food Rescue

In concurrent work to the above, we apply the theoretical findings above to the real-world scenario of *food rescue* [155] in order to see how people perceive, participate in, and react to algorithmic governance. Through a collaboration with 412 Food Rescue, a nonprofit that matches food donations with needy recipients, we build a platform based on virtual democracy that suggests possible destinations for donations based on the learnt preferences of various stakeholders (donors, recipients, volunteers who deliver donations, and 412 Food Rescue employees).

Throughout the process, we solicited stakeholder participation to adjudicate the trade-offs involved in the algorithm's design, balancing equity and efficiency in donation distribution and managing the associated disparate impacts on different stakeholders. Over the course of a year, we had the stakeholders use the WeBuildAI framework to design the matching algorithm, and researched their experiences through a series of studies. The findings suggest that our framework successfully enabled participants to build models that they felt confident represented their own beliefs. In line with our original goals, participatory algorithm design also impacted both procedural fairness and distributive outcomes: participants trusted and perceived as fair the collectively-built algorithm, and developed an empathetic stance toward the organization. Compared to human dispatchers, the resulting algorithm improved equity in donation distribution without hurting efficiency when tested with historic data. Finally, we discovered that the individual model-building raised participants' algorithmic awareness and helped identify inconsistencies in human managers' decision-making in the organization, and that the design of the individual model-building method may influence the elicited beliefs.

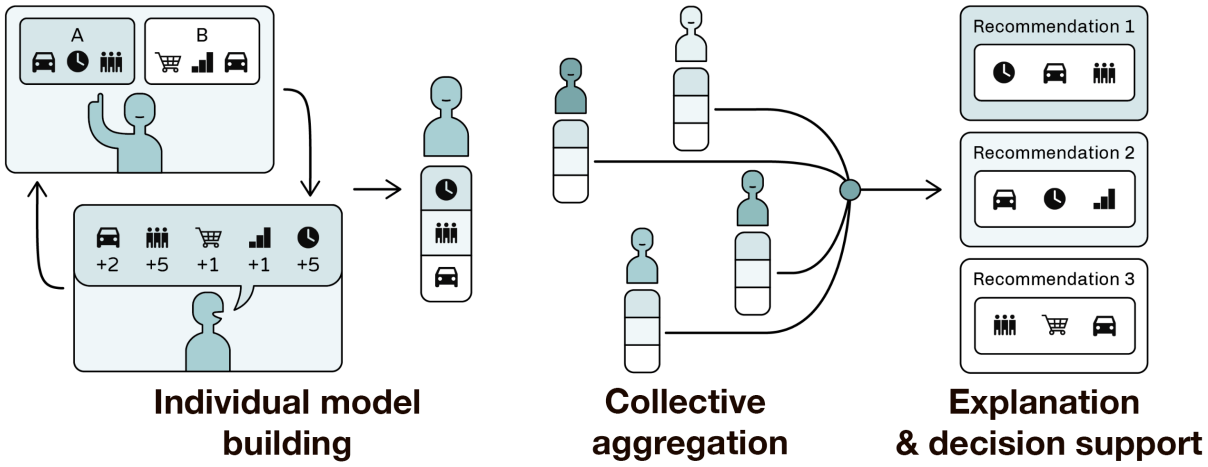


Figure 4.2: The WeBuildAI framework allows people to participate in designing algorithmic governance policy. A key aspect of this framework is that individuals create computational models that embody their beliefs on the algorithmic policy in question and vote on the individual's behalf.

4.2.1 Introduction

Emerging work has called for greater involvement of stakeholders and affected communities in the development of algorithmic systems. These projects have sought to understand the public's expectations of moral behaviors [42; 181; 165] and varying concepts of fairness [158; 157; 233], as well as stakeholders' needs and requirements [241; 7] around Artificial Intelligence (AI) systems; yet translating the results into actual algorithms is difficult, as these studies have often relied on hypothetical moral dilemmas or collected qualitative expectations and opinions that developers and designers need to interpret in order to build the algorithm.

Our vision is to empower people to design algorithmic governance mechanisms for their own communities. We argue that this participatory algorithm design process is a step toward creating algorithmic governance that is effective yet also moral. In traditional participatory governance, stakeholder participation in policy-making improves the legitimacy of a governing institution in a democratic society [111; 113].[2] Participating in service creation has also been shown to increase trust and satisfaction, thereby increasing motivation to use the services [28]. In addition, participation can increase effectiveness. For certain problems, people themselves know the most about their unique needs and problems [111; 171]; participation can help policymakers and platform developers leverage this knowledge pool. Finally, stakeholder participation can help operationalize moral values

[2]By "legitimacy," we refer to Weber's notion that "persons or systems exercising authority are lent prestige" [230]. A policy or action is legitimate when constituents have good reason to support it [112]. In western democratic societies, the legitimacy of governing systems is often established through the public practice of democracy that seeks to earn the consent of the governed by soliciting their input, often through elections, to influence government and public policy.

and their associated trade-offs, such as fairness and efficiency [111]. Even people who agree wholeheartedly on certain high-level moral principles tend to disagree on the specific implementations of those values in algorithms—the objectives, metrics, thresholds, and trade-offs that need to be explicitly codified rather than left up to human judgment.

Enabling stakeholder participation in algorithmic governance raises several fundamental research questions. First, what socio-technical methods will effectively elicit individual and collective beliefs about policies and translate them into computational algorithms? Second, how should the resulting algorithms be explained so that participants understand their roles and administrators can make decisions using the algorithms? How does participation influence participants' perceptions of and interactions with algorithmic governance? Finally, how does the resulting collectively-built algorithm perform?

Our Approach and Contributions

In order to address these research questions, we propose a framework called WeBuildAI that enables people to collectively design an algorithmic policy for their own community (Figure 4.2).[3] By "design," we mean having the community members and stakeholder themselves define the optimization goals of the algorithms, the benefits and costs of the algorithmic governance decisions, and the value principles that they believe their community should embody and operate on. The key aspect of this framework is that individuals create computational models that embody their beliefs on the algorithmic policy in question,[4] and then these models vote on their individuals' behalf. This works like a group of people making a decision together: computational models of each individual's decision-making process make a collective choice for each policy decision. The individual models rank possible alternatives, and the individual rankings are then aggregated via the classic Borda rule. The resulting algorithmic recommendations are explained to support administrative decision-makers.

As a case study, we applied this framework to develop a matching algorithm that distributes donations through collaboration with 412 Food Rescue, a nonprofit that provides an on-demand donation transportation service with volunteer support. The algorithm matches donors with recipient organizations, determining who receives donations and how far volunteers need to drive to deliver donations. We solicited stakeholder participation to adjudicate the tradeoffs involved in the algorithm's design, balancing equity and efficiency in donation distribution and managing the associated disparate impacts on different stakeholders. Over the course of a year, we had the stakeholders—donors, recipient organizations, volunteers, and the 412 Food Rescue staff—use the WeBuildAI framework to design the matching algorithm, and researched their experiences through a series of studies. The findings suggest that our framework successfully enabled participants to build models that they felt confident represented their own beliefs. In line with our original goals, participatory algorithm design also impacted both procedural fairness and distributive outcomes:

[3]We define "community" according to the Merriam-Webster dictionary as a "unified body of individuals," particularly a group linked by a common interest or policy.

[4]By "belief," we mean a "positional attitude," in other words, "the mental state of having some attitude, stance, take, or opinion about a proposition" [205].

participants trusted and perceived as fair the collectively-built algorithm, and developed an empathetic stance toward the organization. Compared to human dispatchers, the resulting algorithm improved equity in donation distribution without hurting efficiency when tested with historic data. Finally, we discovered that the individual model-building process raised participants' algorithmic awareness and helped identify inconsistencies in human managers' decision-making in the organization, and that the design of the individual model-building method may influence the elicited beliefs.

Our paper makes three contributions. First, we offer a framework and methods that enable participatory algorithm design, contributing to emerging research on human-centered algorithms and participatory design for technology. Second, through a case study with stakeholders in a real-world nonprofit, we demonstrate the feasibility, potential, and challenges of community involvement in algorithm design. Finally, our work provides insights on the effects of procedurally-fair algorithms that can further understanding of algorithmic fairness.

4.2.2 Governing Algorithm Design and Participation

Our framework draws from social choice and participatory governance literature to enable participatory algorithm design. In this section, we first lay out normative choices in algorithm design. We then review and identify gaps in participatory design literature and emerging work to introduce stakeholder participation in algorithm design. Finally, we discuss how we leveraged participatory governance literature to inform our framework design.

Normative Choices in Algorithm Design

In line with Aneesh's definition of "algocracy," when "authority becomes embedded in the technology itself" [8] rather than traditional forms of governance, and Danaher's elaborations, we define "governing algorithms" as algorithms that "nudge, bias, guide, provoke, control, manipulate and constrain human behaviour" [84]. All algorithm design choices cannot be addressed by a purely technical approach [130; 232]; particularly in governing algorithms, some design choices require a normative decision, as they affect multiple stakeholders and need to codify critical social values and associated tradeoffs. We describe three such design choices below.

First, increasingly more research has investigated computational techniques to encode social and moral values in algorithms, yet many still rely on fundamental measures and algorithmic "objective functions" that humans must define. Defining these terms is complex. Fairness, for example, broadly defined as treating everyone equally, has multiple definitions and theoretical roots. In prior work, fairness has been defined as equitable distributive outcomes and just, unbiased, non-discriminatory decision-making processes [32]. Fairness is an important value in governing algorithms as algorithms can perpetuate unfair treatment of different populations or stakeholders [84; 239; 109]. Emerging work develops computationally fair algorithms [48; 103], yet applying these techniques to real-world settings still requires human judgment. For example, individual fairness, or treating similar

individuals similarly, requires a definition for "similar individuals" [93].

Second, multiple social values and objectives cannot be satisfied to the same degree, which necessitates making tradeoff decisions. For example, all fairness principles cannot be guaranteed simultaneously [71; 145], so a human decision-maker must determine which fairness definitions an algorithm should use. Similarly, operational efficiency and fairness are often competing values in modern capitalist democracies [183]. Algorithms that aim to achieve both require human judgments about how to balance the two, because there is no fundamental "right" balance and one cannot be determined purely through optimization [30].

Finally, these definitions and values are context-dependent. Recent empirical work on perceptions of "fair" algorithms suggests that different social groups believe in different fairness principles, and even algorithms that embody a fairness principle may not be perceived as fair if the implemented principle is not in accordance with the affected group's beliefs [157]. For example, some groups in the study preferred random allocation that treated everyone equally, and did not consider individual differences to be relevant to task allocation. Other groups desired equity-based allocation, in which the tasks are allocated to satisfy everyone's preferences to a similar degree. Some other groups wanted to consider both preferences and task completion time as fairness factors, so that people work for a similar amount of time and their preferences are satisfied similarly. These findings suggest people believe in epistemically different fairness principles or desire varying ways of operationalizing fairness principles. Real-world examples also suggest that algorithmic software will fail to be adopted if it uses features or objective functions that do not fit the context of the affected community. For example, a "fair" algorithmic school start time scheduling software in Boston received pushback from the community and was ultimately not adopted, because the policymakers' and developers' efforts to decrease racial disparities did not consider important values and constraints of the stakeholders [231]. This body of work suggests that fairness principles must be context-specific, and that algorithmic systems should embody fairness notions derived from the community.

These normative choices in algorithm design are fundamental; how do we understand and formalize context-dependent values? Who should determine these important values and tradeoffs in governing algorithms, and how? Our approach to these questions is inspired by the long line of research on participatory design.

Gaps in Participatory Design and Human-Centered Research on AI

Participatory design originated in Scandinavia in the 1960s with the intention of involving workers in planning job design and work environments. Participatory design was subsequently adopted in the fields of human-computer interaction and engineering [227; 179], and researchers and designers have included "end-users" in design activities for computing systems in a wide range of domains such as workspaces [39], healthcare [23], and robots [90]. In participatory design, the researchers and users of a technology share power and control in determining its technological future [227; 179; 43], so that the stakeholders or populations that the technology will influence have a say in the resulting design, and the technology can better reflect their needs, values, and concerns. More recently, several scholars have

argued that one needs to be more cognizant of the agency and influence of the researchers and designers in "configuring the process participation," and more critical analysis must be done in terms of who initiates participation and who benefits from it [227].

While participatory design has been applied to diverse forms of technology, the research on involving users in the process of designing algorithms or AI is still in its infancy. Rahwan [194] argues for "society-in-the-loop," which stresses the importance of creating infrastructure and tools to involve societal opinions in the creation of AI. Emerging work has also started to explore societal expectations of algorithmic systems such as self-driving cars [42; 181] and robots [165]. This line of work offers an understanding of the public's general moral values around AI through thought experiments, but it is difficult to translate them into actual AI technology as they have often been done in hypothetical moral dilemma situations.

Emerging work seeks to understand participants' values with regard to the fairness of actual AI products, with the goal of representing these values in the final AI design. For example, Zhu et al. proposed Value Sensitive Algorithm Design [241], a five-step design process that starts with understanding the stakeholders and ends with evaluating algorithms' acceptance, accuracy, and impacts, in the context of Wikipedia bots. In this process, designers interpret stakeholder opinions and make the necessary trade-off decisions. Alvarado and Waern organized a participatory workshop for social media curation algorithms in which people were asked to imagine ideal "algorithmic experiences" [7]. Lee et al. and Woodruff et al. conducted interview and workshop studies on what people think "fair" algorithms are in the contexts of donation allocation [158] and online ads [233]. Other scholars systematically investigated perceived fairness of algorithmic decisions in hiring [156], recidivism [91], child welfare services [55], and resource allocation such as task assignment [157] and goods division [159].

To our knowledge, however, little work has sought to formalize subjective concepts of fairness. Furthermore, while these studies provide us with a better understanding of general public and user perceptions of justice and fairness, they do not close the loop on algorithmic developments that respond to these concerns. Our work proposes a method for directly involving end-users or stakeholders of algorithmic services in determining how the algorithms should make decisions. One aspect that differentiates our work is that we offer a tool through which people without algorithmic knowledge can directly specify or "sketch" [59] how they would like the algorithm to behave; we couple this with a method for aggregating different stakeholders' points of view.

Participatory Governance

Our framework draws on the literature on participatory governance. A first step in participatory governance is to determine what governance issues participants will consider and how participation will influence final policy outcomes. User groups, or mini-publics [111], can be configured as open forums where people express their opinions on policies; focus groups can be arranged for specific purposes such as providing advice or deriving design requirements. In full participatory democratic governance, citizen voices are directly incorporated into the determination of the policy agenda. Our framework focuses on this last

form: direct participation in *designing* algorithmic governance. By "direct participation," we mean that people are able to specify "objective functions" and behaviors in order to create desirable algorithmic policies. This direct approach can minimize potential errors and biases that occur when codifying policy ideas into computational algorithms, which has been highlighted as a risk in algorithmic governance [144].

A key aspect of governance is collective decision-making. Our framework builds on social choice theory. Social choice theory involves collectively aggregating people's preferences and opinions by creating quantitative definitions of individuals' opinions, utilities, or welfare and then aggregating them according to certain desirable qualities [206]. Voting is one of the most common aggregation methods, in which individuals choose a top choice or rank alternatives, and the alternatives with the most support are selected. Social choice theory is typically built on an axiomatic approach, formally defining desirable axiomatic qualities and studying voting rules that satisfy them. Indeed, the Borda voting rule satisfies a number of such properties, including monotonicity (pushing an alternative upwards in the votes should not hurt it) and consistency (if two electorates elect the same alternative, their union does too). We adopted a social choice approach specifically because our ultimate design outcome is an algorithm. While we know "quantification" has limitations in capturing nuances in the real world, quantification is an inevitable step in algorithms as they need quantitative inputs. Social choice theory provides a framework for formally reasoning about collective decisions at scale.

Implementing participation in algorithmic governance requires addressing the following challenges. First, how can we enable individuals to form beliefs about policies through deliberation and express these beliefs in a format that the algorithm can implement? Second, how do we consolidate individuals' models? Finally, how do we explain the final decisions so that people can understand the influence of their participation on the resulting policy, and administrators can use the collectively-built governing algorithm? In the next section, we describe our framework and how it addresses these challenges.

4.2.3 The WeBuildAI Framework

Here we lay out the basic building blocks of the WeBuildAI framework, which enables participation in building algorithmic governance through a novel combination of individual belief learning, voting, and explanation. Our framework design draws on the field of political theory, which investigates collective decision-making and effective citizen participation in governance.

The key idea of the framework is to build a computational model representing each individual stakeholder, and to have those models vote on their individuals' behalf. This works like a group of people making a decision together: computational models of each individual's decision-making process make a collective choice for each policy decision.

Individual Belief Model Building

Building a model that embodies an individual's beliefs on policy gives rise to three challenges. First, people need to determine what information, or features, should be used in

algorithms. Second, the individual needs to form a stable policy that applies across a broad spectrum of situations. This process requires people to examine their judgments in different contexts until they reach an acceptable coherence among their beliefs, or reflective equilibrium [85; 196]. Third, people without expertise in algorithms need to be able to express their beliefs in terms of an algorithmic model. We address these challenges by deriving a set of features from people's inputs, and then using both bottom-up machine learning training and top-down explicit rule making.

Feature Selection: The first step is to determine features that people believe should be used by the algorithm to make decisions. People's opinions can be solicited through interviews or surveys. The derived set of features will be used to construct pairwise comparisons between alternatives, or allow people to directly specify weights for each of the features.

Model Building: We use both machine learning and explicit rule specification. By allowing people to use both types of models iteratively, we seek to support deliberation. By building a machine learning model via pairwise comparisons, people can develop a policy that works across various contexts; by explicitly specifying a policy that they have been implicitly forming, participants can consolidate and externalize their beliefs; then by answering new pairwise comparisons questions, they can evaluate whether the rules they have in mind work consistently across contexts.

- *Machine Learning Model.* To train an algorithm that reflects people's decision criteria, the machine learning method uses pairwise comparisons between a pair of alternatives that vary along the features derived from the previous step. Pairwise comparisons have been used to encourage moral deliberation and reach a reflective equilibrium in determining fairness principles [196], and have been used as a way to understand people's judgments in social and moral dilemmas in psychology and economics [83]. This method allows people to become familiar with different contexts, and develop and refine their beliefs.

  We utilize random utility models, which are commonly used in social choice settings to capture choices between discrete objects [168]. In a random utility model, each participant has a true "utility" distribution for each object, and, when asked to compare two potential objects, she samples a value from each distribution. For each participant i, we learn a single vector β_i such that the mode utility of each potential decision x is $\mu_i(x) = \beta_i^T x$. We then learn the relevant β_i vectors via standard gradient descent techniques using Normal loss.

- *Explicit Rule Model.* In this method, participants directly specify their principles and decision criteria as used in expert system design [86]. Human-interpretable algorithmic models [240] such as decision trees, rule-based systems, and scoring models have been used to allow people to specify desired algorithmic behaviors. This approach allows people to have full control over the rules and to specify exceptional cases or constraints. Specifically, for each of the features, participants can specify scores to express how much the algorithm should weight different features.

Model Selection: Once people build their models using the two methods, we visualize the models and show example decisions that each model has made so that people can

understand each model and select the one that best reflects their beliefs.

Collective Aggregation

Once participants have built their models, the next challenge is to construct a collective rule that consolidates the individual models. We address this challenge by leveraging social choice, one of the main theories of collective decision-making, which aggregates peoples' opinions according to certain desirable qualities [206]. Voting is one of the most common aggregation methods. In voting, individuals can specify a top choice or rank alternatives, and the alternatives with the most support are selected. In our framework, we use the Borda voting method due to its relative simplicity and robust theoretical guarantees in the face of noisy estimates of true preferences, as shown in a paper by some of the authors [139].

The Borda rule is defined as follows. Given a set of voters and a set of m potential allocations, where each voter provides a complete ranking over all allocations, each voter awards $m-k$ points to the allocation in position k, and the Borda score of each allocation is the sum of the scores awarded to that allocation in the opinions of all voters. Then, in order to obtain the final ranking, allocations are ranked by non-increasing score. For example, consider the setting with two voters and three allocations, a, b, and c. Voter 1 believes that $a \succ b \succ c$ and voter 2 believes that $b \succ c \succ a$, where $x \succ y$ means that x is better than y. The Borda score of allocation a is $2+0=2$, the Borda score of allocation b is $1+2=3$, and the Borda score of allocation c is $0+1=1$. Therefore, the final Borda ranking is $b \succ a \succ c$.

Once stakeholders create their models, the models are embedded in the AI system to represent the stakeholders; for each algorithmic decision task, each individual model ranks all alternatives, and the ranked lists of all participants are aggregated using the Borda rule to generate the final ranked list.

Algorithm Explanation and Human Decision Support

Finally, the ranked recommendations must be explained to stakeholders to communicate how their participation has influenced the final policy and supported operational decision-making. Communicating the impact of participation can reward people for their effort and encourage them to further monitor how the policy unfolds over time. While the importance of communication is highlighted in the literature, it has been recognized as one of the components of human governance least likely to be enacted [111]. Algorithmic governance offers new opportunities in this regard because the aggregation of individual models and resulting policy operations are documented. A new challenge is how to explain collectively-built algorithmic decisions, an area in which little prior research has been done. We address this challenge by displaying each recommended option's Borda score, its average ranking per stakeholder group, and its "standout" features in order to support the administrators enacting the algorithmic policies.

4.2.4 Case study: Matching algorithm for donation allocation

We applied the WeBuildAI framework in the context of on-demand donation matching in collaboration with 412 Food Rescue [1].

Goals of Participation in Matching Algorithm Design

Organizational Context: 412 Food Rescue is a non-profit that provides a "food rescue" service: donor organizations such as grocery and retail stores with extra expiring food call 412 Food Rescue, and then 412 Food Rescue matches the incoming donations to non-profit recipient organizations. Once the matching decision is made, they post this "rescue" on their app so that volunteers can sign up to transport the donations to the recipient organizations. The service's success depends on the participation of all stakeholders–a continuous stream of donations, recipient organizations' willingness to accept the donations, volunteers' efforts to transport donations, and 412 Food Rescue's operational support and monitoring. The organization has grown successfully for the past few years. They have rescued over three million pounds of food and are expanding their model into food rescue organizations in four other cities, including San Francisco and Philadelphia. The donation allocation policy is at the core of their service operation; while each individual decision may seem inconsequential, over time, the accumulated decisions impact the welfare of the recipients, the type of work that volunteers can sign up for, and the carbon footprint of the rescues.

412 Food Rescue wanted to introduce an algorithmic donation allocation system for two reasons. First, they currently have a few employees per day, known as dispatchers, manually allocating all donations that come in that day. On a busy day, each dispatcher has to manage over 100 donations, which is too many, so the organization wants to reduce dispatcher workload. Second, 412 Food Rescue wishes to improve equity in their donation distribution. The current donation distribution is quite skewed, with 20% of recipient organizations receiving 70% of donations (Figure 4.6a), because allocation decisions are often made for convenience.

Equity-Efficiency Tradeoff and Stakeholder Motivation: In designing this matching algorithm, we used participation to determine the tradeoff between equity and efficiency. In this context, we define "equity" as giving donations to recipients with greater need and "efficiency" in terms of the distance each donation travels from donor to recipient. Balancing equity and efficiency is challenging as this design choice has different impacts on different stakeholders. For example, if the matching algorithm prioritizes efficiency and gives donations to recipients closest to donors, volunteers will benefit from shorter driving times, but the donation distribution may be skewed and recipients in wealthier areas may receive more donations, as donors are often located in wealthier areas. On the other hand, if the matching algorithm prioritizes equity, recipients with greater need may receive more donations, but this may increase the distance that volunteers need to drive, as well as the effort 412 Food Rescue must spend in recruiting the volunteers. Finding a collective solution to this problem is critical to the success of the service, because all stakeholders will be more motivated to continue participating in the service if they feel their needs are

respected.

Stakeholder Participants

Volunteer-Based Participation: We used our framework to build the matching algorithm collectively with 412 Food Rescue's stakeholders. One of the important considerations in participatory governance is determining who participates. A widely-used and accepted method is volunteer-based participation [111], which accepts input from people who will be governed by the system and who choose to participate. Many democratic decisions, including elections, participatory forums, and civic engagement, are volunteer-based. In our application, we used a volunteer-based method with stakeholders directly influenced by the governing algorithm. As our first evaluation of the framework, we chose to work with a small focus group of stakeholders who volunteered to participate in order to get in-depth feedback.

Participation Recruiting and Information: Our research took place over a period of one year. We solicited stakeholder participation to determine how the matching algorithm should weight the factors used to recommend recipient organizations. The stakeholders included donor organizations, recipient organizations, volunteers, and 412 Food Rescue staff. We included the governing entity as a stakeholder because they have a holistic viewpoint on logistics: how the donation is collected, handled and delivered to the recipient organization. The mission of the organization is to reduce food waste and serve food-insecure populations, which overlaps with other stakeholders' goals.

The entire staff that oversees donation matching at the organization participated in the study. Recipients, volunteers, and donors were recruited through an email that 412 Food Rescue staff sent out to their contact list.[5] We replied to inquiry emails in the order in which they arrived, and collected information about respondents' experience with 412 Food Rescue and organizational characteristics in order to ensure diversity. We limited the number of participants from each stakeholder group to 5–8 people, which resulted in an initial group of 23 participants (including V4a and V4b, who participated together) with varying organizational involvement (Table 4.1). Fifteen were female (nine males) and everyone, except one Asian, was white.[6] Sixteen participants answered our optional demographic survey. Two attended at least some college and 14 had attained at least a bachelor's degree. The average age was 48 (Median=50 (SD=16.4); Min-Max:30-70). The

[5]We did not include recipient organizations' clients for several reasons. First, we asked about service operation in this study. Our previous interviews with clients [158] suggest that recipient organizations do not display where their food comes from at the time of distribution. Thus clients generally have no experience with or knowledge of the food rescue process and lack the hands-on experience required to consider disparate impacts on different stakeholders. Because of this, we represented clients' interests via feedback from the staff of recipient organizations who know and serve client populations. Additionally, 412 Food Rescue did not have recipient client contact information for privacy reasons. In the discussion section, we explain how we will seek out a way to expand participation to include groups, including clients, that are not directly involved in the food rescue process.

[6]Our participants were mostly white, which reflects the population of volunteers and non-profit staff in Pittsburgh. This is the result of a volunteer-based method [111]. In our next step, we will implement targeted recruiting of minority populations.

average household income was $65,700 (Median=$62,500 (SD=$39,560); Min-Max:$25,000-$175,000).

Research Process Overview

Our research goal was threefold: we sought to apply the framework to build a matching algorithm, evaluate the usability and efficacy of the framework, and understand the effects of participation. To this end, we used our framework to allow participants to build their own individual models. We conducted think-alouds throughout the data collection procedure to understand participants' thinking processes. We also showed participants the method and results from each step of our framework—for example, how we aggregate individual models and explain the decisions—and conducted interviews to study their understanding and responses to the method. Once participants completed all stages of the framework, we conducted interviews to understand participants' attitudes toward the resulting algorithm and the governing organization, 412 Food Rescue.

Overall, our research resulted in 4–5 individual sessions for each participant and a workshop over the course of a year. Because of the extended nature of the community engagement, 15 participants completed all the individual study sessions, while 8 could participate only in the first couple of sessions due to changes in their schedules or jobs (Table 4.1). Because participants provided research data through think-alouds and interviews in addition to their input for the matching algorithm, we offered them $10 per hour.

Researcher Stance

Our research team included people with diverse backgrounds in human-computer interaction, artificial intelligence, theoretical computer science, information systems, decision science, ethics and design, affiliated with Carnegie Mellon University and University of Texas at Austin. We had a constructive design stance and sought to bring about positive change through the creation of artifacts or systems. Two researchers have conducted research with 412 Food Rescue in the past and one researcher regularly volunteered in homeless shelters and food pantries in Pittsburgh. This relationship and familiarity with public assistance work helped us gain access to the research site.

Analysis

We report how we analyzed qualitative data from all sessions in this section to avoid repetition. All interviews were audio-recorded and transcribed, and researchers took notes throughout the think-alouds and workshop. The data was analyzed following a qualitative data analysis method [185; 82]. Two researchers read all of the notes and interview transcripts and conducted open coding of the transcripts at the sentence or paragraph level on Dedoose.[7] The rest of the research team met every week to discuss emerging themes and organize them into higher levels. As we progressed in our analysis, we drew from the literature on participatory governance [111] and procedural fairness [160; 159] to see

[7] https://www.dedoose.com

whether the themes that we observed were consistent with or different from previous work. After all sessions were completed, we revisited the themes from each session and further consolidated them into the final themes we present in this paper. In Section 4.2.8, we report the number of participants associated with different themes in order to note the relative frequency of different opinions and behaviors in our study. However, as a qualitative study with a small sample size, we note that this should not be taken as an exact weight of whether one opinion is more significant or representative.

4.2.5 Individual Belief Model Building

The first step in building individual belief models is to determine which factors (or features) are relevant and important; we derived these factors from the authors' previous study [158] that examined the 412 Food Rescue stakeholders' concepts of fair donation allocation. A factor that was mentioned most frequently is the distance between donors and recipient organizations. Participants mentioned various other factors that represent the needs of recipient organizations, such as the income level of recipient clients, the food access levels of their neighborhoods, and the size of the recipient organization. Additional factors that were also deemed important were the distributional capabilities of recipient organizations, i.e., how fast they can distribute to their clients, and the temporal regularity in incoming donations. From the factors that participants mentioned, we selected the ones that came up most frequently and had reliable data sources.[8] The selected factors capture transportation efficiency, recipient needs, and temporal allocation patterns (Table 4.2). For example, poverty rate is an indicator of recipients' needs; distance between recipients and donors is a metric of efficiency; and when each recipient last received a donation is a measure of allocation patterns over time.

We conducted three sessions to develop a model to represent each individual in the final algorithm. Participants first completed pairwise comparisons (Figure 4.3a, Session 1) to train algorithms using machine learning. Participants who wanted to elaborate on their models participated in the explicit rule specification session (Figure 4.3b, Session 2). If their belief changed after Session 2, they provided a new set of pairwise comparisons to retrain the algorithm. Participants were later asked to choose one of the two models that represented their beliefs more accurately (Figure 4.4, Session 3).

Machine Learning Model (Session 1)

Pairwise Comparison Scenarios: We developed a web application to generate two potential recipients at random according to the factors (Table 4.2), and asked people to choose which recipient should receive the donation (Figure 4.3a).[9] All participants

[8] We did not use organization types (e.g., shelters and food pantries) or addresses because these aspects may communicate the racial, gender, or age characteristics of recipients and elicit biased answers based on inaccurate assumptions or discrimination.

[9] Improbable combinations of income and poverty (e.g., very high income coupled with very high poverty) were excluded according to the census data. All factors were explained in a separate page that participants could refer to.

completed a one-hour, in-person session where they answered 40-50 randomly generated questions. They were asked to think aloud as they made their decisions, and sessions concluded with a short, semi-structured interview that asked them for feedback about their thought process and their views of algorithms in general. During the research process, the link to the web application was sent to the participants who wished to update their models on their own. In fact, 13 participants chose to answer an additional 50–100 questions after Session 2 to retrain their machine learning models.

Learning Individual Models:

In order to learn individual models, we utilize random utility models, which are commonly used in social choice settings to capture choices between discrete objects [168]. This fits our setting, in which participants evaluate pairwise comparisons between potential recipients. In order to apply random utility models to our setting, we use the Thurstone-Mosteller (TM) model [219; 176], a canonical random utility model from the literature. In this model, the distribution of each alternative's observed utility is drawn from a Normal distribution centered around a mode utility. Furthermore, as in work by Noothigattu et al. [181], we assume that each participant's mode utility for every potential match is a linear function of the match's feature vector. Therefore, for each participant i, we learn a single vector β_i such that the mode utility of each potential match x is $\mu_i(x) = \beta_i^T x$. We then learn the relevant β_i vectors via standard gradient descent techniques using Normal loss.[10] We also experimented with more complicated techniques for learning utility models, including neural networks, SVMs, and decision trees, but linear regression yielded the best accuracy and is the simplest to explain.

Explicit Rule Model (Session 2)

To allow participants to explicitly specify matching rules, we asked them to create a scoring model using the same factors shown in Table 4.2. We used scoring models because they capture the "balancing" of factors that people identified when answering the pairwise questions.[11] We asked participants to create rules to score potential recipients so that recipients with the highest scores would be recommended. Participants assigned values to different features using printed-out factors and notes (Figure 4.3b). We did not restrict the range of scores but used 0-30 in the examples in our instruction. Once participants created their models, they tested how their scoring rule worked with 3-5 pairwise comparisons generated from our web application, and adjusted their models in response. At the end of the session, we conducted a semi-structured interview in which we asked participants to explain the reasoning behind their scoring rules, and describe their overall experience. The sessions took about one hour. Two participants wanted to further adjust their models and scheduled 30 minute follow-up sessions to communicate their changes.

[10] For participants who consider donation type, we learn two machine learning models, one for common donations and one for uncommon donations.

[11] We also experimented with manually-created decision trees, but the models quickly became prohibitively convoluted.

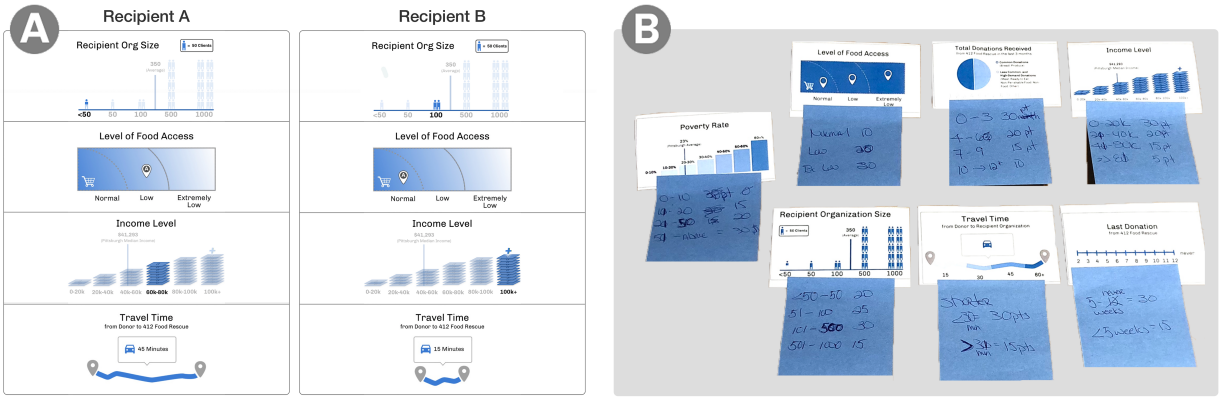


Figure 4.3: Two methods of individual model building were used in our study: (a) a machine learning model that participants trained through pairwise comparisons, and (b) an explicit rule model that participants specified by assigning scores to each factor involved in algorithmic decision-making.

Machine Learning versus Explicit-Rule Models (Session 3)

We asked participants to compare and choose between their machine learning and explicit-rule models, selecting one that best represented their beliefs. To evaluate the performance of the models on fresh data that was not used to train the algorithm, we asked participants to answer a new set of 50 pairwise comparisons[12] before the study session and used them to test how well each model predicted the participants' answers.

To explain the models, we represented them both in graph form that showed the assigned scores along with the input range for each feature (Figure 4.4). In order to prevent any potential bias in favor of a particular method, we anonymized the models ("Model X" or "Model Y"), normalized the two models' parameters (beta values) and scoring rubric using the maximum assigned score in each model, and introduced both models as objects of their creation. In a 60-90 minute session, a researcher walked through the model graphs with the participants, showed the prediction agreement scores between the two models, and presented all pairwise comparison cases in which the two models disagreed with each other or disagreed with participants' choices. For each case, the researcher illustrated on paper how the two models assigned scores to each alternative.

At the completion of these three activities, participants were asked to choose which model they felt best represented their thinking. The models were only identified after their choice was made. A semi-structured interview was conducted at the end asking about their experience and reasons for their final model choice. We also analyzed individual models in terms of the beta values assigned to each factor, or the highest score assigned to each factor. As all the feature inputs were normalized (from 0 to 1), we used the strength of the beta values to rank the importance of factors for each individual.

[12]We used the same set of comparisons for all participants for consistency.

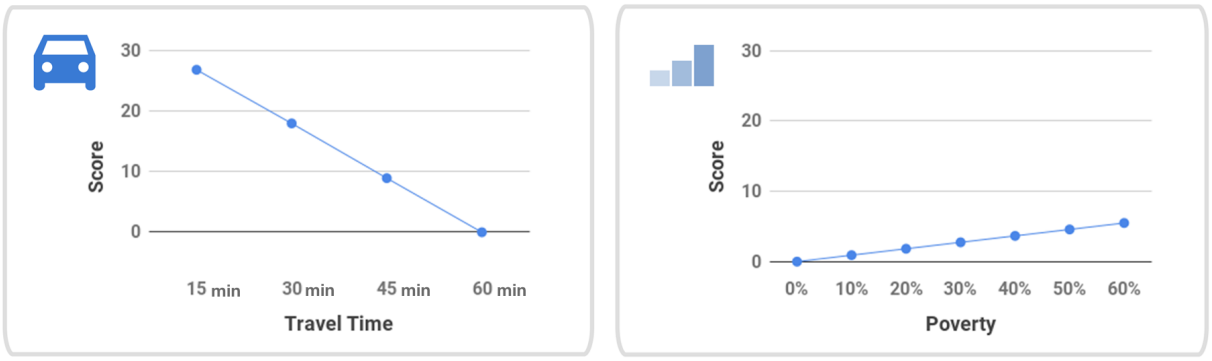


Figure 4.4: Model explanations. Both machine learning and explicit-rule models were represented by graphs that assigned scores according to the varying levels of input features.

Final Individual Models

In total, we trained 23 machine learning models[13] and obtained 15 explicit-rule models. Of the 15 participants who completed all studies and were asked to choose models that better represented their belief, 10 of them chose the machine learning models trained on their pairwise comparisons; the other five chose the models that they explicitly specified.

The machine learning models had higher overall agreement with participant's survey answers than the explicit rule models when tested on 50 new pairwise comparisons provided by each participant, as seen in Table 4.3. However, as our sample size is small, we do not aim to make general claims on which model has better accuracy. In addition, for many, the machine learning model was the one they had built last and therefore reflected their current thinking at the time of comparison; we further elaborate on this in Section 8.1. We also note that we did not observe any differences in participants' perceived accountability in the creation of these models. Both models took an equal amount of participants' time and attention, and participants told us that they felt responsible when making choices and assigning scores.

4.2.6 Collective aggregation

Our framework uses a voting method to aggregate individuals' beliefs. When presented with a new donation decision, each individual's model generates a complete ranking of all possible recipient organizations. The Borda rule aggregates these rankings to derive a consensus ranking and suggest recommendations. We conducted a workshop and interviews to understand participants' perceptions of this method.

[13]We note that there were 8 participants who participated in the first stage of the study but not subsequent stages (Table 4.1). The average cross-validation accuracy of their linear models was quite high, at 0.819.

Method (Workshop)

In an early stage of our research, we conducted a workshop in order to gauge participants' perceptions of the Borda aggregation method and determine the method's appropriateness from a social perspective. Five participants (Table 4.1) who had built their individual models at that time attended the one-hour workshop. All stakeholder groups were represented. We prepared a handout that showed individuals' and stakeholders' average models at the time, and a diagram that explained how the Borda rule worked. The description of the Borda rule given to participants was: "Individuals rank options according to belief. Each option receives a number of points determined by its ranking, with higher-ranked options receiving more points. The points are added up, and the winner is whichever option has the greatest number of points." The words "democratic" or "equal" were not used to avoid potential biases. We facilitated a discussion of how individuals reacted to the similarities and differences between their model and other groups' models, and had individuals discuss whether all the stakeholders' opinions should be weighted equally or differently. For participants who joined our research after this workshop, we asked the same questions about the Borda rule and stakeholder opinion weight in the interview in Session 4.

Varying Stakeholders' Voting Influence

All participants but one believed that the weight given to different stakeholders in the final algorithm should depend on their roles. On average, participants assigned 46% of the voting power to 412 Food Rescue, 24% to recipient organizations, 19% to volunteers, and 11% to donors.[14] Nearly all participants weighted 412 Food Rescue staff as the highest group (n=13 out of 15), as people recognized that they manage the operation and have the most knowledge of the whole system. Donors were weighted the least (or tied for least) by nearly all participants (n=14 out of 15) including the donors themselves, as they are not involved in the process once the food leaves their doors. Recipients and volunteers were weighted similarly because participants recognized that recipient opinions are important to the acceptance of donations, and volunteer drivers have valuable experience interacting with both donors and recipients. In order to translate these weights to Borda aggregation, we allocated each stakeholder group a total number of votes that was commensurate with their weight, and divided up the votes evenly within each group. For example, 412 Food Rescue employees are assigned 46% of the weight; this translates to allocating them 46 votes out of 100 total as a group, where each employee's vote is "replicated" 46/3 times because three 412 Food Rescue employees participated in our study.

4.2.7 Explanation and decision support

Once recommendations are generated, the decision support interface presents the top twelve organizations, accompanied by explanations, to support the human decision-maker who matches incoming donations to recipients. We used the explanations to demonstrate to participants how their opinions had been incorporated into the algorithm's decision-making.

[14]This is based on the input from participants that participated in the workshop and/or Session 4.

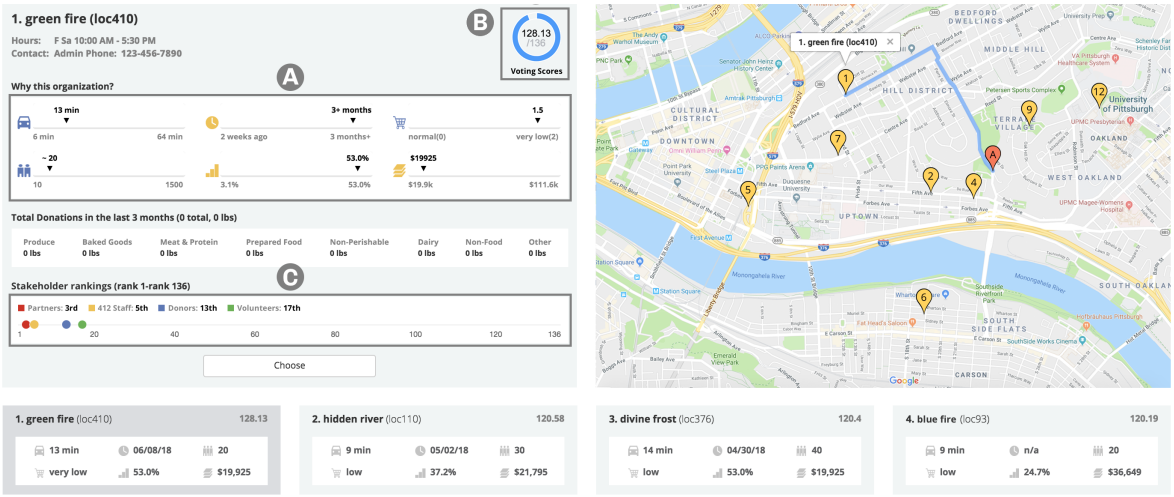


Figure 4.5: The decision support tool explains algorithmic recommendations, including the nature of stakeholder participation, stakeholder voting results, and characteristics of each recommendation. The interface highlights the features of the recommended option that led to its selection (marked by A), the Borda scores given to the recommended options in relation to the maximum possible score (marked by B), and how each option was ranked by stakeholder groups (marked by C). All recipient information and locations are fabricated for the purpose of anonymization.

We also explained average stakeholder models to participants so that they could learn about others' models.

Design of Decision-Support Tool

The interface of our decision support tool is shown in Figure 4.5. The tool was designed with other considerations, such as choice architecture [217], but they are beyond the scope of this paper. We focus instead on the explanation of decisions made by collectively-built algorithms.

- Decision Outcome Explanation (marked by A in Figure 4.5): We used an "input influence" style explanation [33]. Features are highlighted in yellow when an organization is in the top 10% of recipient organizations ranked by that factor. For example, poverty rate is highlighted because the selected organization is in the top 10% of recipients when ranked from highest to lowest poverty rate.
- Voting Score (marked by B in Figure 4.5): The Borda score for each organization is displayed. It shows this option's score in relation to the maximum possible score that an option could receive (i.e., scores when every individual model picks this option as its first choice). This voting score can indicate the degree of consensus among participants.
- Stakeholder Rankings (marked by C in Figure 4.5): Stakeholder rankings show how each stakeholder group ranked the given organization on average. It is a visual re-

minder that all stakeholder groups are represented in the final algorithm and gives the decision-maker additional information about the average opinion of each stakeholder group.

We implemented the interface by integrating it into a customer relations management system currently in use at 412 Food Rescue. Algorithms were coded in Ruby on Rails, the front-end interface used Javascript and Bootstrap, and the database was built with Postgres. The distances and travel times between donors and recipients were pre-computed using the Google Maps API and Python. We used donor and recipient information from the past five months of donation records in the database. On average, the algorithm produced recommendations for each donation in five seconds.

Method (Session 4)

We conducted a one-hour study with each participant to understand how the decision support and explanation influenced their perceptions of the matching algorithms and their attitude toward 412 Food Rescue. In order to generate summary beta vectors for each stakeholder group, we normalized the beta vectors for all stakeholders in the group and took the pointwise average. This yields a summary beta vector where the value of each feature roughly reflects the average weight that stakeholders in the same group give to that feature.

We first showed participants the graphs of their individual models and graphs of the averaged models for each stakeholder group, and asked participants to examine similarities and differences among these models. We next had participants interact with the decision support tool run on a researcher's laptop. The researcher walked participants through the interface, explaining the information and recommendations, and asked them to review the recommendations and pick one to receive the donation. After each donation, participants were asked their opinions of the recommendations, the extent to which they could see their models reflected in the results, and their general experience. We concluded with a 30 minute semi-structured interview in which we asked how participation influenced their attitude toward algorithms and 412 Food Rescue. We also asked participants to reflect on the overall process of giving feedback throughout our studies.

4.2.8 Findings: the impact of participatory algorithm design

In the previous sections, we described how stakeholders used the WeBuildAI framework to build the matching algorithm for 412 Food Rescue over multiple sessions and a workshop. We now report the qualitative findings from observations, think-alouds, and interviews to describe the impacts of the WeBuildAI framework and participation.

Participants' Experience with the WeBuildAI Framework

Overall, the individual belief-elicitation step of the framework—using both machine learning and explicit rule specification methods and visualizing the learned models—successfully enabled participants to build an individual model that represented their beliefs on how the

algorithm should make a matching decision. Participants perceived the automatic aggregation method based on the Borda rule as a nuanced, democratic approach; the decision support tool and explanation allowed them to understand how algorithmic recommendations were made. **Effects of Individual Model Building Methods on Elicited Beliefs**:

Participants told us that performing pairwise comparisons and subsequently specifying explicit rules helped them develop and consolidate their beliefs into a set of principles that they could apply consistently in different decision contexts. Answering pairwise comparison questions helped familiarize participants with the problem setting; however, some participants commented that they felt like they were applying internal rules inconsistently, particularly in their first few questions. Explicitly specifying scores for each feature helped them reconcile their conflicting beliefs. For example, V1 told us that she originally used organization size inconsistently, sometime favoring smaller organizations or bigger organizations, but when creating a rule, she determined that organization size should not matter. When she answered the new set of pairwise comparison questions to retrain the machine learning model, she further evaluated whether she could consistently apply her belief, i.e., that the organization size does not matter, to different contexts and whether she encountered any new situations in which she would need to further refine her rule.

In choosing the model to use in the final matching algorithm, the most important factor for all participants, except one, was how closely each model represented their beliefs. In Session 3, 10 out of 15 participants chose their machine learning models. For many, this was the model they had built last and therefore that reflected their thinking at the time of comparison. Others felt that the machine learning model had more nuance in the way different factors were weighted, and some valued the linearity of the model compared to their manual rules, which were often step-wise functions. Explicit-rule models were chosen by five participants. For four of these participants, their explicit-rule model did a better job of weighing all of the factors that mattered to them and screening off unimportant factors. In other words, machine learning models learned rules that they disagreed with—for example, a machine learning model may give linearly increasing weight to larger organization sizes.

On the contrary, for one participant, the procedural difference in the two methods was why he chose the explicit-rule model. R2 trusted the reflective process of specifying a model and did not trust his pairwise answers nor the machine learning model built from them, even though the accuracy of the machine learning model was 90%, compared to 76% for the model that he created. He believed that determining policy should be based on defining principles, rather than case-by-case decisions; for this reason, he wanted to build a rule and follow the outcomes from the rule.

An unexpected finding was that the methods' procedural differences seemed to influence which aspects participants focused on at the time of decision-making and, in some cases, the rules that participants made. Creating a scoring model from a top-down approach seemed to evoke a higher level of construal [220], eliciting an abstract level of thinking that was absent when answering pairwise comparisons. Many participants stated answering pairwise comparisons felt emotional because it made them think of real-world organizations. For example, V1 said that developing explicit scoring rules felt "robotic"; R3 said that he felt that creating the scoring model was easier than the pairwise comparisons because

it took the emotion out of the decision-making process. For an administrative decision-maker, F3, answering pairwise questions made her focus on day-to-day operational issues like travel time because she related the questions to real-world decision-making. This contrasted with her explicit-rule model, which favored equity-related factors like income and poverty. When comparing the models in Session 3, she told us that she focused on idealistic matching that prioritized equity when she was specifying scoring rules. In the end, she chose her machine learning model, stating that while her explicit-rule model was appealing as a way of pushing herself beyond her operational thinking, she deemed travel time and last donation date most important in practice.

Responses to the Borda-Based Aggregation: Participants appreciated that the Borda method gave every recipient organization a score (n=5) and that it embodied democratic values (n=4).[15] In the workshop, F1 felt that giving every organization a score captured the subtleties of her thinking better than other methods, such as considering only the top-ranked organization: *"I appreciate the adding up [of] scores. Recognize the subtleties."* V3 also stated that being able to rank all recipients is *"more true to...[being] able to express your beliefs."* R1 approved of the method, saying, *"It's very democratic,"* relating it to a form of human governance. Two other individuals, D2 and D4, also related the method to voting systems in the US. D4 recognized that some US cities in California recently used a similar voting method for their mayoral election. It is worthwhile to note that, when we asked about potential alternatives, participants expressed difficulty thinking of them (n=3). For example, R2 said, *"I guess I don't know what the alternative way to do it would be, so I'm okay with it."*

Responses to the Decision Support Interface: Participants were almost universally appreciative of the fact that the system keeps a human dispatcher in the loop to make the final decision rather than automating the decision entirely. While some participants (F1 and R5) acknowledged that full automation could be more efficient than a human-in-the-loop process, most participants expressed that having a human dispatcher overseeing the process was important as they might have knowledge of additional decision factors outside the scope of the algorithm. F3 expressed that the combination of human and computer decision-making elements was "magical" in that it combined the objective data of an algorithm with human elements *"that the computer will never know... like so and so at this place loves peaches and they make peach pies."* Others (e.g., R2) expressed that the algorithm could enable human decision-making in a way that reduces bias or favoritism on the part of the dispatcher, thereby making the decisions of the organization more fair and objective.

Participants were interested in the stakeholder rankings and asked to see more information. Given that the top twelve results often did not show the first choice for any stakeholder group, several participants wanted to see the first choice for each stakeholder group in addition to the voting aggregation scale (n=7). Participants appreciated that the stakeholder rankings showed opinions that might differ from those of 412 Food Rescue dispatchers (n=4). V6, who was concerned that 412 Food Rescue staff did not heavily weight

[15]We note that the description of Borda given to participants described a scoring process and did not include words such as "voting" and "democracy" as reported in Section 6.1.

factors that were important to her, was pleased that the voter preference scale illustrated the difference between her stakeholder group's average model and 412 Food Rescue's average model. She hoped that the staff would see that their thinking differed from other stakeholders and perhaps reconsider their decisions in order to be more inclusive of other groups' opinions. 412 Food Rescue staff were interested in the information as well and F3 mentioned that, while she would not solely base her decisions on stakeholder ranking information, she might use it as a tiebreaker between two similar organizations.

Participation and Perceptions of Algorithmic Governance

In a manner consistent with theories on procedural justice [160; 159] and participatory policy-making [111], participants believed that having control over the algorithm through participatory algorithm design made the resulting algorithm fair, and this process improved their attitudes toward the organization as a whole.

Procedural Fairness in Participatory Algorithm Design: All participants mentioned that the fact that the organization was putting a priority on fairness, being open to new ideas, and including multiple stakeholder groups improved their perceived fairness and trust of both the matching algorithm and the organization itself. For example, one participant said, *"These are everybody's brain power who were deemed to be important in this decision... it should be the most fair that you could get."* Some expressed that participation expanded the algorithm's assumptions beyond those of the organization and developers (n=6). V6 noted that it is easy for organizations to remain isolated in their own viewpoints and that building an algorithm based on collective knowledge was more trustworthy to her than *"412 [Food Rescue] in a closed bubble coming up with the algorithm for themselves."* V3 echoed this sentiment, stating that participation was *"certainly more fair than somebody sitting at a desk trying to figure it out on their own."* At 412 Food Rescue, F2 stated that *"getting input from everyone involved is important"* to challenge organizational assumptions and increase the effectiveness of their work. Other participants noted that all stakeholders have limited viewpoints that can be overcome with collective participation (n=3). R1 felt the algorithm would be fair only *"if you took the average of everybody. ...[My model] is only my experience. And I view my experience differently than the next place down the road. And my experience is subjective."*

Empathetic Stance toward the Governing Organization: Participation in algorithm design led many participants to increase the degree to which they viewed 412 Food Rescue positively and develop a more empathetic stance toward the organization (n=8). For some participants, this happened because participation exposed the difficulty of making donation matching decisions and made them realize that there might not be a perfect solution, which in turn made them thankful for the work of the organization (n=4). For example, after experiencing the burden of making the matching decision and seeing how similar the recommended recipients can be in the interface, D2 and V3 both expressed gratitude for 412 Food Rescue. Participants also expressed appreciation for the organization's concern for fairness and the effort needed to continually make such decisions. This shift in perception is particularly important because it can improve people's tolerance for and understanding of tradeoffs in governance decisions.

The participatory algorithm design also increased some participants' motivation to engage with the organization (n=4). Many participants appreciated that their opinions were valued by the organization enough to be considered in the algorithm building process and expressed that they may increase their involvement with the organization in the future either through increased volunteer work (V3 and V6) or donation acceptance (R2).

Reactions to Other Stakeholders' Models: While sharing other stakeholders' models is not a requirement of our framework, in this work, we showed the models to participants in order to get feedback on the fully transparent implementation of our framework.[16] We report how participants responded to similarities and differences in stakeholder models.

In individual models, all participants considered efficiency and equity factors. For example, all stakeholder group models valued distance as one of the top three factors and favored organizations that were deemed to be in greater need. Reviewing the models, participants expressed feeling assured that they shared these guiding principles with other participants (n=8). For example, all prioritized higher as opposed to lower poverty, and lower as opposed to higher food access. R7 was pleased to note that all participants were *"on the same page"* and concluded that *"no matter what group or individuals we're feeding, [we] have the same regard for the food and the individuals that we're serving."*

A main source of disagreement among models was how the factors were balanced. 412 Food Rescue Staff tended to weight travel time and last donation significantly more than the other factors. Donors, recipients, and volunteers tended to give all factors other than organization size relatively equal importance. Participants also had divided views on organization size, arguing for larger or smaller organizations, and did not prioritize this factor compared to others. In responses, participants acknowledged these differences and sought to make sense of others' assumptions. For example, R1, referencing how important travel time was to her, mentioned that hers is more of a *"business model"* whereas others were more altruistic, more heavily weighting factors like income and food access. Some participants were even pleased to see differences in the models (n=3). R3 was pleased that other participants were considering unique viewpoints. Likewise V4 and R1 both stated that it was natural to expect differences between stakeholders, as everyone has unique experiences, and that *"this is the point of democracy"* (V4).

However, one participant, V6, was concerned that 412 Food Rescue staff did not weight heavily her most important factors such as food access, income, and poverty. While she said that the algorithm was "fair" as it was collectively created, her trust in the organization was lowered as a result, because she inferred that they believe in different principles. She also raised a concern about other participants' input qualities. It took her significant effort to develop a model that accurately represented her views, and she could not judge whether other participants were *"thoughtful enough to really put the effort into their models and capture their own emotions with it."* She concluded that she still trusted the algorithm, but appreciated having human oversight of the final decision.

[16] Participants also told us that they were curious about other stakeholders' beliefs.

Participation and Awareness of Algorithms and Organizational Decision-Making

Our findings suggest that participating in algorithm design improved algorithmic awareness at an individual level, as well as awareness of inconsistencies in decision-making practices at an organizational level.

Increased Algorithmic Awareness: At an individual level, participating in algorithm design changed participants' attitudes toward algorithms.[17] They felt they better understood what an algorithm was and had more appreciation for the kinds of decisions that algorithms could make. For some participants, seeing how the two models predicted their answers in our study session made them rethink their initial skepticism and begin to trust the algorithm. V1, who in earlier studies expressed doubt that an algorithm could be of any use in such a complex decision space, stated at the end of Session 3 that he now "wholeheartedly" trusted the algorithm, a change brought about by seeing the work that went into developing his models and how they performed. F3 expressed that before participating, *"the process of building an algorithm seemed horrible"* given the complexities of allocation decisions. Seeing how the process of building the algorithm was broken down *"into steps ... and just taking each one at a time"* made the construction of an algorithm seem much more attainable. For D2, interacting with the researchers who were building the algorithm gave him an awareness of the role human developers play in determining algorithms. He said that, after this process, his judgment of an algorithm's fairness in other algorithmic systems would be based on *"how it was developed and who's behind it and programmed [it] and how it's influenced."* D2 felt that the final algorithm for 412 Food Rescue was fair because he came to know and trust the researchers over the course of his participation.

Improved Awareness of Inconsistency in Organizational Decision-Making: The process of eliciting individual models allowed participants from the governing organization to be more aware of internal inconsistencies in decision-making within their organization, and provided an opportunity for them to revisit their own assumptions about other stakeholders. Guided only by the broad goals of the organization's mission, the employees previously made matching decisions according to their own criteria and interpretations of that mission. By externalizing their decision-processes into computational belief models, the employees were able to formalize their own decision-making processes, and see how their models meshed with or differed from other employees' processes, which brought hidden assumptions to the surface. For example, after seeing other employees' models, they discovered that some employees prioritized mid-sized organizations whereas others prioritized larger organizations, and employees differed in the ways they weighted poverty, income, and food access.

Moreover, seeing other stakeholders' models allowed employees to compare their assumptions about other stakeholders with the models actually made by the stakeholders. One common assumption held by the staff was that volunteers would prioritize travel time, but our volunteer stakeholders had diverse models, varying from one that predominantly weighted travel time to one that gave equal weights to travel time and recipient organizations' needs. When F2 saw that volunteers did not weight travel time as highly as she

[17] None of our participants had a background in programming.

had thought, she questioned her evaluation of travel time: *"Maybe [volunteers] don't care as much. I think you end up hearing from the people who care... It's like that saying with customer service: Only complain when something's happened."* This reflection opens up the possibility that the organization could seek to appeal to diverse volunteer motivations and tailor recruiting methods accordingly.

4.2.9 Evaluation of Algorithmic Outcomes

Our qualitative findings in the previous section show the procedural effect of participatory algorithm design, but what outcomes do collectively-built algorithms produce? In this section, we evaluate the algorithm's performance on various metrics.

Evaluation Goal

In the literature on policy-related algorithmic systems, the status quo—current human decision-making practice—is deemed to be an appropriate baseline for comparison to measure the algorithmic tool's efficacy; thus, we compare our algorithm with current human decision-making at 412 Food Rescue. One major reason that the organization wanted to introduce the algorithmic allocation system was to improve equity in donation allocation made by organizational staff and distribute the donations to a larger set of recipients. Indeed, the skewness of their current distribution of donations (i.e., 20% of the organizations receiving 70% of the donations (Figure 4.6a)) is not the result of conscious strategy, but rather the result of, for example, the memory bias of human decision-makers selecting recipients that they have given donations to recently.

Dataset

The final matching algorithm included 23 individual models (Section 4.2.5) that generated complete rankings of possible recipients for each incoming donation; the rankings were then aggregated using the Borda method with the stakeholder weights provided in Section 6. We ran this collectively-built algorithm on historical allocation data from 412 Food Rescue containing a total of 1,760 donations from 169 donors over the course of five months (March–August 2018).[18] There were 380 eligible recipient organizations in the database, and 277 of those received donations in the timeframe we considered.[19] We compared our algorithm (AA) with two benchmarks: human allocations recorded in historical data (HA), and a random algorithm that selected a recipient uniformly at random (RA). In the simulations for our algorithm and the random algorithm, we applied some of the real-world constraints that influenced human dispatchers' decisions: for any given donation, we filtered out recipients that did not handle the donation type or were not open for at least 2 hours between the incoming donation time and 6 pm.

[18]The original data set had 1,862 donations from 177 donors given to 305 recipient organizations. 412 Food Rescue staff told us that 28 of the recipient organizations were either backup recipient organizations or became inactive at the time of the evaluation, thus we excluded them from the data.

[19]46 recipients were added during the course of the five months, and for each day, we filtered out organizations based on the date when the recipient organizations were added in algorithm testing.

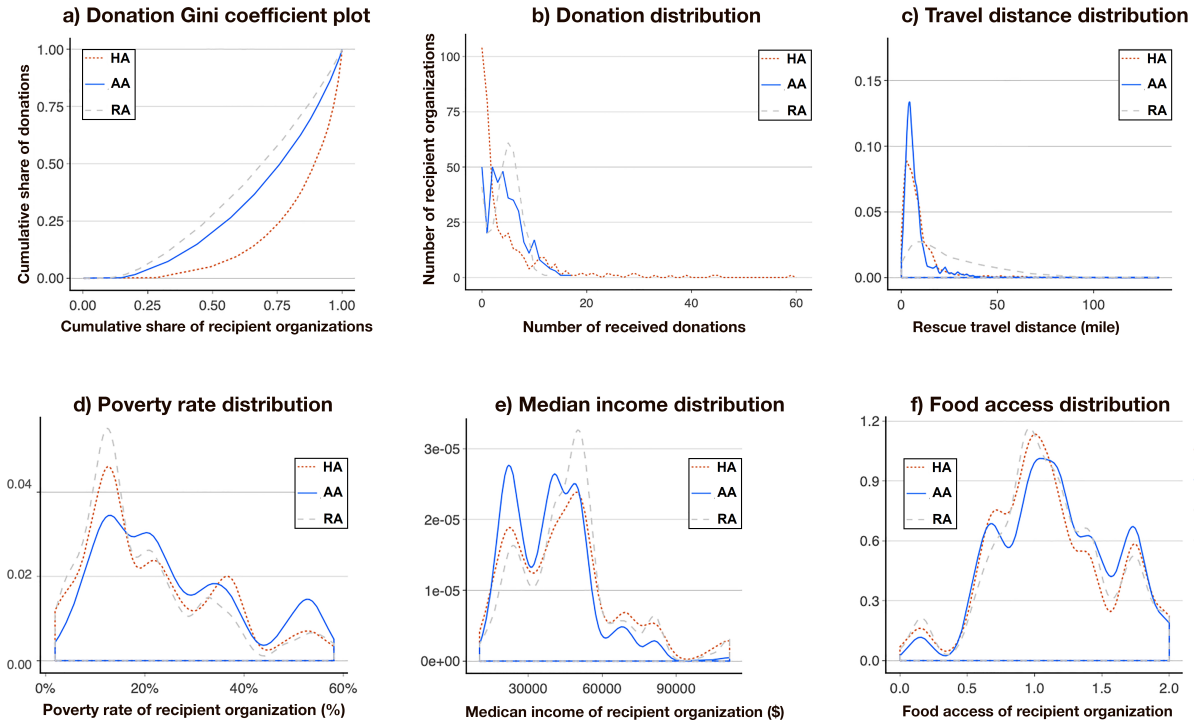


Figure 4.6: The performance of our algorithm (AA) versus the human allocation (HA) and a uniformly random allocation (RA), on various metrics.

Results

The results indicate that our algorithm can make donation allocations more equitable compared to human allocation without hurting efficiency (Figure 4.6).

Number of Donations Allocated to Recipient Organizations: Our algorithm resulted in a more equal donation distribution compared to human allocation, as illustrated in Figure 4.6b. As the human donation distribution is skewed, we conducted a Mann-Whitney U test, a nonparametric test that does not require the data to be normally distributed, to compare the number of donations allocated to recipient organizations.[20] The results show that algorithmic allocation was significantly more equally distributed than human allocation (AA Median = 4 donations (SD = 3.73), Min-Max:0-20; HA Median = 2 donations (SD = 7.26) Min-Max:0-59, U = 57814, p <.00000001).

We also conducted a Gini coefficient analysis, a standard economic inequality measure of income [117] or other kinds of resources [12]. A Gini index of zero means perfect equality, with everyone getting the same number of donations, and an index of 100 means maximum inequality, with one organization receiving all donations. Algorithmic allocation resulted in a Gini index of 42, which was lower than the Gini index of 68 in human allocation; this indicates that the algorithmic allocation was more equal. The random allocation algorithm achieved a Gini index of 32, which intuitively is close to the minimum possible, subject

[20]The convention is to report medians as the data is not normally distributed.

to the constraints. Graphically, as seen in Figure 4.6a, the closer the allocation line is to the diagonal line $y = x$, the fairer the allocation. Additionally, the x-axis is ordered from lowest to highest, so, for instance, our results show that the lowest 50% of all recipient organizations received about 5% of all donations under a human dispatcher, but received about 20% of all donations under our algorithm.

Poverty, Income, and Food Access of Recipients: When considering poverty, income, and food access levels, random allocation can be seen as uniformly sampling from the poverty, median income, and food access rates of all recipients because these features are completely recipient-specific. As illustrated in Figure 4.6d, Figure 4.6e, and Figure 4.6f, the human dispatcher's decisions closely followed the underlying population distributions, but our algorithm donated to recipients with higher poverty rates, lower median incomes, and worse food access. A Mann-Whitney U test shows that the algorithmic allocation gave donations to areas with higher poverty rates (Median = 21.6%, SD = 14.44%) significantly more than human allocation (Median = 18.3%, SD = 13.73%, U = 1303400, p < .00000001). Indeed, Figure 4.6d shows that the human and the random algorithm gave more donations to areas with 10%-15% poverty rates, whereas our algorithms gave more donations to areas with about 50% poverty rates. Algorithmic allocation also gave more donations to recipients with lower income (Median = \$40,275, SD = \$16,312) than human allocation did (Median = \$42,255, SD = \$22,037, U = 1773200, p < .00000001), and the same pattern is observed to a lesser degree in the recipients' access to food levels (AA Median: 1.15 (SD=0.42), HA Median: 1.06 (SD=0.44), U=1414400, p =.0002; 0=Normal access, 2=Extremely low access).

Distance and Efficiency: One of the concerns of the organization was that distributing the donations more equitably could lead to longer and less efficient donation allocation. Our simulation results suggest that algorithmic allocation did not increase rescue distance, as illustrated in Figure 4.6c. A Mann-Whitney U test shows that the distance of rescues under algorithmic allocation, whose median is 5.5 miles, is significantly shorter than under human allocation, whose median is 6.15 miles (U = 1646900, p = 0.001).

4.2.10 Discussion

In this paper, we envision a future in which people are empowered to build algorithmic governance mechanisms for their own communities. Our framework, WeBuildAI, represents one way to realize this goal. We have implemented and evaluated a system of collective algorithmic decision-making, contributing to the emerging research agenda on algorithmic fairness and governance by advancing understanding of the effects of participation.

Summary of the Research Questions and Results

We summarize our results in response to the research questions raised in the introduction.

What socio-technical methods will effectively elicit individual and collective beliefs about policies and translate them into computational algorithms? How should the resulting algorithms be explained so that participants understand their roles and administrators understand their decisions?

- The WeBuildAI framework successfully enabled participants to build models that they felt confident represented their own decision-making patterns. Participants understood graphical representations of individual models (Figure 4.4) and felt that collective aggregation via the Borda rule was fair. The decision support helped organizational administrators and other stakeholders understand how the final recommendations were made.
- Our findings suggest the elicitation method design could influence elicited beliefs. The top-down explicit-rule method may have promoted idealistic beliefs, while the bottom-up pairwise comparison-based machine learning method may have promoted realistic beliefs that accounted for emotions and constraints associated with tasks.

How does participation influence participants' perceptions of and interactions with algorithmic governance?

- Participation not only resulted in new technology design but also affected participating individuals and organizations [227; 111]. Our participants reported greater trust in and perceived fairness of the matching algorithm, the governing institution, and administrative decisions after participating. Some participants were more motivated to use the services, felt respected and empowered by the governing institution, and reported a greater empathy for difficulties in the organization's decision-making process.
- Our participatory algorithm design, particularly the individual model building method, increased participants' algorithmic awareness and literacy. Through the process of translating their judgments into algorithms, they gained a new understanding and appreciation of algorithms. The method also revealed inconsistencies in employee decision-making in the governing organization, and made employees revisit their assumptions of other stakeholders.

How does the resulting collectively-built algorithm perform?

- The comparisons of the collectively-built matching algorithm and human allocation, using five months of historic data, suggest that the matching algorithm makes donation allocations more even, and gives more donations to recipient organizations in areas with higher poverty, lower income, and low access to food, without increasing the transportation distance.

Contributions to Research on Human-Centered Algorithmic Systems

Fairness and Moral Behavior in AI: In response to recent scholarly and journalistic work that has pointed out the need for "fair" algorithms, much research has been done to devise computational techniques that guarantee fairness in algorithmic outcomes. Our work offers a method for building procedurally-fair governing algorithms [159]. Our findings also offer empirical evidence of the effects of procedural fairness from the perspectives of both those who are affected by algorithms and those who use algorithms; the framework not only increased perceived fairness and trust of the algorithm but also influenced the organization by making the disparate effects of the algorithm more salient in their daily operation.

Our work also suggests that ongoing research seeking to understand people's moral concepts for algorithms and AI needs to be more cognizant of the design of the stimuli. (Some studies use more illustrative, vivid descriptions, whereas others use abstract textual descriptions.) Previous work in experimental moral psychology suggests that the vividness and realism of stimuli influences participants' answers. Consistent with this literature, our work suggests that the top-down versus bottom-up approach of building an algorithm may elicit different levels of construal, resulting in qualitatively different algorithmic models. It is important to choose an elicitation method and level of abstraction appropriate for the task context, and to take a reflective approach so that people can be aware of those situational effects and build a model in accordance with their beliefs.

Community Engagement in Algorithm Design: Our work contributes to recent research that calls for community engagement in AI design by offering a method to leverage varying stakeholders' participation directly in the design of the algorithm. By working with real-world stakeholders with various educational and economic backgrounds to build an algorithm that operates a service, we demonstrate the feasibility and potential of community involvement in algorithm design. At the outset of our research, we were unsure whether participants would feel confident and comfortable enough to express their beliefs on algorithms, and were concerned they might mistrust AI due to negative representations in popular media. It has been a rewarding experience to see participants not only expressing their beliefs, but also gaining trust in and becoming empowered through algorithmic systems. AI systems should be designed to facilitate these changes.

Levels of Participation in Algorithmic Governance

In this section, we define levels of participation in algorithmic governance. We discuss the upsides and downsides of different forms of governance and when collective participation is appropriate, reflecting on our research.

Closed, Non-Participatory Governance: Institutions can design a governing algorithm without involving stakeholders by drawing from their existing data and assumptions. This form of governance is cost-effective compared to participatory governance, which requires effort and resources in soliciting and synthesizing participation. Closed governance is appropriate when there are legitimate metrics for algorithm design. For example, it might be appropriate if the goal is solely to minimize the volunteers' travel time. In our research, the organization was open to stakeholder participation because the staff were unclear on how to balance efficiency and equity in their daily operations. Additionally, closed governance may not inherently earn stakeholders' trust; it works best when the governing institution has already established trust with those being governed. Otherwise the algorithmic decisions may be challenged, mistrusted, or not adopted.

Mediated, Indirect Participatory Governance: Another form of governance is the mediated use of participants' input, resulting in participants' indirect influence on final algorithmic policy. In this form, stakeholders provide input to inform the designers and policymakers, who later design and implement the governing algorithms. The input can be collected through interviews or tools such as individual belief modeling, as in our framework. This form allows the governing organizations to operate on more accurate

stakeholder assumptions, and communicating about the stakeholders' involvement can cultivate trust and increase the chances of adoption by those who are governed. This form is most appropriate when the organization seeks to use participatory feedback while retaining full control of the algorithm's design.

Direct Participatory Governance: In fully participatory algorithmic governance, stakeholders' participation is directly implemented in the final algorithm. In this form, participants feel most empowered and responsible, according to both existing literature and our work. However, the governing organization has less control over the final algorithm design. Direct participatory governance is most appropriate in contexts where stakeholders' trust and motivation to participate in the governing organization are critical, when a high level of procedural fairness is required, or in organizations and communities that are already self-governed, such as Reddit.

Extension of the WeBuildAI Framework and Future Work

Our application of the framework to 412 Food Rescue is a case study that implements participatory governance in one context. Our framework can be used and extended to support both mediated and direct participatory governance, and potentially for other algorithmic governance situations that involve normative design decisions and associated tradeoffs. For example, our framework could be used to create governing algorithms that allocate public resources or contribute to smart planning services, placement algorithms in school districts or online education forums, or hiring recommendation algorithms that balance candidate merit with equity issues. Extending our framework to new contexts requires addressing several challenges.

Individual Model Building as a Design Tool: Our findings suggest that the process of building individual models of algorithmic policy has many benefits. Externalized models provide a concrete place for starting a conversation about similarities and differences among the stakeholders or staff members of the organization. Designers and policymakers can use the models to inform algorithm design, or as an auditing or evaluation metric to assess the algorithm's effects from diverse stakeholders' perspectives. However, our research only used about 8-10 features that people could understand. Further research will be needed to apply the individual modeling method to algorithms with hundreds of features or more complex features. New techniques will be needed to explain and combine the features into a set that people can process.

Collectively Aggregated Decisions for Direct Participatory Governance: Our framework can be applied to enable direct participatory governance, particularly in contexts in which trust, motivation, and perceived fairness matter, and, in its current implementation, contexts that do not require instantaneous decisions (within, say, less than a second).

One challenge, though, is to determine who participates and whether participation needs to be regulated. Opening up an algorithm to participation means that some participants may potentially hold opinions that are not socially acceptable. One way to avoid this is to limit participation so that democratic control of algorithms is subject to the constraints of public reason [195; 31]. This ensures that the behavior of algorithms is justified

by a universally agreed-upon subset of principles. Future work would need to investigate how to broaden participation while respecting diversity within public reason, and devise an ethical way to determine the boundaries of participation.

Another challenge is ensuring the quality of participation, particularly when participation occurs at scale. Techniques used in crowdsourcing for quality assurance could be adopted to judge the quality of participation based on the amount of time and number of iterations people use in creating their models. Anecdotally, in our study, we observed that the machine learning model's accuracy was low when participants told us that they were applying rules inconsistently. Further work needs to investigate whether model accuracy can be another metric.

When people participate in building systems, those systems become more transparent to them and they gain a deeper understanding of how the systems work. While this is one of the main sources of trust, one potential concern is that people will use this knowledge to game and strategically manipulate the system. To clarify, we do not mean that the potential manipulation of the systems by the disempowered is a risk. We aim to create benefits for all those in need, and we believe the system could be at risk if some individual parties skew the results to maximize their own benefits when all participating individuals have a similar level of need. Indeed, one of the main topics of research in computational social choice [48] is the design of voting rules that discourage strategic behavior—situations where voters report false preferences in order to sway the election towards an outcome that is more favorable according to their true preferences. However, this is not likely to be an issue for our framework because each individual does not have direct control over the final algorithm behavior. One may try to manipulate one's pairwise comparisons or specify preferences to obtain a model that might lead to preferred outcomes in very specific situations, but the same model would play a role in multiple, unpredictable decisions. The relation between their models and future outcomes is so indirect that it is virtually impossible for individuals to benefit by behaving strategically. That said, future work would need to evaluate this question in the real world.

Promoting Representative Participation: One of our goals in designing this participatory framework is to empower stakeholders who typically do not have a say in the algorithms that govern their services, communities, or organizations. By empowering, we mean providing a method or tool that allows people to influence and control a system that they themselves use or an institution to which they belong [75; 97]. This shared power between users and developers, or individuals and governing parties, could increase the self-efficacy [24] and motivation [75] of stakeholders. Empowerment is one of the traditional values of HCI research and practice [198].

However, recently scholars have also pointed out that "material empowerment," or the technical tool itself [198] is not enough to enable people to make positive effects on social problems; one needs to devise solutions that also account for legal, social, and economic constraints [198; 175]. Our framework provides a tool that can enable stakeholders to participate in algorithm design, but it in and of itself will not necessarily result in equal empowerment of all stakeholders. Including representation from communities that are underserved or disadvantaged is a critically important challenge to address in future work. While many in these communities may technically have the opportunity to participate, they

may face barriers like time or resource constraints that limit their access to participation. For our context with 412 Food Rescue, we acknowledge for example that volunteers must have access to at least two relatively scarce commodities: access to a private vehicle and free time. Furthermore, recipient organizations often do not have reliable contact information for their clients, who may not have regular access to email or cell phone service. This poses a practical barrier to participant recruitment. In addition to technological design interventions like those we put forward in this paper, social and economic infrastructure will be necessary to ensure equal participation of all stakeholders.

Limitations

Like any study, our work has limitations that readers should consider. Our study evaluated people's experiences with participation, as well as their attitudes toward and perceptions of the resulting algorithmic systems. As our next step, we will deploy the system in the field in order to understand long-term effects and behavioral responses. In the deployment, we will also consider additional evaluation measures for the algorithm, such as stakeholder satisfaction. Additionally, in developing our framework, we intentionally used a focused group of participants to get in-depth insights and feedback on our tools and framework. As we implement our next version, we will examine participation with a larger group of people, including recipient organizations' clients, by developing an educational component and targeted recruiting methods. We will also explore the possibility of running an open system, where people can join at any time or update their models by providing more data. We also acknowledge that despite our best efforts to base our design choices on participants' input gained through interviews (for example, who the stakeholders are, what factors to use), our views might have influenced our analysis of participants' inputs. Our plan to have an online system where participants can further comment on the selected features, stakeholders, and evaluation measures may mitigate this in the future. Finally, our framework needs to be tested with other contexts and tasks that involve different cultures and group dynamics. We are particularly interested in the effects of participation when collective opinions are polarized. On the one hand, it might be the case that a participatory, voting-based approach would be the only way to find a consensus solution. On the other hand, additional techniques—such as public deliberation through an open forum—might be needed to bring together polarized parties to ensure the efficacy of the resulting algorithms. Future work would need to investigate this question further.

4.3 Conclusions

4.3.1 Virtual Democracy, in Theory

Our theoretical and empirical results identify Borda count as an especially attractive voting rule for virtual democracy, from a statistical viewpoint. However, Borda count is also compelling in terms of usability and explainability.

In more detail, in our implemented donor-recipient matching system, clicking on a recommended alternative displays an explanation for why it was ranked highly by Borda

count, which consists of two components. First, we show the alternative's average position in the predicted preferences of each of the four stakeholder groups. Note that this information determines the Borda score of the alternative, given the weight of each stakeholder group.[21] Second — this is the more novel component — we show specific features in which the recommended alternative stands out. This is interesting because classic social choice theory does not have features for alternatives, and we are able to give this type of explanation precisely because our alternatives are represented as vectors of features (which is crucial for the application of learning-to-rank algorithms).

Based on the results presented in this paper, as well as these additional insights, we use Borda count in our implemented virtual-democracy-based system.

4.3.2 Virtual Democracy, in Practice: WeBuildAI

Increasingly, algorithms make decisions influencing multiple stakeholders in government institutions, private organizations, and community services. We envision a future in which people are empowered to build algorithmic governance mechanisms for their own communities. Toward this goal, we proposed the WeBuildAI framework. In this framework, stakeholders build an algorithmic model that represents their beliefs about ideal algorithm operation. For each decision task, each individual's model votes on alternatives, and the votes are aggregated to reach a final decision.

As a case study, we designed a matching algorithm that operates 412 Food Rescue's on-demand transportation service, implementing the framework with their stakeholders: donors, volunteers, recipient organizations, and 412 Food Rescue's staff. We then evaluated the resulting algorithm with historical donation data, which showed that our algorithm leads to a more even donation distribution that prioritizes organizations with lower income, higher poverty rate, and lower food access clients compared to human allocation decisions. Our findings suggest that the framework improved the perceived fairness of the allocation method. It also increased individuals' awareness of algorithmic technology as well as the organization's awareness of the algorithm's impact and employee decision-making inconsistencies.

Our study demonstrates the value and promise of using the WeBuildAI framework as a design tool in order to achieve human-centered algorithmic governance. Future work needs to investigate mechanisms to expand the application of the framework and its boundary conditions, as well as ways to overcome existing socioeconomic and institutional barriers to enabling wider participation.

However, there are some limitations to our case study. First, although the staff appreciated the implementation of our virtual democracy framework with the Borda count, it is unclear whether this is because of the well-designed interface or the use of the Borda count as the aggregation method. While the Borda count is easy to explain in certain ways (e.g., in the Borda count, the sum of scores is all that matters, and we can easily compute the average Borda scores for each group of stakeholders individually), perhaps using another reasonable algorithm would also lead to an explainable and practicable interface.

[21] These weights were decided by the stakeholders themselves.

Additionally, we noticed that some participants viewed features non-monotonically. In particular, multiple participants had single-peaked preferences with respect to recipient size, where they valued mid-size organizations the highest due to the amount of food (one carload) that would be delivered, and their utilities tapered off on both sides. This suggests that linear models are inappropriate. Indeed, because of this feedback, we explored polynomial models as well as decision trees, SVMs, and neural networks, but found that linear models performed the best due to the relative scarcity of data we collected.

One other area for improvement is in the data collection stage. We allowed participants to only choose their more preferred destination in each pairwise comparison, but it could be more instructive to allow them to express a wider range of preferences. For instance, allowing them to specify the strength of their preference (strong, weak, indifferent) could significantly reduce cognitive load and result in more informative and consistent preferences.

Finally, we note that although we included recipient organizations in our stakeholder groups, we did not source comparisons from the people who consume the food donations. It would be interesting and valuable to include them in future work to get a more holistic set of opinions from all stakeholders in the 412 Food Rescue ecosystem.

Role		Studies Involved
412 Food Rescue.*		
F1		Sessions 1-4
F2		Sessions 1-4
F3		Sessions 1-4, w
Recipient organizations.	(Clients served monthly, client neighborhood poverty rate)	
R1	Human services program manager (N=150, 13%)	Sessions 1-4
R2	Shelter & food pantry center director (N=50, 20%)	Sessions 1-4
R3	Food pantry employee (N=200, 53%)	Sessions 1-4
R4	Animal shelter staff	Session 1
R5	Food pantry staff (N=500, 5%)	Sessions 1-4
R6	After-school program employee (N=20, 33%)	Session 1, w
R7	Home-delivered meals delivery manager (N=50, 11%)	Sessions 1-4
R8	Food pantry director (N=200, 14%)	Sessions 1-2
Volunteers.		
V1	White male, 60s	Sessions 1-4, w
V2	White female, 30s	Session 1
V3	White female, 70s	Sessions 1-4, w
V4	White female, 70s (V4a), white male, 70s (V4b) †	Sessions 1-4
V5	White female, 60s	Sessions 1-4
V6	White female, 20s	Sessions 1-4
Donor organizations.		
D1	School A dining service manager	Session 1
D2	School B dining service manager	Sessions 1-4
D3	Produce company marketing coordinator	Session 1
D4	Grocery store manager	Sessions 1-4
D5	Manager at dining and catering service contractor	Session 1
D6	School C dining service employee	Session 1, w

Table 4.1: Participants. Sessions indicate the study sessions that they participated in: *w* represents a workshop study. *Info excluded for anonymity. † A couple participated together.

Factor	Explanation
Travel Time	The expected travel time between a donor and a recipient organization. Indicates time that volunteers would need to spend to complete a rescue. (0-60+ minutes)
Recipient Size	The number of clients that a recipient organization serves every month. (0-1000 people; AVG: 350)
Food Access	USDA-defined food access level in the client neighborhood that a recipient organization serves. Indicates clients' access to fresh and healthy food. (Normal (0), Low (1), Extremely low(2)) [225]
Income Level	The median household income of the client neighborhood that a recipient organization serves (0-100K+, Median=$41,283) [224]. Indicates access to social and institutional resources [200].
Poverty Rate	Percentage of people living under the US Federal poverty threshold in the client neighborhood that a recipient organization serves. (0-60 %; AVG=23% [224])
Last Donation	The number of weeks since the organization last received a donation from 412 Food Rescue. (1 week–12 weeks, never)
Total Donations	The number of donations that an organization has received from 412 Food Rescue in the last three months. (0-12 donations) A unit of donation is a carload of food (60 meals).
Donation Type	Donation types were common or uncommon. Common donations are bread or produce and account for 70% of donations. Uncommon donations include meat, dairy, prepared foods, etc.

Table 4.2: Factors of matching algorithm decisions. The ranges of the factors are based on their real-world distributions.

	D2	D4	F2	F3	R1	R2	R3	R5	R7	V1	V3	V4	V5	V6
ML	**0.86**	0.78	**0.92**	**0.92**	**0.90**	0.90	**0.78**	**0.94**	0.74	**0.90**	**0.92**	**0.78**	0.56	0.68
ER	0.68	**0.68**	0.68	0.86	0.80	**0.76**	0.70	0.92	**0.74**	0.76	0.82	0.82	**0.80**	**0.88**

Table 4.3: Accuracy of the Machine Learning (ML) model and the Explicit-Rule (ER) model. Bold denotes the model the participant chose as the one that better represented their belief after seeing both models' explanations (Figure 4.4) and their predictions on the 50 evaluation pairwise comparisons. F1 chose the machine learning model but did not complete additional survey questions to calculate model agreement, so the result is not included in this table.

Human beings, who are almost unique in having the ability to learn from the experience of others, are also remarkable for their apparent disinclination to do so.

Douglas Adams.

5 Impartial Ranking

In this chapter, we study rank aggregation algorithms that take as input the opinions of players over their peers, represented as rankings, and output a social ordering of the players (which reflects, e.g., relative contribution to a project or fit for a job). To prevent strategic behavior, these algorithms must be *impartial*, i.e., players should not be able to influence their own position in the output ranking. We design several randomized algorithms that are impartial and closely emulate given (non-impartial) rank aggregation rules in a rigorous sense. Experimental results further support the efficacy and practicability of our algorithms.

Based on our theoretical findings, we also develop *HirePeer*, a novel alternative approach to hiring at scale. HirePeer leverages peer assessment to elicit honest assessments of fellow workers' job application materials, which it then aggregates using an impartial ranking algorithm. We perform three studies that investigate both the costs and the benefits to workers and employers of impartial peer-assessed hiring. We find, to solicit honest assessments, algorithms must be communicated in terms of their impartial effects. Second, in practice, peer assessment is highly accurate, and impartial rank aggregation algorithms incur a small accuracy cost for their impartiality guarantee. Third, workers report finding peer-assessed hiring useful for receiving targeted feedback on their job materials.

5.1 Impartial Ranking, in Theory

We now turn to designing voting rules that satisfy certain axiomatic desiderata. First, we examine a setting in which the set of voters exactly corresponds to the set of alterna-

tives; that is, voters must evaluate themselves in order to return a complete ranking. In this setting, we wish to design voting rules that do not incentivize voters to strategically misreport their preferences in order to alter the results of the aggregation in their favor.

Our work is primarily motivated by online labor markets, such as Upwork or Freelancer. In the bigger markets, employers typically receive dozens of applications for a job, but employers do not have the knowledge required to accurately evaluate applicants. For the past year we have been building a prototype of a new online labor market, where applicants for a job—who are well-suited to evaluate applications for that same job—rank each other. We would like to implement a mechanism that aggregates these rankings into a single ranking that is then shown to the employer.

However, the foregoing application has a clear problem, which gets in the way of applying standard rank aggregation rules: strategic behavior. Specifically, in these relatively high-stakes scenarios, it is likely that a player would try to improve his own position in the output ranking by manipulating his reported ranking. For example, he might weaken a strong contender for the top position by ranking him last. Our goal, therefore, is to design rank aggregation rules that are *impartial*, in the sense that the position of a player in the output ranking is completely independent of the report of that player. However, it is easy to prove that there are no deterministic rank aggregation rules that are both impartial (according to our definition) and Pareto efficient [177].[1] Therefore, we restrict our design space to randomized algorithms.

5.1.1 Our Approach and Results

On a high level, our approach is to design *randomized* rank aggregation rules that are impartial and closely emulate standard rank aggregation rules that are not impartial. Specifically, we focus on providing impartial approximations to the important class of *pairwise* rules, which, as input, only require information about the fraction of players ranking any one player above another. Our theoretical results crucially depend on the notion of approximation—or measure of error—in question.

In Section 5.1.5, we introduce the k-PARTITE algorithm, which, in a nutshell, randomly partitions the players into subsets, builds a probability distribution over the positions of members of one subset based on the aggregate opinion of members of other subsets, and then generates a distribution over rankings that is consistent with these distributions over positions. We prove that k-PARTITE is impartial, and, when used in conjunction with any pairwise rule, it provides small *backward error* with respect to that rule: With high probability, k-PARTITE places each player in the same position that the given pairwise rule would have placed him had the input rankings been slightly perturbed.

In Section 5.1.6, we present the COMMITTEE algorithm. It randomly chooses a subset of players, who serve as the eponymous committee. Each committee member is positioned based on the opinions of other committee members, and then all other players are ordered by the committee. The key idea is that, to avoid conflicts and achieve impartiality, each committee member has slots that are reserved for him, and he is inserted into the reserved

[1] The latter property means that if everyone ranks one player above another, so does the output ranking.

slot that most closely matches the aggregate opinion of other committee members. We prove that COMMITTEE provides *mixed error* guarantees with respect to any given pairwise rule, that is, with high probability, COMMITTEE places each player in a position that is *close* to where the given pairwise rule would have placed him had the input rankings been slightly perturbed. Taking on some *forward error*—a mismatch between the positions—allows for improved backward error compared to k-PARTITE.

In Section 5.1.7, we empirically measure the performance of our impartial algorithms with respect to the popular *Kemeny rule*, which is defined via a natural optimization objective. The experimental results demonstrate that our impartial algorithms, when coupled with the Kemeny rule, output near-optimal rankings with respect to the Kemeny objective, despite the impartiality constraint.

5.1.2 Related Work

At this point there is a significant body of work on the design of impartial mechanisms [87; 6; 128; 44; 214; 29; 104; 164; 35], including several papers in major AI conferences [154; 17]. We only elaborate on the papers that are most closely related to ours.

The paper of de Clippel et al. [87] introduced the notion of impartiality, in the context of dividing *credit* for a joint project. Specifically, the output of their mechanism is the fraction of the total credit each player receives, and impartiality means that a player cannot affect his own share of the credit. This mechanism is deployed on the fair division website `Spliddit.org`, where one of the suggested applications is ordering authors on scientific papers. However, an impartial credit division mechanism does *not* induce an impartial ranking mechanism, because, when players are sorted by credit, a player can improve his own position by decreasing another player's share.

Berga and Gjorgjiev [29] study the impartial rank aggregation problem from an axiomatic viewpoint, but focus on deterministic rules and a stronger notion of impartiality. Their results suggest that deterministic impartial rank aggregation methods are severely limited, and support our focus on randomized algorithms.

On a technical level, our k-PARTITE algorithm is reminiscent of an algorithm of Alon et al. [6], in that it randomly partitions the players into subsets, and the outcome of players in one subset is only determined by players in other subsets. But the details of the algorithm, and its analysis, are completely different.

5.1.3 Preliminaries

In this section we introduce terminology and notations that are standard in computational social choice [47], as well as the formal instantiation of the concept of impartiality in our setting.

Rankings and Aggregation

For any $k \in \mathbb{N}$, let $[k] = \{1, \ldots, k\}$. Our setting involves a set of players $[n] = \{1, \ldots, n\}$. The opinions of players are represented as rankings over $[n]$, which we think of as *permu-*

tations. Let Π represent the set of all permutations of $[n]$, and let Π^n represent the set of all *input profiles*. For any $\sigma \in \Pi$, let $\sigma(j)$ be the player at position j in σ and let $\sigma^{-1}(i)$ be the position of player i in the ranking σ (where position 1 is the highest and position n is the lowest).

A *deterministic rank aggregation rule* (also known as a *social welfare function*) is a function $f : \Pi^n \to \Pi$, which takes in an input profile and returns a ranking. A *randomized* rank aggregation rule returns a probability distribution over rankings. We sometimes find it convenient to slightly abuse notation and think of the domain of a rank aggregation rule as $\Pi^n \times 2^{[n]}$—for $\vec{\sigma} = (\sigma_1, \ldots, \sigma_n)$ and $X \subseteq [n]$, $f(\vec{\sigma}, X)$ is the application of the rule to the input profile $(\sigma_i)_{i \in X}$.

Pairwise Rank Aggregation Rules

An input profile $\vec{\sigma} = (\sigma_1, \ldots, \sigma_n)$ induces a *pairwise comparison matrix* $A(\vec{\sigma})$, where

$$A(\vec{\sigma})_{ij} = \frac{|\{k \in [n] : \ \sigma_k^{-1}(i) < \sigma_k^{-1}(j)\}|}{n}.$$

In words, the (i, j) entry is the fraction of players who rank i above j. Let Ω be the set of pairwise comparison matrices. Therefore, we can think of $A : \Pi^n \to \Omega$ as a function that takes in an input profile and returns its associated pairwise comparison matrix. As before, we will also use the notation $A(\vec{\sigma}, X)$, for a subset of players $X \subseteq [n]$, to denote the pairwise comparison matrix associated with the rankings of the players in X.

Some rank aggregation rules only require the information encoded in the pairwise comparison matrix to compute their output. Formally, a deterministic *pairwise* rank aggregation rule is a function $f : \Omega \to \Pi$. We denote the class of all deterministic pairwise rules by $\mathcal{P}$.[2]

We pay special attention to two popular pairwise rules:

- *The Borda Rule:* Given $\vec{\sigma} \in \Pi^n$, the score of each player i is $\sum_{j=1}^{n}(n - \sigma_j^{-1}(i))$ (that is, each player awards $n - k$ points to the player in position k), and the players are ranked by non-increasing score. It may not be immediately apparent that Borda is a pairwise rule—proving this well-known fact is left to the curious reader as an easy exercise.
- *The Kemeny Rule:* The *Kendall tau distance* d_{KT} between two rankings $\sigma, \tau \in \Pi$ is the number of pairs of players on which the two rankings disagree. Given $\vec{\sigma} \in \Pi^n$, the Kemeny rule returns a ranking in $\operatorname{argmin}_{\tau \in \Pi} \sum_{i=1}^{n} d_{KT}(\tau, \sigma_i)$. Computing the output of the Kemeny rule is hard, but can be done in practice using integer programming or heuristic algorithms [77].

Other well-known rules, such as Copeland and Maximin, are also pairwise.

[2] We do not consider randomized pairwise rules in this section. Strictly speaking, we do not require this determinism, but we assume it as it makes the proofs more transparent.

Impartiality

Recall that we are interested in designing rank aggregation rules that are impartial, that is, no player i can affect his probability of being ranked in position j, for all $i, j \in [n]$. Formally:

Definition 5.1. *A (possibly randomized) rank aggregation rule f is* impartial *if for all $i \in [n]$, all input profiles $(\sigma_1, \ldots, \sigma_n) \in \Pi^n$, and all $\tilde{\sigma}_i \in \Pi$, it holds that $\vec{x} = \vec{y}$, where x_j is the probability i is ranked in position j in $f(\sigma_1, \ldots, \sigma_{i-1}, \sigma_i, \sigma_{i+1}, \ldots, \sigma_n)$, and y_j is the probability i is ranked in position j in $f(\sigma_1, \ldots, \sigma_{i-1}, \tilde{\sigma}_i, \sigma_{i+1}, \ldots, \sigma_n)$.*

In an alternative model, we may assume that each player i has a value v_{ij} for being ranked in position j, and then impartiality would mean no player can affect his *expected* value for the outcome, regardless of his value function. Although this definition may seem weaker than Definition 5.1 at first glance, it is easy to verify that the two definitions are, in fact, equivalent.

5.1.4 Measures of Error

Given that our goal is to approximate rank aggregation rules, the measure of error is critical to the statement of the formal problem. To define appropriate notions, we adapt concepts that are standard in scientific computing (e.g., in numerical stability analysis): forward error, backward error, and mixed error. We view these imported definitions as part of our conceptual contribution.

Definition 5.2. *Let f be a rank aggregation rule. A rank aggregation rule g is said to have (Δ_P, Δ_F)* forward error *with respect to f if for every input profile $\vec{\sigma} \in \Pi^n$, the probability that for all $i \in [n]$ it holds that*

$$\frac{|f(\vec{\sigma})^{-1}(i) - g(\vec{\sigma})^{-1}(i)|}{n} < \Delta_F$$

is at least $1 - \Delta_P$.

Intuitively, a low amount of forward error implies that every player i is placed near his correct rank (as determined by f) with high probability. Unfortunately, as the next theorem states, impartial rank aggregation rules cannot approximate the Borda rule. Since Borda is a pairwise rule, the theorem rules out the possibility of approximating all pairwise rules.

Theorem 5.3. *For all $n \geq 2$ and $\varepsilon > 0$, there exists no impartial rank aggregation rule g that gives a $(1/2 - \varepsilon, 1/3)$ forward error with respect to the Borda rule f.*

Proof. For $n = 2$, a direct analysis (which we omit) gives the result. Let us therefore consider only the case $n \geq 3$. Let g be an impartial rank aggregation rule.

Suppose we have the input profile $\vec{\sigma}$ where $i \neq 2$ gives the ranking $(i-1, \ldots, n, 1, \ldots, i-2)$. Note that if player 2 continued this trend and gave the ranking $1, \ldots, n$ then all players would have the same Borda score.

Now let us consider player 2 in more depth, and define the probability vector $\vec{x} \in [0, 1]^n$, where x_i denotes the probability player 2 will be in position i when g determines the

ranking. By impartiality we know that $\vec{x}$ does not depend on the ranking of player 2. As $\vec{x}$ is a probability vector, we must have one of the following.

Case 1: The first $\lfloor n/2 \rfloor$ entries of $\vec{x}$ sum to at most $1/2$. In this case, if player 2 has the ranking $(2, 1, 3, 4, 5, \ldots, n)$ in $\vec{\sigma}$, then $f(\vec{\sigma})^{-1}(2) = 1$.

Case 2: The last $\lfloor n/2 \rfloor$ entries of $\vec{x}$ sum to at most $1/2$. In this case, if player 2 has the ranking $(1, 3, 2, 4, 5, \ldots, n)$ in $\vec{\sigma}$, then $f(\vec{\sigma})^{-1}(2) = n$.

In either case, we find that with probability at least $1/2$, g will place 2 in a position at distance at least $\lfloor n/2 \rfloor$ from f's placement. That is, with probability at least $1/2$ we have $|f(\vec{\sigma})^{-1}(2) - g(\vec{\sigma})^{-1}(2)| \geq \lfloor n/2 \rfloor \geq n/3$, giving at best a forward error of $(1/2, 1/3)$. □

With this impossibility in hand, we set our sights on an alternate error measure, which is well defined only with respect to *pairwise* rank aggregation rules. For this definition and throughout this section, we use the Frobenius norm and denote $\|A\|_\infty = \max_{i,j} |A_{i,j}|$.

Definition 5.4. *Let $f \in \mathcal{P}$. A rank aggregation rule g is said to have* (Δ_P, Δ_B) backward error *with respect to f if for every input profile $\vec{\sigma} \in \Pi^n$ the probability that for all $i \in [n]$ there exists a matrix $\tilde{A} \in \Omega$ such that*

1. $\left\|A(\vec{\sigma}) - \tilde{A}\right\|_\infty < \Delta_B$, *and*
2. $f(\tilde{A})^{-1}(i) = g(\vec{\sigma})^{-1}(i)$,

is at least $1 - \Delta_P$.

Intuitively, a low amount of backward error implies that every player i is placed in a rank that had the players altered their opinions slightly, i would be in the correct rank (according to f) with high probability.

Finally, we define a third measure of error, which, in a sense, is a union of the two previous notions.

Definition 5.5. *Let $f \in \mathcal{P}$. A rank aggregation rule g is said to have* $(\Delta_P, \Delta_B, \Delta_F)$ mixed error *with respect to f if for every input profile $\vec{\sigma} \in \Pi^n$, the probability that for all $i \in [n]$ there exists a matrix $\tilde{A} \in \Omega$ such that*

1. $\left\|A(\vec{\sigma}) - \tilde{A}\right\|_\infty < \Delta_B$, *and*
2. $\frac{\left|f(\tilde{A})^{-1}(i) - g(\vec{\sigma})^{-1}(i)\right|}{n} < \Delta_F$,

is at least $1 - \Delta_P$.

5.1.5 The k-Partite Algorithm

We now introduce and analyze our first impartial rule, k-PARTITE, which is formally given as Algorithm 2. As it appears somewhat opaque, it is best to understand its ideas when we assume that all the X_i are the same size, i.e., k divides n, $|X_i| = n/k$, and $\gamma_i = k$ for all $i \in [k]$. Slight adjustments are made when this is not the case, which for purposes of intuition can be safely ignored.

First, players are randomly split into k groups of equal size $X_1, \ldots, X_k$, and then each such group separately ranks all n players producing rankings τ_i. The crux of the algorithm is the construction of the matrix Z, which, in turn, is the sum of $Z^{(i)}$ matrices. Intuitively, the $Z^{(i)}$ matrix represents X_i's contribution to Z, and its (a, b) entry indicates the probability that a should be placed in position b overall. Specifically, each player not

in X_i is placed in his exact position dictated by τ_i with probability $1/k$, and in all positions that the players in X_i themselves were assigned to in τ_i with probability $1/(n(k-1))$. This information is encoded as the only non-zero entries in $Z^{(i)}$—each column then sums to $1/k$, each row representing a player in X_i is zero, and all other rows sum to $1/(k-1)$. As we show in the full version of the paper, Z is doubly stochastic (its rows and columns sum to 1); hence we can apply the Birkhoff-von Neumann Theorem [34; 228] to sample from this distribution and remain faithful to the probabilities.

input: $f \in \mathcal{P}$ and $\vec{\sigma} \in \Pi^n$

Randomly split all n players into k groups $X_1, \ldots, X_k$ where $|X_i| \in \{\lfloor n/k \rfloor, \lceil n/k \rceil\}$
for $i = 1, \ldots, k$ **do**
  $\tau_i \leftarrow f(\vec{\sigma}, X_i)$
  $\gamma_i \leftarrow n/|X_i|$
  Let $Z^{(i)} \in \mathbb{R}^{n \times n}$ where

$$Z^{(i)}_{a,b} \leftarrow \begin{cases} \frac{1}{\gamma_i} & \text{if } a \notin X_i \text{ and } \tau_i(b) = a \\ \frac{1}{\gamma_i(\gamma_i - 1)|X_i|} & \text{if } a \notin X_i \text{ and } \tau_i(b) \in X_i \\ 0 & \text{otherwise} \end{cases}$$

end for
$Z \leftarrow \sum_{i \in [k]} \frac{\frac{n}{|X_i|} - 1}{k-1} Z^{(i)}$
Sample a ranking σ such that a is ranked in position b with probability $Z_{a,b}$
return σ

Algorithm 2: k-PARTITE

Our goal is to prove the following theorem, which states the guarantees of k-PARTITE.

Theorem 5.6. *k-PARTITE is impartial, and, for every $f \in \mathcal{P}$ and $\vec{\sigma} \in \Pi^n$, if $k = \lfloor (n/\ln n)^{1/3} \rfloor$, it gives at most*

$$(4/k, 4/k) \in \left(O\left(\left(\frac{\ln n}{n} \right)^{1/3} \right), O\left(\left(\frac{\ln n}{n} \right)^{1/3} \right) \right)$$

backward error with respect to f.

Note that, in particular, the error goes to 0 as n grows. Turning to the proof, it is clear that the algorithm is impartial because of the inability of any player i to affect the i^{th} row of the Z matrix. We therefore focus on establishing the error bound. To this end, we first prove several lemmas.

Lemma 5.7. *If t players $X = \{x_1, \ldots, x_t\}$ are sampled without replacement from $[n]$ with input profile $\vec{\sigma} \in \Pi^n$, then*

$$\mathbb{P}\left[\|A(\vec{\sigma}) - A(\vec{\sigma}, X)\|_\infty \geq \varepsilon \right] < n^2 \exp\left(-\frac{t\varepsilon^2}{2} \right).$$

Proof.

$$\begin{aligned}
&\mathbb{P}\left[\left\|A(\vec{\sigma}) - A(\vec{\sigma}, X)\right\|_\infty \geq \varepsilon\right] \\
&\qquad \leq \sum_{i<j} \mathbb{P}\left[\left|A(\vec{\sigma})_{i,j} - A(\vec{\sigma}, X)_{i,j}\right| \geq \varepsilon\right] \\
&\qquad \leq \sum_{i<j} 2\exp\left(-\frac{t\varepsilon^2}{2}\right) = 2\binom{n}{2}\exp\left(-\frac{t\varepsilon^2}{2}\right) \\
&\qquad < n^2 \exp\left(-\frac{t\varepsilon^2}{2}\right),
\end{aligned}$$

where the first transition follows from the union bound, and the second transition follows from Hoeffding's Inequality. □

Lemma 5.8. *For every $f \in \mathcal{P}$, $\vec{\sigma} \in \Pi^n$, and $\varepsilon > 0$, k-*PARTITE *gives at most*

$$\left(1 - \left(\frac{k-2}{k-1}\right)\left(1 - n^2 k \exp\left(-\frac{\lfloor n/k \rfloor \varepsilon^2}{2}\right)\right), \varepsilon\right)$$

backward error to f.

Proof. Observe that

$$\begin{aligned}
&\mathbb{P}\left[\exists i \in [k] \text{ s.t. } \left\|A(\vec{\sigma}) - A(\vec{\sigma}, X_i)\right\|_\infty \geq \varepsilon\right] \\
&\qquad \leq \sum_{i=1}^{k} \mathbb{P}\left[\left\|A(\vec{\sigma}) - A(\vec{\sigma}, X_i)\right\|_\infty \geq \varepsilon\right] \\
&\qquad \leq \sum_{i=1}^{k} n^2 \exp\left(-\frac{|X_i| \varepsilon^2}{2}\right) \\
&\qquad \leq \sum_{i=1}^{k} n^2 \exp\left(-\frac{\lfloor n/k \rfloor \varepsilon^2}{2}\right) \\
&\qquad = n^2 k \exp\left(-\frac{\lfloor n/k \rfloor \varepsilon^2}{2}\right),
\end{aligned}$$

where the second inequality follows from Lemma 5.7.

Further observe that, by Lines 5 and 7 of k-PARTITE, for any player a he is placed

directly where one of the X_i places him with probability

$$\begin{aligned}\sum_{i\in[k]:\ a\notin X_i} \frac{\frac{n}{|X_i|}-1}{k-1}\frac{1}{\gamma_i} &= \sum_{i\in[k]:\ a\notin X_i} \frac{\frac{n}{|X_i|}-1}{k-1}\frac{1}{\frac{n}{|X_i|}} \\ &= 1-\frac{1}{n(k-1)}\sum_{i\in[k]:\ a\notin X_i}|X_i| \\ &\geq 1-\frac{1}{n(k-1)}\sum_{i=1}^{k}|X_i| \\ &= 1-\frac{1}{n(k-1)}n \\ &= \frac{k-2}{k-1}.\end{aligned}$$

Now, if for all $i \in [n]$, $\|A(\vec{\sigma}) - A(\vec{\sigma}, X_i)\|_\infty < \varepsilon$, and each player a is placed in the position that some X_{i_a} places him, then we can set $\tilde{A} = A(\vec{\sigma}, X_{i_a})$ for all $a \in [n]$ to satisfy the conditions of backward error. Moreover, these events are independent. We conclude that k-PARTITE gives at most

$$\left(1-\left(\frac{k-2}{k-1}\right)\left(1-n^2k\exp\left(-\frac{\lfloor n/k\rfloor\,\varepsilon^2}{2}\right)\right),\varepsilon\right)$$

backward error, as stated. □

Proof of Theorem 5.6. From Lemma 5.8 it suffices to show that if we have $\varepsilon = 4/k$,

$$1-\left(\frac{k-2}{k-1}\right)\left(1-n^2k\exp\left(-\frac{\lfloor n/k\rfloor\,\varepsilon^2}{2}\right)\right)\leq\frac{4}{k}.$$

Observe that

$$\begin{aligned}&n^2k\exp\left(-\frac{\lfloor n/k\rfloor\,\varepsilon^2}{2}\right) \\ &\quad\leq n^2k\exp\left(-\frac{(n/k-1)\varepsilon^2}{2}\right) \\ &\quad= n^2k\exp\left(-\frac{(n/k-1)(4/k)^2}{2}\right) \\ &\quad= n^2k\exp\left(\frac{8}{k^2}\right)\exp\left(-\frac{8n}{k^3}\right) \\ &\quad\leq n^2\left(n^{1/3}\right)\exp\left(\frac{8}{2^2}\right)\exp\left(-\frac{8n}{\frac{n}{\ln n}}\right) \\ &\quad= e^2n^{-17/3} \\ &\quad\leq n^{-2}.\end{aligned}$$

Thus we see that

$$\begin{aligned}
&1 - \left(\frac{k-2}{k-1}\right)\left(1 - n^2 k \exp\left(-\frac{\lfloor n/k \rfloor \varepsilon^2}{2}\right)\right) \\
&\quad = \frac{1}{k-1} + \left(\frac{k-2}{k-1}\right)\left(n^2 k \exp\left(-\frac{\lfloor n/k \rfloor \varepsilon^2}{2}\right)\right) \\
&\quad \leq \frac{1}{k-1} + n^{-2} \\
&\quad \leq 2/k + 2/k \\
&\quad = 4/k.
\end{aligned}$$

□

A natural question is why we insist on what appears to be a somewhat convoluted algorithm, instead of a more natural approach such as the impartial NAIVE-BIPARTITE, formally given as Algorithm 3. The reason is that this algorithm does not even guarantee tolerable *mixed* error in general. Indeed, consider $f \in \mathcal{P}$ that is defined as follows. Let $X \subseteq \{2, \ldots, n\}$ be the set of players such that at least one player ranks i above 1; return the ranking starting with the players of X ordered lexicographically, followed by the players of $[n] \setminus (\{1\} \cup X)$ ordered lexicographically, and player 1 inserted into position $\lfloor n/3 \rfloor$ overall (shifting appropriately). Now consider the input profile where i reports the ranking $(i, 1, 2, \ldots, i-1, i+1, \ldots, n)$. Then NAIVE-BIPARTITE will always return a ranking where player 1 is placed first or second—as he will always top his set. This means that the algorithm cannot even provide a mixed error of $(1/2, 1, 1/4)$.

input: $f \in \mathcal{P}$ and $\vec{\sigma} \in \Pi^n$

1: Randomly split the n players into two sets X and Y where $|X| = \left\lceil \frac{n}{2} \right\rceil$ and $|Y| = \left\lfloor \frac{n}{2} \right\rfloor$
2: $\tau_1 \leftarrow f(\vec{\sigma}, X)$ restricted to the players only in Y
3: $\tau_2 \leftarrow f(\vec{\sigma}, Y)$ restricted to the players only in X
4: σ interlaces τ_1 and τ_2, that is,

$$\sigma(i) \leftarrow \begin{cases} \tau_1\left((i+1)/2\right) & \text{if } i \text{ is odd} \\ \tau_2\left(i/2\right) & \text{if } i \text{ is even} \end{cases}$$

5: **return** σ

Algorithm 3: NAIVE-BIPARTITE

5.1.6 The Committee Algorithm

k-PARTITE demonstrates that there exist impartial rules that accurately imitate any $f \in \mathcal{P}$. Observe, however, that the algorithm is somewhat hamstrung by the fact that a player must

be (with high probability) ranked in *exactly* the location that a small perturbation of the input rankings would give.

To allow more flexibility, we focus on mixed error, and consider Committee, given as Algorithm 4. Intuitively, this algorithm selects a random committee $X = \{x_1, \ldots, x_k\}$, which then determines the entire ranking. First, for each committee member x_i, we determine their rank using only the rankings given by the remaining $k-1$ members. However, as directly placing each committee member in this fashion may cause collisions (i.e., multiple members may be assigned the same rank) we restrict placement of x_i to only the positions $i, i+k, i+2k, \ldots$ Specifically, we assign x_i to the closest such position to the rank given to x_i by the other committee members. There are then k of the n positions assigned. Second, the committee ranks all of the n players, and the non-committee members are placed in the order ranked by the committee in the remaining $n-k$ slots.

input: $f \in \mathcal{P}$ and $\vec{\sigma} \in \Pi^n$

Randomly select a subset $X = \{x_1, \ldots, x_k\} \subseteq [n]$
for $i = 1, \ldots, k$ **do**
  $c \leftarrow \arg\min_{j \in \{i, i+k, \ldots\}} \left| j - f\left(\vec{\sigma}, X \setminus \{x_i\}\right)^{-1}(x_i) \right|$
  $\sigma(c) \leftarrow x_i$
end for
$\tau \leftarrow f(\vec{\sigma}, X)$
$j \leftarrow 1$
for $i = 1, \ldots, n$ **do**
  if $\tau(i) \notin X$ **then**
    while $\sigma(j)$ is occupied **do**
      $j \leftarrow j + 1$
    end while
    $\sigma(j) \leftarrow \tau(i)$
  end if
end for
return σ

Algorithm 4: Committee

The algorithm yields the following guarantees

Theorem 5.9. Committee *is impartial, and, for every* $f \in \mathcal{P}$, $\vec{\sigma} \in \Pi^n$, *and* $\varepsilon > 0$, *if* $k = 1 + \frac{2}{\varepsilon^2} \ln\left(\frac{n^3}{\varepsilon}\right)$, *it gives at most* $(\varepsilon, \varepsilon, (k+1)/n)$ *mixed error with respect to* f.

Importantly, this theorem allows for an incomparable error to Theorem 5.6. That is, we can reduce the backward error so long as we are willing to take on some forward error. For example, setting ε appropriately gives at most $\left(n^{-2/5}, n^{-2/5}, 2/n + (34/5)n^{-1/5} \ln n\right)$ mixed error.

As in the case of k-partite, the impartiality of Committee is obvious, because the position of each committee member is only determined by other committee members, and the position of non-committee members is determined by committee members. The proof

of Theorem 5.9 therefore focuses on establishing the stated mixed error guarantee; it is relegated to the full version of the paper.

5.1.7 Experiments

Our theoretical results indicate that impartial rules like k-PARTITE and COMMITTEE are likely to yield rankings that are close, in a sense (backward error or mixed error), to the output of a given rank aggregation rule. In this section we investigate a more natural metric, which is beyond the reach of our theory, and empirically demonstrate that our rules perform well with respect to this metric, too.

In our experiments, we focus on the Kemeny rule (see Section 5.1.3), as it is defined via an optimization problem, so we can use its objective function as our measure. Specifically, we interpret the Kemeny rule as maximizing the number of *agreements* with the input rankings, that is, given $\vec{\sigma} \in \Pi^n$, it chooses a ranking $\tau \in \Pi$ that maximizes

$$\mathrm{Kem}(\tau, \vec{\sigma}) = \sum_{i=1}^{n} \left(\binom{m}{2} - d_{KT}(\tau, \sigma_i) \right). \tag{5.1}$$

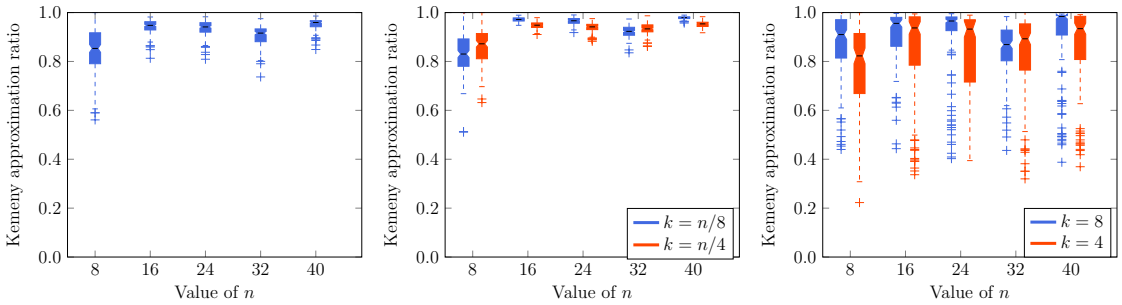
We quantify the error of an impartial rule by comparing how well it does with respect to measure (5.1) with the performance of the optimal ranking returned by the Kemeny rule. In more detail, let f be the Kemeny rule, and let g be an impartial rule; given an input profile $\vec{\sigma}$, we are interested in the *Kemeny approximation ratio* $\mathrm{Kem}(g(\vec{\sigma}), \vec{\sigma})/\mathrm{Kem}(f(\vec{\sigma}), \vec{\sigma})$; this ratio is upper-bounded by 1 due to the definition of the Kemeny rule. We use the number of agreements, instead of the number of disagreements, as our measure because in cases where the number of disagreements is very small, the ratio would be misleadingly large.

The input profiles are generated according to the popular Mallows model [166]. In this model, there is a base ranking of the alternatives τ^*, and rankings are drawn i.i.d. from a probability distribution over Π, defined by

$$\Pr[\sigma \mid \tau^*] = \frac{\phi^{d_{KT}(\sigma, \tau^*)}}{\sum_{\sigma' \in \Pi} \phi^{d_{KT}(\sigma', \tau^*)}},$$

for a *dispersion* parameter $\phi \in [0, 1]$. Note that $\phi = 1$ corresponds to uniformly random rankings (and therefore input rankings disagree on pairs of alternatives with probability $1/2$), whereas $\phi = 0$ means, by convention, that all rankings coincide with the base ranking, that is, there is unanimous agreement. We empirically study the Kemeny approximation ratio of the impartial rules NAIVE-BIPARTITE, COMMITTEE, and k-PARTITE, for multiple values of ϕ, each of which represents a different level of agreement.

Throughout our experiments, we let $k = n/4, n/8$ for COMMITTEE and let $k = 4, 8$ for k-PARTITE. The intuition behind these choices is that the size of the initial committee and each subset in the partition should grow with n, and choosing these values of k works reasonably in practice. We ran experiments with $n \in \{8, 16, 24, 32, 40\}$ players and $\phi \in \{0, 0.1, 0.3, 0.5, 0.7, 0.9, 1\}$.



(a) Naive-bipartite (b) Committee (c) k-partite

Figure 5.1: Kemeny approximation ratio of three impartial mechanisms for $\phi = 0.3$. The median of each boxplot is marked with a black line, the edges of each box denote the quartile values, and the whiskers extend to data within 1.5 times the interquartile range from the edges of each box.

Our results for the three impartial rules—Naive-bipartite, Committee, and k-partite–and $\phi = 0.3$ are shown in Figures 5.1a, 5.1b, and 5.1c, respectively; the results for other values of ϕ, which can be found in the full version of the paper, are qualitatively similar. In each figure, the x axis shows the values of n, and the y axis shows the Kemeny approximation ratio obtained by the impartial rule. All three impartial rules perform well as n increases. As a baseline, a straightforward calculation shows that with $\phi = 0.3$, a ranking drawn from the Mallows model would agree with the base ranking (which is typically the output of Kemeny when the input profile is drawn from Mallows) with probability 0.77 on any given pair of alternatives, and that probability is 0.5 if the latter ranking is replaced with a uniformly random ranking, so the (impartial) rule that chooses a ranking uniformly at random gives a Kemeny approximation ratio of $0.5/0.77 = 0.65$.

On a high level, the three impartial rules achieve excellent Kemeny approximations despite their very different theoretical guarantees. But k-partite has by far the highest variance. This phenomenon is due to the fact that if the position of a player is chosen from a setting in which he was not placed in his exact place as prescribed by players in some partition, he is essentially placed in a random location in the final ranking. In Naive-bipartite and Committee, this is not an issue, as players are always placed in some sense close to a position in which a subset of players believes they belong. As can be seen in the full version of the paper, this phenomenon is more pronounced for lower values of ϕ because placing players in a random location is penalized more heavily when the population is generally sharply clustered around a certain ranking.

Furthermore, the performance of our impartial algorithms depends on the specific choice of k (except for Naive-bipartite, which does not depend on any parameter k). The following observations can be seen in the full version of the paper. Committee performs better with the smaller value of k, but this effect lessens as ϕ increases (i.e., it helps more when players have generally similar opinions). With small ϕ, because most committees will agree on a consistent ranking, the additional error from inserting players into larger buckets leads to a noticeable difference. However, as players start to disagree more, the

benefit of getting better estimates from larger committees counteracts the insertion error. k-PARTITE acts similarly: the larger value of k leads to better performance for low ϕ, but this effect again lessens as ϕ increases. This is because each player is placed exactly where one of the groups places him with probability $(k-2)/(k-1)$, which increases with k, but being placed in one of these positions is most beneficial when players generally agree. As opinions become increasingly random and diffuse, groups disagree more strongly about where to place a specific player.

5.2 Impartial Ranking, in Practice: HirePeer

Expert crowdsourcing (e.g., Upwork.com) provides promising benefits such as productivity improvements for employers, and flexible working arrangements for workers. However, to realize these benefits, a key persistent challenge is effective hiring at scale. Current approaches, such as reputation systems and standardized competency tests, develop weaknesses such as score inflation over time, thus degrading market quality. In conjunction with the theoretical work described above, we develop *HirePeer*, a novel alternative approach to hiring at scale that leverages peer assessment to elicit honest assessments of fellow workers' job application materials, which it then aggregates using an impartial ranking algorithm [148]. We perform three studies that investigate both the costs and the benefits to workers and employers of impartial peer-assessed hiring. We find, to solicit honest assessments, algorithms must be communicated in terms of their impartial effects. Second, in practice, peer assessment is highly accurate, and impartial rank aggregation algorithms incur a small accuracy cost for their impartiality guarantee. Third, workers report finding peer-assessed hiring useful for receiving targeted feedback on their job materials.

5.2.1 Introduction

Expert crowdsourcing is on the rise. From 2009 and 2013, one of the largest platforms for expert crowdsourcing, Upwork.com (previously oDesk), witnessed an 800% increase in the number of paying employers [3]. Yet as more employers and workers move to expert crowdsourcing, a critical challenge remains: employers struggle to hire effectively and efficiently at scale. On Upwork, for instance, it takes employers approximately three days to screen, interview, and hire every candidate [223]. Relative to the duration of an expert crowdsourcing task, this cost in time and effort is enormous, encouraging employers to adopt a satisficing strategy (i.e., hiring workers who are "good enough" instead of finding the most qualified candidate overall) [216]. This cost also damages workers' prospects: when employers cannot confidently identify qualified applicants, they offer lower wages to offset their risk of low-quality results; such depressed wages consequently turn away qualified workers, or workers may respond to lower payment with lower quality work [207]. Indeed, in other markets, such large costs for hiring have been shown to dissuade employers from hiring workers entirely [212]. This may cause online crowdsourcing markets to degrade over time. This section investigates a new scalable method to hire expert workers quickly and accurately.

Perhaps the most widely adopted method today to address the large costs of screening applicants is reputation systems. These systems aggregate a candidate's prior task performance, as assessed by past employers, into a score. Although reputation systems are widely adopted by platforms, they bring with them their own set of challenges to effective hiring at scale, which worsen over time. For instance, online reputations become inflated over time: the (social) cost of giving negative feedback is higher than positive feedback [129]. As a result, norms shift over time, and reputation inflation worsens, reducing reliability.

While ongoing work continues to improve existing approaches to address some of these limitations, this section instead presents an entirely new approach to hiring at scale. Our approach is based on a widely used technique to address the need for accurate assessments of open-ended material at massive scale: peer assessment. To date, peer assessment remains the gold standard of review, as seen in its use to assess quality in top-tier academic conferences [229], grant reviewing [73], and more recently massive online classrooms [151]. We investigate: can crowd experts peer-assess each others' job materials to identify qualified candidates? Specifically, we investigate if peer assessment can generate a ranked list of all job applicants from which the employer can make final hiring decisions.

As might be apparent, the conflicts of interest that arise in a hiring setting are the central challenge in realizing this approach. Specifically, because all crowd experts applying to a task presumably would like to take the job, they have an incentive to rate other applications *strategically*, to make themselves look more attractive to the employer. This section describes a system, HirePeer, that overcomes these conflicts. Overcoming conflicts requires both algorithms that can aggregate judgments such that participants derive no benefit from strategic assessments (*impartial* algorithms), and a careful consideration of human-centered components of this process.

First, we investigate whether automatic impartial aggregation of worker assessments of open-ended work is necessary in real-world hiring settings with conflicts of interest. Our first study creates an environment within Amazon Mechanical Turk with conflicts of interest through carefully designed incentives. It then demonstrates the need for impartial algorithms, and the necessity of communicating the presence of such impartial algorithm to participants. We find an effective introduction does not need rely on explaining a complicated randomized algorithm, but rather on the psychology of choice. In a between-subjects randomized experiment ($n = 170$), we find a *consequence explanation* results in the least amount of strategic behavior [172]. On the other hand, we find communication based on a "policing" framing to be ineffective.

Second, we investigate HirePeer's real-world implications for employers. Importantly, we find peer assessment is feasible for hiring in expert crowdsourcing, with accuracies of more than 90% compared to non-conflicted expert judgments (such as those made by employers). We then examine the cost of impartial peer assessment by analyzing the accuracy of three impartial aggregation algorithms [136] and find that, in practice, impartiality comes at a small price. In a between-subjects randomized experiment ($n = 150$), we find impartial peer assessment, in a setting that utilizes the consequence explanation introduced in this section, results in a 8% decrease in accuracy compared to peer assessment where impartiality is not guaranteed.

Finally, we explore worker-oriented implications of peer-assessed hiring. Specifically we

look at, if, and how, expert crowdworkers might benefit from peer assessment and feedback. We conduct a case study to deploy HirePeer in a real-world expert crowd hiring setting, where crowd experts complete an open-ended, complex task. This case study suggests peer-assessed hiring benefits crowd experts by a) exposing them to how other applicants assembled resumés and applications, b) introducing them to new skills to develop in the future, and c) giving them targeted feedback on their job materials.

In short, this section has three contributions. First, it introduces peer assessment as a new, scalable, and accurate approach to hiring in expert crowdsourcing marketplaces, instantiated in a system *HirePeer*. Second, through a real-world deployment of three impartial mechanisms, it quantifies the tradeoff between guaranteeing impartiality and accurate ranking. Third, it presents a brief exploration of how workers may benefit from peer-assessed hiring.

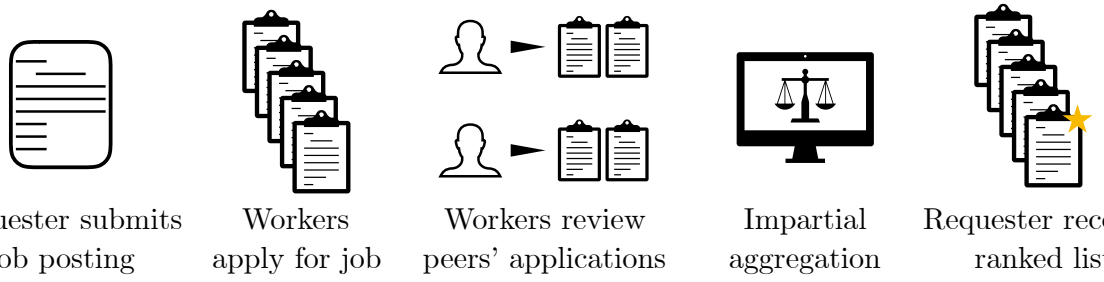


Figure 5.2: HirePeer's workflow of impartial peer-assessed hiring for expert crowdsourcing

5.2.2 Related Work

This section primarily draws on three bodies of literature: a) existing interventions for large-scale hiring on online marketplaces, b) online peer assessment in education, and c) impartial mechanism design.

Platform-specific reputation systems are perhaps the most widely-adopted approach to facilitate hiring in expert crowdsourcing. Although reputation systems are intended to signal worker trustworthiness and facilitate transactions between strangers, they suffer from reputation inflation [129]—eventually, employers almost always award high feedback scores to employees.

Peer review remains the gold standard for assessing open-ended materials, as evinced by its wide adoption in academia to judge paper submissions [229] and by the NSF to review grants [73]. More recently, *online* peer assessment has been introduced in educational settings; in both massive online open courses (MOOCs) and in large physical classrooms, peer assessment has proved to be an effective way to scale accurate assessments of open-ended complex work [70; 226]. However, applications of scalable online peer assessment outside of the classroom remain limited.

Realizing peer-assessed hiring requires careful consideration for how to effectively handle conflicts of interest at scale (all workers who apply to a task would like to be chosen

for the task). Recently, Kahng et al. presented three impartial[3] algorithms (called NAIVE-BIPARTITE, COMMITTEE, and k-PARTITE) which aggregate pairwise comparisons to generate a ranked list [136]. While all three impartial mechanisms have strong theoretical guarantees, we explore their performance in a real-world setting.

5.2.3 HirePeer: System description

A requester using HirePeer posts her task to the labor platform (e.g., Upwork) as usual. However, instead of applying to the job directly, workers who are interested in the task are notified to apply to the task on the HirePeer website (see Figure 5.2). When applicants have completed their job application, they are then asked to review a machine-selected set of other applications. To reduce inadvertent biases in evaluation, reviewing is double-blind [151]. Before workers start reviewing, they are notified their assessments will be aggregated with an impartial mechanism.

Because prior work shows pairwise comparisons encourage attention to non-superficial features and lead to more accurate assessment [63], workers conduct pairwise comparisons of peers' anonymized job materials. An expert-generated rubric for the specific task type guides evaluation—our current system has rubrics for web design, data visualization, etc. The rubric contains a) domain-specific criteria, b) more general criteria that are important in an expert crowdsourcing context like communication and timeliness of task completion, and c) qualitative textual feedback on job materials. An expert rubric allows us to collect accurate assessments from both novice and expert workers [54]. Feedback on application materials is later shown to both the task requester and to the applicant.

Once peer assessments have been collected, they are aggregated by the impartial mechanism. Importantly, our mechanisms aggregate assessments into a ranked list (rather than merely choosing a subset of qualified candidates). Armed with this ranked list and the qualitative feedback on each application, the requester can hire the best suited applicant on the crowdsourcing platform.

5.2.4 Study 1: Is an impartial algorithm necessary?

While there have been many theoretical papers on the design of impartial algorithms [87; 136], little work has been done on effectively communicating the presence of impartial algorithms to users. Such an introduction is not only important given increased calls for algorithmic transparency across the community, but also because participants may behave strategically (i.e., attempt to boost their own position) if they do not realize their assessments are aggregated impartially.

If participants behave non-strategically in general, then it may be unnecessary to communicate the impartial mechanism at all (in fact, the mechanism itself may be unnecessary except to thwart the occasional strategic behavior). But if participants engage in strategic behavior, it is important to investigate:

[3] A ranking mechanism is *impartial* if no participant can affect her position in the final ranking [136].

Research Question 1: For accurate assessments, should the presence of an impartial algorithm be communicated to participants?

If strategic behavior is commonplace, then communicating an impartial mechanism may discourage it if participants believe that strategic behavior has no benefit to them. It is likely that different ways of communicating impartial mechanisms may differ in their effectiveness at discouraging strategic behavior; so our study also investigates:

Research Question 2: Which framing of impartial algorithms best discourages strategic participant behavior?

Changing behavior without technical explanations

If impartial mechanisms are to be deployed widely to non-experts, it would be desirable for explanations to not rely on mathematical understanding. We consider two ways of doing so: a) by describing consequences, and b) by leveraging psychological theories of choice to nudge behavior. In particular, we leverage the effects of different "framings," or methods to describe the same situation, that emphasize different attributes. Different framings of game-theoretic tasks result in drastically different outcomes: Tversky and Kahneman found basic tenets of rational behavior can be violated with simple word changes in task instructions [222]. These results were later corroborated in diverse, real-world applications on Amazon Mechanical Turk [184]. Thus, we investigate whether using a framing approach is even more beneficial than describing potential consequences, as it not only it does not require participants to have knowledge of algorithms or mathematics, but also it relies on fundamental and systematic human biases.

Three Ways to Communicate Impartiality

We consider three different ways to communicate impartiality. First, we consider a *consequential* explanation. To discourage strategic behavior, we describe the consequences of using an impartial algorithm: "The ranking you generate will not affect the final aggregated ranking of your item as we use an impartial algorithm." Note that prior work suggests that such an approach may not completely prevent strategic behavior, but may reduce it. For example, Mazar et al. suggest that when consequences of "dishonest" (i.e., strategic) actions are well-known, such as while claiming exaggerated income tax exemptions, people only behave dishonestly to a small extent, as doing so allows them to preserve their positive self-image [172].

We also consider two framing-based approaches. First, we consider a *policing* approach, which is the most common technique in the related literature [57]. Participants in this condition were told, "To prevent you from cheating, we implemented an impartial algorithm." Second, we consider an *responsibility externalization* framing, based on Greenwald's theory of the totalitarian ego, specifically *beneffectance* [122]. This theory suggests while people perceive themselves to be responsible for desired outcomes (such as performing a kind act), but responsibility for undesired outcomes is externalized to others (e.g., traffic leading to aggressive driving). As such, this theory suggests participants see themselves to be honest, but may be concerned that others may behave strategically. Participants in this setting

were told, "For your protection, we prevent other workers from cheating using an impartial algorithm."

Participants and experimental setup

We conducted a between-subjects randomized experiment on Amazon Mechanical Turk (AMT) to test which of three communications of an impartial mechanism minimized strategic behavior compared to our control condition ($n = 170$). We used AMT as an experimental setting for two reasons: first, it can be challenging to discern strategic behavior from low quality work on AMT [131], providing a rich experimental setting to evaluate decision making; second, AMT is a representative sample of a typical online labor market, and has been shown to be a reliable environment for behavioral studies [170].

Participants were randomly assigned to one of four between-subjects conditions. The control condition made no mention of an impartial mechanism, and instead simply reminded participants to read instructions carefully (this has been shown in previous crowd work to have no effect). The other three conditions described the algorithm as above (with consequences, policing, or responsibility externalization). We displayed each in a reminder (in bold) at the bottom of the task instructions on AMT, depending on which condition a participant was randomly assigned. We also included this reminder a second time, immediately before the task.

Task structure and strategic behavior

The experiment used a simple task with known ground truth, to simplify evaluation, while still leaving room for well-defined strategic behavior.

Task We collected eight product reviews from Amazon for the bestselling mobile phone when this study was conducted: the Samsung Galaxy. The reviews were collected to have large differences in quality (the numbers of upvotes for the reviews differed by orders of magnitude). We then introduced typos into each review. Unbeknownst to the participants, all participants edited the same review across all conditions, which was at position #6 in ground-truth (where product review #8 was lowest in quality).

Participants were first asked to proof-read these reviews, and fix typos. Each participant then ranked eight product reviews from the Amazon product page (i.e., without introduced typos), and their edited review, in terms of quality. The product reviews, including their own, were presented to participants in order of true quality, measured by the number of upvotes on Amazon.

The task took at most 15 minutes, and participants were paid $10 USD per hour (before bonuses, described below).

Incentives for strategic behavior Participants were notified the rankings provided by all study participants would be aggregated (similar to peer-assessed hiring), and they would receive a bonus if their review landed in one of the top five positions in the aggregated ranking (a similar incentive structure to peer-assessed hiring). Specifically, the bonus structure was $5 USD if their review landed in position 1, $4 USD for position 2, and so on, and bonuses were awarded as promised. Because most workers in AMT's labor

pool participate to earn money, this task's incentive structure aligns with participant motivations, and is therefore an ecologically valid way to create a similar incentive structure to peer-assessed hiring [131]. Each participant edited the same review, compared it to the the same ground truth ranking of reviews, and had the same incentive to manipulate their report.

This incentive structure also allows for only one kind of strategic behavior: exaggerating the ranking for the edited review, by placing it above position #6. It also allows for a measure of strategic behavior: how much higher than position #6 they placed their review (as reviews differed in quality by orders of magnitude).

Comparison to peer-assessed hiring This task design has critical similarities to hiring. First, ranking edited reviews is similar to ranking job materials, e.g., resumes; and the ranking is similarly subjective, allowing for strategic interpretation. Similarly, there is a strong incentive to rank oneself higher.

The task differs from peer-assessed hiring in that participants are only comparing one artifact, instead of the multiple used in hiring, such as resumes, work experience, etc. Such a comparison would be even more subjective, but allows for similar strategic behavior. Second, our task has bonuses for even small strategic behaviors. The hiring scenario would be more analogous to having a very large bonus for position #1 (i.e., being hired), and vanishing bonuses for other positions. Our task design is necessary because we seek to measure the degree of strategic behavior.

Result: Need for introduction of impartial algorithm

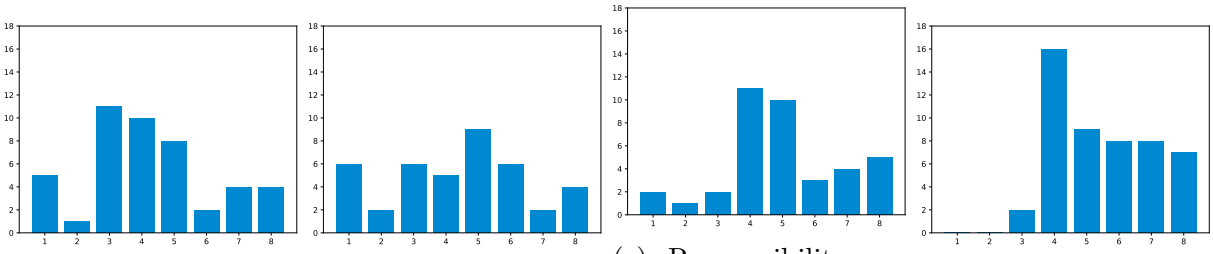
Participants spent a median duration of 9.5 minutes to complete this task. In the control condition, participants had a significantly lower average rank (mean = 4.2, ground truth = 6, $F(1, 166) = 15.3, p < 0.001$). In other words, control participants exaggerated their assessment by 30%, suggesting an impartial algorithm (and its effective communication) are necessary.

Result: Consequence explanation most effective

As shown in Figure 5.3, participants exposed to the consequence explanation exaggerated the ranking of their product review an average of 10% ($p < 0.01$), far less than the total possible, and lower than both the control and other framing-based explanations. This is similar to the reuslts of Mazar et al., where participants engaged only in limited strategic behavior when consequences were known.

5.2.5 Study 2: Is peer assessment for hiring accurate?

Study 1 demonstrated the need to communicate an impartial framing, and an effective way to do so. Study 2 investigates the real-world performance of impartial ranking. Impartial rank-aggregation methods guarantee their outcomes are resilient to strategic assessments (i.e., artificially inflating a worker's own position), but in theory, impartiality comes at a cost to accuracy [136]. This is because an impartial aggregation algorithm, by design,



(a) Control (b) Police (c) Responsibility externalization (d) Consequence

Figure 5.3: From Study 1, histogram of review placement for each framing condition; x: position, y: frequency. A skew to the right suggests less strategic behavior. Consequence explanation resulted in the least strategic behavior.

ignores some information (for instance, NAIVE-BIPARTITE disregards 75% of comparisons in expectation to ensure impartiality).

In practice, the effect on overall accuracy is context dependent. On the one hand, the final ranking may be more accurate if participants report more accurate assessments (because manipulation is no longer beneficial). However, if the strategic manipulation without such a mechanism is small enough, the loss of information may result in lower real-world accuracy. Furthermore, if participant outcomes are not dependent on their own assessments, some participants may put in less effort in creating accurate assessments. In this study, we investigate:

Research Question 3: Does peer assessment result in more accurate ranking of applicants in an expert hiring setting?

Research Question 4: What is the net cost in accuracy for impartial guarantees of ranked aggregation?

Coefficients	β	F	p-value
Intercept (control)	4.2667	15.336	<2e-16
Police	0.1083	0.267	0.78971
Responsibility Externalization	0.7333	1.783	0.07630
Consequence	1.2333	3.216	0.00156

Table 5.1: From Study 1, consequence description leads to the least amount of strategic behavior. β coefficients are the average difference in rank from control condition (positive is less strategic behavior).

Participants and recruitment

We conducted a two-condition between-subjects experiment on AMT with 50 participants per condition. Workers who had previously taken part in our studies were not allowed to participate. This study was conducted on AMT because the platform allowed us to readily hire a large number of workers, as required for our experimental design below.

Task structure

Study 1 used a simplified task structure to make strategic behavior readily apparent. This study uses our HirePeer system introduced before, and asks for multiple paired-comparisons, instead of a ranking task.

Since multiple comparisons can be composed into a (partial) ranking, the two tasks are similar in the strategic behavior they support. (However, we acknowledge that participants may not see as readily how best to behave strategically while comparing two artifacts created by peers.)

To simulate the hiring scenario, we wanted a "job" that most AMT workers would believe they were qualified for, and had subjective selection criteria but did not require specific domain skills. Furthermore, because AMT is a micro-task market where workers are not looking for long-term employment, we wanted tasks that did not require workers to commit to long-term work, yet offered a significant monetary reward.

Therefore, our task asks crowd-workers to write feedback to newcomers to AMT. This is a task that is subjective, does not require specialized domain skills, and is something expert AMT workers might believe they are qualified for. To ensure participants felt they were qualified, participants were required to have a Master's Qualification on AMT: an indication of consistently high quality performance and familiarity with AMT. Along with potential bonuses, the task paid up to $20, which is a significant monetary reward on the platform.

Task design: Participants were first asked to write several paragraphs of advice for AMT newcomers. The task instructions stated, "In your advice paragraphs, share tips on how to be successful, mistakes you made that you recommend they avoid, and other information you think a new Turker would find helpful."

Then, they assessed a randomly selected subset of other peers' work (i.e., their peers' advice). Concretely, at most four hours after the first phase, participants completed 50 randomly-generated pairwise comparisons among pieces of advice written by peers in the same condition, deciding which piece of advice in each comparison was higher quality (where quality was defined as more actionable and specific). Repeated pairwise comparisons were permitted (and outputs were used for quality control). At the end of both phases, participants were asked to complete a 13-question survey to understand perceptions of trust, fairness, and effort. We also captured how long they spent writing advice.

Incentive structure: Participants received a bonus if their advice piece placed in the top ten spots of the overall ranking, out of 50 total spots ($10 USD for position one, $9 USD for position two, and so on). There were two conditions. The *impartial* condition used the *consequence explanation* from Study 1. The control condition did not include

this explanation, and instead reminded participants to pay attention to instructions (as in Study 1).

Collecting ground truth: Ground-truth ranking for each condition's advice was generated by asking 50 non-conflicted workers—25 per condition—to compare pieces of advice. This is similar to ground-truth collection in other peer assessment evaluations [151]. Non-conflicted participants were both Master Turkers and completed over 10,000 accepted human intelligence tasks (HITs) to establish a high level of expertise in the task. Non-conflicted participants conducted 50 pairwise comparisons for which piece of advice (generated from conflicted participants) was of higher quality, where quality was defined as actionable and specific. All non-conflicted participants evaluated the same 50 comparisons to generate ground-truth. Note that this method yields *ground truth comparisons*, rather than a ground-truth ranking. Ranking the 50 pieces of advice would be a prohibitively time- and effort-intensive task.

Data analysis

First, we read all responses to ensure they were sensible; all but three responses across conditions were grammatically correct and included actionable advice. These responses were kept for the following analysis. The quality of advice was similar across conditions: 1,044 characters in control vs. 1,143 characters in impartial; length is correlated with quality [150]. Median time spent writing advice (9.5 minutes control vs. 6.5 minutes impartial) did not differ in a statistically significant manner. This suggests no differences in participant recruitment across conditions.

To create rankings, we used jackknife resampling, similar to other peer assessment evaluation work [151]. In each condition, first we chose 35 of the 50 conflicted participants without replacement and sampled 25 of their pairwise comparisons, also without replacement. Because impartial algorithms are *randomized*, we ran each impartial rank-aggregation algorithm 50 times on each set of assessments to capture the variability of results. Similarly, we repeated the process of choosing participants and assessments 25 times for each condition to capture the variability caused by choosing particular assessments. This process as a whole resulted in 1250 bootstrapped rankings across conditions. We then used bootstrap significance tests introduced by Politis and Romano [190] for accuracy measures.

To evaluate the accuracy of our ranking mechanisms, we measured the agreement between the complete ranking output by each mechanism and the non-conflicted comparisons. First, given the output of a ranking mechanism, we extracted the 50 pairwise comparisons seen by non-conflicted participants from the output of the peer assessment process. Then, we assigned the output a score that measures how well the ranking agrees with the non-conflicted comparisons. The score is equal to the total number of non-conflicted participants who agree with the relative ordering of the 50 pairwise comparisons in the output ranking divided by the total number of non-conflicted participants in the majority opinion for all 50 pairwise comparisons. Note that the score is calculated relative to the majority of non-conflicted participants; this allows us to penalize mechanisms less for confusing the order of alternatives that non-conflicted participants are less sure about (i.e., which have

only a slim majority among expert opinions) and to penalize mechanisms more for disagreeing with the order of alternatives that non-conflicted participants heavily agree with (i.e., alternatives with a solid majority consensus among non-conflicted participants).

Result: Peer assessment with conflicts of interest is accurate

To generate rankings without guaranteeing impartiality, we used the Kemeny rule [140], a standard method to generate rankings from an incomplete set of comparisons. Overall, the aggregated peer assessed ranking was highly similar to non-conflicted participant judgements. Even without aggregating peer assessments in an impartial manner, the accuracy was 96.6% using our metric above; see Table 5.2. This suggests peer assessed hiring could form the basis for scalable expert hiring.

Result: Guaranteeing impartiality leads to a modest loss in ranking accuracy

We compared the performance of the Kemeny rule with no framing (the *control* condition) to rankings generated from data from the *impartial* framing condition with impartial aggregation. The accuracy of ranked aggregation decrease by 8% (96.6% in control/non-impartial, vs. 88.8% in impartial); see Table 5.2. In other words, the theoretical guarantees of impartiality come at a cost of 8% in accuracy in our experimental setup.

What is an 8% loss in practice? If non-conflicted participants generating ground-truth assessments are 75% in agreement on average, as was the case in our study, and perform 20 comparisons each, then with 20 candidates a 6.67% loss in accuracy corresponds to two switches in the true ranking (e.g., switching candidates in the 10th and 11th position with each other, and the third and fourth positions with each other), and a 10% loss is equivalent to three such switches. Depending on the stakes, this loss in accuracy (and the resulting increase in employer time to hire) may be acceptable.

Aggregation Mechanism	Average Accuracy
Kemeny	0.9665*
NAIVE-BIPARTITE	0.8884
COMMITTEE	0.8044
k-PARTITE	0.7831

Table 5.2: From Study 2, (NAIVE-BIPARTITE) aggregation led to a reduction of accuracy by 8%, as compared to aggregation of assessments from control condition with the Kemeny rule; each entry represents average accuracy for each condition and related aggregation. All other rows represent aggregations of assessments from experimental (i.e., impartial) conditions.

Question	Average Likert Score
I will make changes to resumé	4.6
I will learn a new skill	3.6
The feedback helped me	5.0
I put in effort	4.2
I was honest	4.8
My peers put in effort	3.6
My peers were honest	4.0
My effort affects my ranking	4.0
I enjoyed the process	5.0

Table 5.3: From Study 3, average Likert scores from post-use survey; 1: strongly disagree, 5: strongly agree. Even in a competitive hiring setting, expert crowd workers perceived peer assessment to be helpful, enjoyable, and were inclined to iterate on their job materials.

Result: Consequence explanations catalyze beliefs that assessment effort is unrelated to final ranking

Participants in the impartial condition were significantly more likely to believe their effort did not impact the final ranking of their advice piece (Control median: 4, Impartial median rating: 2, 7-point Likert scale; Wilcoxon $Z = 612.5$, $p < 0.01$) (No other survey responses differed significantly across conditions). This is interesting because the impartial framing makes no mention of how effort affects ranking. In fact, to be effective, the impartial mechanism relies on worker assessments to be honest and effort-full. It seems likely that because of this belief, participants in the impartial condition put in less effort into comparisons, slightly decreasing accuracy.

In sum, Study 2 suggests peer assessment is an accurate alternative to hiring based on expert assessment. The benefits to employers, such as decreased time to hire, and lesser reliance on worker reputations are potentially enormous. Employers can also guard themselves against individual strategic assessment at a small cost (8%) to accuracy. Next, we turn to how peer-assessed hiring may affect workers.

5.2.6 Do workers benefit from peer-assessed hiring?

In the classroom, peer assessment improves students' self-reflection [151], iteration on work [152], and development of criteria for goodness that are better aligned with experts [63]. Do these benefits transfer to workers in peer-assessed hiring? Furthermore, what reactions do expert crowd workers have to peer-assessed hiring more generally? In short, we investigate:

> **Research Question 5**: What benefits of peer assessment in education transfer to peer-assessed hiring?

To address this research question, we conducted a case study for hiring on Upwork.com; an expert crowdsourcing platform for programmers, designers, and other expert professions. Note this case study is meant to be suggestive, rather than evaluative. If participants

reported none of the benefits of classroom peer-assessment, then this may not be an aspect to study further in future work. On the other hand, if participants reported some benefits (as we found), these findings may better inform and focus further research. First, to inform the design of this study, we ran two small pilots: hiring for a data visualization project and a Django development task. For this present case study, we hired expert crowd workers for the task of creating a banner ad for one of our research group's software tools, and included details about this study alongside the job description. Eleven Upwork professionals applied to this task. We describe results from the five participants who completed every stage in our protocol. We acknowledge that because of the high attrition rate, collected feedback may be biased.

Consenting participants submitted their anonymized applications to HirePeer (witnessing the impartial framing). Then, they conducted three randomly generated pairwise comparisons among their peers' job application materials. Since our system asks for comparisons, we modified the comparison-based user interface developed by Cambre et al., to ensure that assessment was scaffolded effectively [63]. After submitting these comparisons, each participant filled out a post-use survey similar to Study 2 to gather their feedback on HirePeer. The survey consisted of Likert questions to measure perceptions of effort and truthfulness of both themselves and their peers and free-response questions about overall experiences from the process.

Additionally, workers were rewarded for ranking their peers, and we ran impartial algorithms on their comparisons in order to select a winner who was invited to the task and paid for it separately.

Result: Feedback generation and reception helpful to identify new skills and improve job materials

Consistent with peer assessment literature in the classroom, multiple participants stressed the peer assessment process made them more mindful about writing a coherent and convincing application [204]. One participant stated HirePeer "helped me a lot to organize my mind and write the right things," and another wrote HirePeer "was a good exercise in application writing." Interestingly, all participants were receptive to feedback received from peers (selection bias may factor into this). Concretely, participants reported they "liked comparing proposals,", that "receiving feedback of other freelancers is a great one", and also noted no other platforms integrate this feature. One participant reported "topics that were included on the proposal [peer's resumés]...helped me a lot." Additionally, participants were slightly more likely to want to learn a new skill after this process (table 5.3).

Result: Not all participants completed assessment

Five of the 11 participants completed all steps of the review process and the post-use survey. This attrition rate is similar to peer-assessment in large-scale online courses [149], yet could be explored in more detail in future work. In addition, while our sample size is too small to draw statistical conclusions, participants who did complete our task "somewhat agreed" their effort did in fact impact their final placement (average Likert 4.0). We explore the

emergent relationship between effort and impartiality in the discussion section, and how future work might rigorously investigate this.

5.3 Conclusions

5.3.1 Impartial Peer Ranking, in Theory

Pairwise comparisons.

In practice, players would often be asked to compare pairs of players, rather than giving a complete ranking. Crucially, our results seamlessly extend to that setting. Indeed, standard measure concentration inequalities show that a small number of random caparisons are sufficient to accurately estimate the pairwise comparison matrix (see, e.g., Procaccia and Shah 2016). Because we focus on pairwise rank aggregation rules, this means that the input to the rule is qualitatively unaffected by the transition from complete rankings to pairwise comparisons.

Strong impartiality.

A natural notion of impartiality that is stronger than ours (call it *strong impartiality*) requires that a player would not be able to affect the *subset* of players that are ranked above himself. In the context of online labor markets, for example, the rationale is that an employer is more likely to select the applicant in position k if the applicants in positions $1, \ldots, k-1$ are relatively weak, so it is not just the applicant's position that determines his chances of getting the job. Unfortunately, strong impartiality seems too stringent to admit reasonable rules. In fact, we can prove that no strongly impartial rule can give a $(1/3, 1/7, 1/10)$ *mixed* error with respect to Borda, but this statement requires the additional assumption that strongly impartial randomized rules are distributions over strongly impartial deterministic rules.[4] We believe that a similar statement holds without the additional assumption.

Flipping the quantifiers.

Our definitions of backward error and mixed error include the words "for all $i \in [n]$ there exists a matrix $\tilde{A} \in \Omega$." That is, for each player we can find a pairwise comparison matrix close to the original one, such that our rule puts i in a position that is identical or close to that in which the given rule would put i on the input $\tilde{A}$. The definitions would be even more compelling if the quantifiers were flipped, i.e., "there exists a matrix $\tilde{A} \in \Omega$ such that for all $i \in [n]$." Under this alternative formulation, we are not allowed to tailor $\tilde{A}$ to i, but rather there is one pairwise comparison matrix that achieves the desired property for every i. It remains an open problem whether our theorems (or variants thereof) still hold under

[4]Note that Committee and Naive-bipartite can be represented as distributions over impartial (not strongly impartial) deterministic rules, but k-partite cannot.

these more demanding notions of error, and, if not, whether these notions are feasible at all.

5.3.2 Impartial Peer Ranking, in Practice: HirePeer

Peer-assessed hiring in expert crowdsourcing is a novel alternative approach to hiring that is likely to engender many emergent effects that future work could investigate.

Practical peer-assessed hiring of experts

Even in the conflicted setting of hiring, we found scalable peer assessment can be accurate. While Study 1 shows that workers are likely to inflate their own assessment without impartial framing, Study 2 shows that the aggregated assessment of peers is highly correlated with non-conflicted expert assessors, even without using impartial aggregation. With such high agreement (96%), it seems reasonable to suggest that peer-assessed hiring can offer an alternate, scalable method to hiring crowd-experts. In particular, peer-assessed hiring can even empower non-expert employers to accurately hire qualified employees.

Collusion and privacy concerns

This work is limited in its notion of strategic behavior: although impartial mechanisms ensure any participant cannot affect her final position, it is still possible to manipulate the order of *other* applicants by reporting strategically. For instance, a coalition of workers (e.g., friends) could collectively manipulate their final placement by always selecting each others' proposals. Future work may investigate mechanisms that are resilient to collusion in their guarantees.

Another salient concern is that of anonymity. When the pool of applicants is small enough, participants may be able to identify competitors from their de-identified profiles. However, these concerns are less applicable to the expert crowdsourcing space, where the applicant pool is large, and typically has no means of communicating with each other.

Amplifying pedagogical benefits of peer review in hiring

We present initial observations that peer assessment benefits from the classroom may transfer to expert crowdsourcing. Future work may incorporate several existing interventions to improve feedback quality, such as providing tiered rubrics and banks of exemplar feedback to reuse. Furthermore, while the small sample for the case study allowed initial, qualitative observations, future work could study these benefits at larger scale with a more diverse population and investigate if pedagogical benefits evolve over time: if a crowd expert is not selected for a job, can peer-assessed hiring help them land the next job?

Incentivizing Effort

Importantly, impartial algorithms do not necessarily incentivize effort on the part of participants. Because any agent's report does not affect the distribution of her ranking, purely

self-interested agents do not have any incentive to put effort into evaluating pairwise comparisons and could, theoretically, be just as well off reporting random noise as their true opinions. We did not observe this phenomenon in our (small) pilot study, and agents in general invested effort into their comparisons, but this requires further study.

One potential approach to incentivizing effort in impartial mechanisms is to offer small monetary bonuses for accurate comparisons, where accuracy is evaluated based on a small set of ground-truth comparisons performed by an impartial and knowledgeable party. As long as these monetary bonuses are not too large (i.e., the maximum bonus is less than the perceived difference between positions in the final ranking), agents will be incentivized to put in effort and impartial guarantees would still hold. However, implementing this scheme requires additional funds on the part of the mechanism designer as well as specialized expertise on the part of the "spot-checker."

Democracy cannot be static. Whatever is static is dead.

Eleanor Roosevelt.

6 Conclusion

The word "democratize" has multiple meanings. One potential definition is "introduce democratic principles to;" another is "make broadly accessible." In this thesis and beyond, I hope to conduct research that encapsulates both of these meanings. This involves exploring how computer science can help make democracy more efficient, fair, and robust; it also includes incorporating democratic ideas in new settings in order to address difficult problems in computer science and beyond.

Broadly, all projects presented in this thesis can be thought of as partial answers to the question, "How can we give people more direct and provably robust control over decision-making processes?" For instance, liquid democracy allows voters to choose delegates whose opinions, at least in theory, align more directly with their own, participatory budgeting allows citizens to allocate a portion of their city's budget toward public projects they deem to be personally important to them, and multiwinner elections allow citizens to elect more broadly representative committees instead of choosing one person to represent a heterogeneous electorate. Virtual democracy and impartial peer ranking each provide a framework for incorporating peoples' opinions in arenas where they were previously ill-suited due to a lack of infrastructure or misaligned incentives.

The technical contributions throughout this thesis can be viewed as an attempt to design democratic systems with theoretical properties that align with traditional desiderata in computer science. For liquid democracy, we explore how an increased flexibility of delegation changes the robustness and weight distribution of the voting system; for participatory budgeting, we examine how to ensure fairness and efficiency in a more flexible budgeting system; in the case of multiwinner elections, we attempt to ensure proportionality while allowing for variable-sized committees that are responsive to voters' opinions. Likewise, for virtual democracy, we seek to find a voting rule that is robust to the sort of machine learning errors that are inevitable when trying to predict peoples' opinions, and for impartial peer ranking, we design a system that balances impartiality and accuracy in

order to convince rational and self-interested parties to report actionable information for peer ranking.

It is also important to note that none of the democratic paradigms exist in a vacuum. While we have focused our attention on each individually, there is significant room for synergistic interactions among them. For instance, using liquid democracy in conjunction with participatory budgeting could allow for additional flexibility in allocating funds toward public projects, and the general approach central to virtual democracy—namely, learning models of people and letting the models vote—is broadly applicable in many different contexts where voting is expensive but individual opinions are the best source of information, which is a theme in participatory design in artificial intelligence.

Limitations and Future Work

However, there exists a gap between theory and practice, and the work in this thesis is not yet implementable in the real world. Although our work in virtual democracy and impartial peer ranking has a significant HCI component that shows promise, further work must be done in order to implement and deploy our results. Part of this divide stems from the assumptions necessary for theoretical approaches. In order to prove theorems, we must assume a well-defined model of human behavior and identify clear objectives to optimize; in practice, humans are often irrational, do not have clear objective functions in mind, or may not even know their own utility function for various alternatives. However, theoretical results are still a useful tool for directional analysis, and I hope to continue working on theoretical projects that will have a tangible impact on the world.

In a sense, there is also a gap between theory and theory. Computational social choice provides a useful and robust theoretical framework in which to study human decision-making that intuitively operates in two stages. In the first stage, voters are allowed to express preferences over a predetermined, fixed set of alternatives in a specific format (e.g., approval votes, complete rankings, or partial rankings). In the second stage, a predetermined, fixed mechanism aggregates the reported preferences and produces an output. However, computational social choice assumes that voters' utilities are fixed throughout the process (i.e., it cannot take deliberation into account), provides no guidance about how the set of alternatives is to be chosen, and only allows voters to communicate their opinions to the mechanism in very specific formats. While some simplification is necessary in order to obtain tractable and clean mathematical results, the scope of computational social choice is quite limited. In particular, I am particularly interested in exploring models of deliberation that take into account the dynamics of changing opinions based on social interaction among voters, which traditional computational social choice approaches does not encapsulate.

Finally, I would like to stress that the distinction between using computer science to help democracy and using democracy to help computer science is not, by any means, well-defined, and the topics discussed in this thesis are by no means an exhaustive set of topics at the intersection of democracy and computer science. Rather, there are many other democratic processes and problems in computer science that would benefit from

cross-pollination, and I am excited to continue exploring this line of work in the future.

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