

SPAM Tolerance for Pauli Error Estimation

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CMU-CS-25-149

December 2025

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*Submitted in partial fulfillment of the requirements
for the Master's degree in Computer Science.*

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Keywords: Quantum learning, Pauli channel, Population Recovery

To my brother, Vivikt.

Abstract

The Pauli channel is a fundamental model of noise in quantum systems, motivating the task of Pauli error estimation. We present an algorithm that builds on the reduction to Population Recovery introduced in [FO21]. Addressing an open question from that work, our algorithm has the key advantage of robustness against even severe state preparation and measurement (SPAM) errors. To tolerate SPAM, we must analyze Population Recovery on a combined erasure/bit-flip channel, which necessitates extending the complex analysis techniques from [PSW17, DOS17]. For n -qubit channels, our Pauli error estimation algorithm requires only $\exp(n^{1/3})$ unentangled state preparations and measurements, improving on previous SPAM-tolerant algorithms that had 2^n -dependence even for restricted families of Pauli channels. We also give evidence that no SPAM-tolerant method can make asymptotically fewer than $\exp(n^{1/3})$ uses of the channel.

Acknowledgments

I would like to express my deepest gratitude to my advisor, Ryan O'Donnell, for nurturing my interest in TCS and quantum computation, introducing me to the world of research, and guiding me every step of the way. I am truly grateful for his constant encouragement and the clarity and perspective he brings to every discussion.

I am incredibly grateful to David Woodruff for serving on my committee and for providing invaluable mentorship and insight over the past year.

Finally, I would like to thank my parents, sister, and brother for being my greatest source of inspiration and strength.

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Chapter 1

Introduction

Quantum benchmarking, or the task of characterizing the noise of a quantum device, is a fundamental challenge in building reliable, scalable quantum systems in the Noisy Intermediate-Scale Quantum (NISQ) era [EHW⁺20]. Although full process tomography provides a complete description of an unknown noise channel, its exponential complexity makes it impractical for large-scale systems. As a result, there is strong interest in noise models that can be learned efficiently while accurately capturing noise in experimentally relevant settings.

One of the most widely used theoretical models for noise is the n -qubit *Pauli channel*. This mixed-unitary channel applies n -qubit Pauli operators to an input n -qubit quantum state according to a fixed probability distribution, referred to as the *Pauli error rates*. *Pauli error estimation* is the task of learning the Pauli error rates of an unknown channel to precision ϵ in some metric. Since there are 4^n Pauli error rates, yet we seek subexponential complexity, a natural choice for the precision metric is ℓ_∞ . Under this metric, it suffices to find and estimate the largest $1/\epsilon$ Pauli error rates.

Pauli channels arise naturally as effective noise models in a variety of settings. More importantly, the *randomized compiling* technique introduced in [WE16] converts general quantum noise to a Pauli channel, reinforcing the central role of Pauli noise in benchmarking protocols. Beyond characterizing noise in experimental devices, Pauli error estimation has broad applications in quantum error correction and fault tolerance; its importance for optimizing quantum codes and decoders, as well as for tailoring fault-tolerant schemes, is discussed in greater detail in [FW20].

Given the significance of Pauli error estimation in diagnosing and modeling errors in present-day experimental setups, we are practically motivated to design algorithms that are experimentally simple to implement. In particular, we focus on protocols that do not involve entangling operations: that is, algorithms that prepare unentangled states, pass them through the channel, and perform unentangled measurements, followed by classical postprocessing. Moreover, for relevance to real-world quantum implementations, it has long been considered desirable [KLR⁺08] to develop algorithms that are robust against *state preparation and measurement* (SPAM) errors. See, e.g., [CLO⁺23] for more on practical aspects of Pauli channel estimation in the presence of SPAM. Developing an algorithm for Pauli error estimation that is both resource-efficient and SPAM-robust remains an important challenge and is the focus of this work.

1.1 Our result

In this work, we give the first entanglement-free Pauli error estimation algorithm that is robust to arbitrary SPAM errors and achieves subexponential dependence on n . Here we briefly describe the setting (see Chapters 2 and 3 for more complete definitions), and then state our main theorem.

Setup: the channel. We assume access to an n -qubit Pauli channel \mathcal{E} with unknown error rates π . That is, π is a probability distribution on $\{I, X, Y, Z\}^n$, and \mathcal{E} may be modeled as drawing $P \sim \pi$ and then applying the n -qubit Pauli unitary P .¹ In fact, for a completely general n -qubit channel, there is a natural notion of its ‘‘Pauli error rates’’, arising from randomized compiling [WE16]. Our algorithm can easily be extended to learn Pauli error rates in this more general setting too; see Remark 3.0.5.

Setup: SPAM. We only consider algorithms that employ single-qubit state preparation and single-qubit measurements. Moreover, we assume these are subject to SPAM errors, governed by ‘‘retention parameters’’ $r_{\text{prep}}, r_{\text{meas}} \in [0, 1]$, known to the algorithm (via prior calibration). As discussed in Chapter 3, several different natural operational interpretations of SPAM are all mathematically equivalent to the following model: after a state is prepared, it is passed through the depolarizing channel $\rho \mapsto r_{\text{prep}} \cdot \rho$; and, before it is measured, it is passed through the channel $\rho \mapsto r_{\text{meas}} \cdot \rho$. In particular, $r_{\text{prep}} = r_{\text{meas}} = 1$ corresponds to no SPAM.

We may now state our main result:

Theorem 1.1.1. *In the above setup, let $r = r_{\text{prep}} \cdot r_{\text{meas}}$ be the overall SPAM retention parameter, and write $r = 1 - \delta$, which may be thought of as the overall rate of SPAM. Let a precision parameter $0 < \epsilon \leq 1/2$ be given, and assume $\delta \leq .99$. Then there is an algorithm with the following properties:*

- For $m = \begin{cases} \exp\left(O\left((\delta n)^{1/3} \ln^{2/3}(1/\epsilon)\right)\right) & \text{if } \delta \gg \frac{\ln(1/\epsilon)}{n} \\ \text{poly}(n/\epsilon) & \text{if } \delta \ll \frac{\ln(1/\epsilon)}{n} \end{cases}$, it prepares m unentangled n -qubit states, where each qubit is chosen uniformly at random from $\{|0\rangle, |i\rangle, |+\rangle\}$.
- After passing these through the channel, it performs unentangled measurements on the resulting qubits, in only the $\{|0\rangle, |1\rangle\}$, $\{|i\rangle, |-i\rangle\}$, and $\{|+\rangle, |-\rangle\}$ bases.
- After efficient classical postprocessing, it returns an estimate of the Pauli error rates $\tilde{\pi}$ such that $\|\tilde{\pi} - \pi\|_\infty \leq \epsilon$ with high probability.

Notably, even in the presence of *low* SPAM error, which corresponds to the case where $\delta \ll \frac{\ln(1/\epsilon)}{n}$, our algorithm achieves sample complexity and runtime $\text{poly}(n/\epsilon)$. This matches, up to polynomial factors, the bounds previously established for the SPAM-free setting. The algorithm presented in [FO21], which assumes no SPAM error, has overall runtime $O(n \log(n/\epsilon) \cdot \epsilon^{-3})$ and sample complexity $m = O(\epsilon^{-2}) \log(n/\epsilon)$.

¹We use **boldface** to denote random variables.

We remark that although m is subexponential in n our algorithm, a dependence of $\exp(n^{1/3})$ seems reasonably plausible in real-world settings with $50 \leq n \leq 1000$, whereas prior work’s dependence of $2^{\Omega(n)}$ is infeasible in such settings. Moreover, as we discuss below, there is strong evidence that *any* Pauli error estimation algorithm that tolerates even a tiny amount of SPAM must use the channel on the order of $\exp(n^{1/3})$ times.

1.2 The connection with Population Recovery, and the difficulty of SPAM

Let us first recall how the Pauli error estimation problem would be solved if: (i) entangling operations were allowed; (ii) perfect state preparation and measurement is assumed. In this case, there is a simple algorithm based on superdense coding [BW92]: Prepare n Bell pairs in registers A_1B_1, \dots, A_nB_n , pass $A_1 \cdots A_n$ through the channel, and then measure each A_iB_i pair in the Bell basis. It is easy to show that if the channel applies n -bit Pauli operator \mathbf{P} (which it does with probability $\pi(\mathbf{P})$), then $\mathbf{P} \in \{I, X, Y, Z\}^n$ can be perfectly “read off” from the measurement outcomes. Thus with entangling operations allowed and no SPAM, the problem reduces to a classical task: learning an unknown probability distribution on $[4]^n$ to ℓ_∞ -precision ϵ , which is easily done with $O(1/\epsilon^2)$ samples [Can20].

Suppose now that we consider a more realistic setting, with a slight amount of state preparation or measurement noise (or both). In this setting, *even if we allow entangling operations*, the problem becomes much more difficult. For example, suppose we are even *promised* that the Pauli channel is only supported on $\{I, X\}^n$ Pauli strings. In this case, the superdense coding strategy is unnecessary: without SPAM, we could simply repeatedly pass $|0\rangle^{\otimes n}$ through the channel, measure in the $\{|0\rangle, |1\rangle\}$ basis, and be reduced to the classical task of learning an unknown probability distribution on $\{0, 1\}^n$. But suppose there is some small constant probability $p > 0$ that each measurement outcome is flipped (with a similar observation being possible for state preparation error). Then we precisely have an instance of the classical *Population Recovery* problem [DRWY12, WY16]: learning a probability distribution on $\{0, 1\}^n$ when the samples are passed through a binary symmetric channel with bit-flip probability p . For this task it is known [PSW17, DOS17] that *any* algorithm that succeeds must use at least $\exp(\Omega(n^{1/3} \ln^{2/3}(1/\epsilon)))$ samples. Thus even in this essentially classical version of the Pauli error estimation problem, any slight positive rate of SPAM error means that one cannot asymptotically improve on the “ m ” in our Theorem 1.1.1.

1.3 Prior work, and our improvements

A main prior work on SPAM-tolerant Pauli error estimation is by Flammia and Wallman [FW20]. Among other results, they gave a SPAM-tolerant Pauli error estimation algorithm that succeeds in the harder task of estimating the Fourier spectrum of the Pauli error rates. Its total complexity, however is $\tilde{O}(\frac{1}{\Delta} \cdot 2^n / \epsilon^2)$, where Δ is the “spectral gap”. In addition to having fully exponential

complexity in n , the dependence on Δ may be unfavorable, as this quantity may be arbitrarily small, or even 0 for some simple channels. Some improvements to this result were made in [CZSJ22], at the cost of adding entanglement with n ancilla qubits into the protocol.

By contrast, our approach builds on the subsequent paper by Flammia and O’Donnell [FO21], which was the first to make a connection between Pauli error estimation and Population Recovery. This work gave a novel method, using *no* entangling operations, that in the SPAM-free setting reduced Pauli error estimation to another form of Population Recovery: learning a probability distribution on $\{0, 1\}^n$ in which the samples are passed to the learner through a “Z-channel” with crossover probability $\frac{1}{3}$ (see Definition 2.2.2). Z-channels had not previously been studied in the context of Population Recovery, but it was observed in [FO21] that prior Population Recovery algorithms for the binary erasure channel apply equally well for the Z-channel. By using such prior algorithms [DRWY12, MS13], Flammia–O’Donnell obtained the same result Theorem 1.1.1 but with two differences:

- they used the channel only $m = O(\log(n/\epsilon)/\epsilon^2)$ times;
- however, they assumed no SPAM errors.²

In this work, we first show that for the very general SPAM model we allow in Theorem 1.1.1, the Pauli error estimation problem can be reduced to yet another variant of Population Recovery: namely, one that *combines* $\frac{1}{3}$ -rate Z-channel noise with δ -rate BSC noise. This kind of channel is more general than all the ones previously studied for Population Recovery. Then, the main effort is to give a Population Recovery algorithm for this general channel; for this, we need to extend the approach from [PSW17, DOS16], which in turn requires generalizing certain complex analysis theorems from [BE97].

²Actually, [FO21] were able to tolerate a very limited form of “measurement error”; specifically, one where each measurement is either perfect, or else reports “error” with probability at most $\frac{1}{4}$. They showed that such special measurement errors had the effect of increasing the Z-channel crossover probability up to $\frac{1}{2}$, the threshold below which prior BEC Population Recovery algorithms worked just as efficiently.

Chapter 2

Preliminaries

2.1 Quantum preliminaries

Definition 2.1.1. The four 1 -qubit Pauli operators are defined as

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.1)$$

with σ_0 being the identity operator, and σ_1, σ_2 , and σ_3 corresponding to rotations on the Bloch sphere by 180° degrees about the x - y -, and z -axes, respectively. More generally, an n -qubit Pauli operator is defined as $\sigma_C = \bigotimes_{i=1}^n \sigma_{C_i}$, where C denotes a string in $\{0, 1, 2, 3\}^n$.

Definition 2.1.2. We define the operation \oplus on $\{0, 1, 2, 3\}$ by $i \oplus j = k$ iff $\sigma_i \sigma_j = \sigma_k$ (up to phase). Equivalently, \oplus is bitwise xor when $\{0, 1, 2, 3\}$ are regarded as 2-bit base-2 numbers. We extend \oplus to operate coordinatewise on $\{0, 1, 2, 3\}^n$.

Definition 2.1.3. An n -qubit Pauli channel is defined as

$$\rho \mapsto \sum_{C \in \{0,1,2,3\}^n} \pi(C) \cdot \sigma_C \rho \sigma_C^\dagger, \quad (2.2)$$

where ρ denotes the input n -qubit quantum state, and π is a probability distribution on $\{0, 1, 2, 3\}^n$. We refer to the parameters $\pi(C)$ as the *Pauli error rates*.

Definition 2.1.4. The orthonormal eigenbasis for the Pauli operator $\sigma_1 = \sigma_x$ will be denoted by $|\chi_+^1\rangle, |\chi_-^1\rangle$. Note that this basis corresponds to the two unit vectors on the Bloch sphere pointing along the positive/negative x -axis, denoted by $|+\rangle, |-\rangle$ respectively. Similarly, the eigenbasis for $\sigma_2 = \sigma_y$ is $|\chi_+^2\rangle, |\chi_-^2\rangle$, aka $|i\rangle, |-i\rangle$; and, the eigenbasis for $\sigma_3 = \sigma_z$ is $|\chi_+^3\rangle, |\chi_-^3\rangle$, aka $|0\rangle, |1\rangle$ respectively.

2.2 Population Recovery and classical channels

We first recall communication channels, in the sense of classical information theory:

Definition 2.2.1. Let Σ, Γ be finite alphabets. A *classical channel* is a stochastic map $\Lambda : \Sigma \rightarrow \Gamma$. We extend to $\Lambda : \Sigma^n \rightarrow \Gamma^n$ by letting Λ act independently on each symbol.

The most typically studied examples are the following, both with $\Sigma = \{0, 1\}$:

- the *Binary Symmetric Channel* with crossover probability p (abbreviation: BSC_p), where $\Gamma = \{0, 1\}$ and Λ flips the bit with probability p ;
- the *Binary Erasure Channel* with erasure probability p (abbreviation: BEC_p), where $\Gamma = \{0, 1, ?\}$ and Λ preserves each bit with probability $1 - p$ and replaces it by $?$ with probability p .

[FO21] also showed the relevance of the following channel for Pauli error estimation:

Definition 2.2.2. The *Z-channel* with crossover probability p (abbreviation: Z_p) has $\Sigma = \Gamma = \{0, 1\}$; it always maps 0 to 0, and it maps 1 to 0 with probability p .

Now we recall the classical problem of learning a distribution from noisy data, introduced in [DRWY12, WY16] under the name “Population Recovery”:

Definition 2.2.3. Let $\Lambda : \Sigma \rightarrow \Gamma$ be a classical channel. In the associated *Population Recovery* problem, the goal is to learn an unknown probability distribution \mathcal{D} on Σ^n to a given ℓ_∞ precision of $\epsilon > 0$ (with high probability). The model is that the learner can get independent samples from Γ^n distributed as $\Lambda(\mathbf{x})$, where $\mathbf{x} \sim \mathcal{D}$.

Population Recovery has been well studied for the BSC and BEC channels (see, e.g., [DRWY12, WY16, BIMP13, MS13, LZ15, LZ17, DST16, PSW17, DOS17]), as well as for the binary deletion channel.

A simplified version of Population Recovery is the following:

Definition 2.2.4. In the *Individual Recovery* variant of Population Recovery, a particular string $b \in \Sigma^n$ is given and the only task is to estimate $\mathcal{D}(b)$ to additive precision ϵ (with high probability).

It is well known [DRWY12, PSW17] that Population Recovery efficiently reduces to Individual Recovery via a branch-and-prune strategy; e.g., Section 4.2 in [FO21] shows the following:

Proposition 2.2.5. *For a given channel on constant-sized alphabets, there is a reduction from Population Recovery to Individual Recovery that loses only an $O(n \log(n/\epsilon))$ factor in terms of efficiency.*

2.3 Recapping [FO21]

We use the algorithmic plan for Pauli error estimation from [FO21]. Following their exposition, two characters are introduced: Alice the learner, and Charlie the “channel operator”. In particular, we think of Charlie as implementing the Pauli channel with error rates π according to

the following process: When Alice wants to pass a state $|\psi_A\rangle$ through the channel, she sends it to Charlie, who picks a random $\mathbf{C} \sim \pi$, and then returns $\sigma_{\mathbf{C}} |\psi_A\rangle$ to Alice. In fact, our algorithm will only have Alice passing random n -qubit states composed of $|+\rangle, |i\rangle, |0\rangle$ through the channel. Thus the setup can be summarized as below:

Definition 2.3.1. To learn a Pauli channel defined by error rates π , Alice repeatedly performs “random nontrivial probes” as follows:

1. Alice picks a uniformly random string $\mathbf{A} \in \{1, 2, 3\}^n$.
2. Alice prepares an unentangled n -qubit state $|\psi_A\rangle$, where the j th qubit is $|\chi_+^{A_j}\rangle$.
3. Alice sends $|\psi_A\rangle$ to Charlie. Charlie draws a string $\mathbf{C} \in \{0, 1, 2, 3\}^n$ according to π , and sends back $\sigma_{\mathbf{C}} |\psi_A\rangle$.
4. Alice performs an unentangled measurement on $\sigma_{\mathbf{C}} |\psi_A\rangle$, where the j th qubit is measured in the $|\chi_{\pm}^{A_j}\rangle$ basis. She obtains a readout $\mathbf{R} \in \{+, -\}^n$, which we can relabel to be from $\{0, 1\}^n$.

In the description of Definition 2.3.1, Alice picks her random string \mathbf{A} first, and then Charlie (independently) draws his random string \mathbf{C} second. However, it is equivalent and more helpful to imagine Charlie “secretly” drawing \mathbf{C} first, and then Alice picking \mathbf{A} next (without knowledge of \mathbf{C}); thus, we think of the probe based on \mathbf{A} as acquiring information about Charlie’s outcome $\mathbf{C} = C$. With this interpretation in mind, it is not hard to see the following fact [FO21, Fact 17]:

Fact 2.3.2. Fix an outcome $C \in \{0, 1, 2, 3\}^n$ for Charlie’s draw. Now when Alice performs a random nontrivial probe, the coordinates of her readout \mathbf{R} are independent, and for each $1 \leq j \leq n$:

1. If $C_j = 0$, then $\mathbf{R}_j = 0$ with probability 1.
2. If $C_j \neq 0$, then with probability $\frac{1}{3}$ we have $\mathbf{R}_j = 0$, and with probability $\frac{2}{3}$ we have $\mathbf{R}_j = 1$.

(Briefly, this is because if Charlie will apply the identity Pauli σ_0 to a qubit, it will never flip from a $+1$ eigenvector to a -1 eigenvector; but, if Charlie will apply a non-identity Pauli σ to a qubit, there is a $\frac{2}{3}$ chance that Alice chose its state to be a -1 eigenvector for σ .)

Another way to say Fact 2.3.2 is that given C , the readout \mathbf{R} is distributed as $\Xi(C)$, where Ξ is defined as follows:

Definition 2.3.3. We define the classical channel

$$\Xi : \{0, 1, 2, 3\} \rightarrow \{0, 1\}, \quad 0 \mapsto 0, \quad 1, 2, 3 \mapsto \begin{cases} 0 & \text{with probability } \frac{1}{3}, \\ 1 & \text{with probability } \frac{2}{3}. \end{cases} \quad (2.3)$$

Thus Alice’s strategy of random nontrivial probes transforms the Pauli channel estimation task into the Population Recovery problem with channel Ξ .

We next observe another fact, [FO21, Observation 16]:

Fact 2.3.4. Fix any $B \in \{0, 1, 2, 3\}^n$. Let $C \in \{0, 1, 2, 3\}^n$ be the outcome of Charlie’s draw. If Alice does a random nontrivial probe with $\mathbf{A} \in \{1, 2, 3\}^n$, and then she flips each bit \mathbf{R}_j for which σ_{A_j} and σ_{B_j} anticommute, then \mathbf{R} is distributed as $\Xi(B \oplus C)$.

This means that for any $B \in \{0, 1, 2, 3\}^n$ of Alice's choice, she can simulate the Population Recovery problem (with channel Ξ) under the “ B -altered distribution” $\pi^{\oplus B}$, defined by $\pi^{\oplus B}(C) = \pi(B \oplus C)$. Finally, suppose that in this scenario she can efficiently estimate $\pi^{\oplus B}(0^n) = \pi(B)$ for B 's of her choice. This means she can solve the Individual Recovery problem for π under Ξ , hence the general Population Recovery problem (by Proposition 2.2.5).

In summary, [FO21] show that Pauli error estimation efficiently reduces to the following Individual Recovery task: Given samples from $\Xi \circ \pi$, estimate $\pi(0^n)$ to additive precision ϵ . Finally, it is easy to observe that in this scenario, there is no harm in merging the Pauli symbols 1, 2, 3 into the bit 1; this does not change $\pi(0^n)$, and it converts the channel Ξ into $Z_{\frac{1}{3}}$, the binary Z-channel with crossover probability $\frac{1}{3}$. This completes the recap of how [FO21] reduces (SPAM-free) Pauli error estimation to Individual Recovery of $\pi(0^n)$ under noise channel $Z_{\frac{1}{3}}$.

Chapter 3

Modeling SPAM

Operationally, SPAM errors might occur in a variety of natural ways. To take state preparation errors for example, when an algorithm tries to prepare a certain state $|a\rangle$, several things might happen: $|a\rangle$ might be replaced by a random pure state at a fixed angle from $|a\rangle$; or, $|a\rangle$ might be replaced with a random mixed state ρ having a fixed fidelity with $|a\rangle$; or, $|a\rangle$ might be replaced with the maximally mixed state with a certain probability. Mathematically, these are all equivalent to $|a\rangle$ being passed through a depolarizing channel. We therefore make the following definition:

Definition 3.0.1. We model single-qubit state preparation error, with “retention parameter” $r_{\text{prep}} \in [0, 1]$, by assuming that when $|a\rangle$ is intended, the actual produced state is $r_{\text{prep}} \cdot |a\rangle\langle a| + (1 - r_{\text{prep}}) \cdot \frac{1}{2}I$, where $\frac{1}{2}I$ corresponds to the maximally mixed single-qubit state. Equivalently, the produced state has expected fidelity $\frac{1}{2} + \frac{1}{2}r_{\text{prep}}$ with the intended state.

Remark 3.0.2. Our model assumes for simplicity that we have the same retention parameter for every intended state $|a\rangle$. We also assume that the learning algorithm knows the parameter r_{prep} from prior calibration. The same remarks hold for Definition 3.0.3 below.

Similarly, for modeling measurement errors, there are several natural operational possibilities: when measuring ρ against an intended outcome $|a\rangle$, one might actually get the result of measurement against some other random outcome $|b\rangle$ making a certain angle from $|a\rangle$; or, ρ might get perturbed to a random state with a certain expected fidelity before the correct measurement is made, etc. Once again, these possibilities are mathematically equivalent to ρ being passed through the depolarizing channel. We therefore define:

Definition 3.0.3. We model single-qubit measurement error, with “retention parameter” $r_{\text{meas}} \in [0, 1]$, by assuming that when ρ is measured in an intended orthonormal basis $\{|a\rangle, |a^\perp\rangle\}$, the outcome is as if the state $r_{\text{meas}} \cdot \rho + (1 - r_{\text{meas}}) \cdot \frac{1}{2}I$ were measured instead.

The algorithm under SPAM. We may now consider how the combined SPAM error affects our algorithm. As described in Section 2.3, when learner Alice makes random nontrivial probes to the Pauli channel defined by error rates π , we may imagine that Charlie first draws $\mathcal{C} \sim \pi$, obtaining some outcome $C \in \{0, 1, 2, 3\}^n$. Given C , everything else occurs in an unentan-

gled fashion across qubits: For each $j \in [n]$, Alice prepares a random $|\chi_+^{A_j}\rangle$, state preparation error replaces this by $r_{\text{prep}} \cdot |\chi_+^{A_j}\rangle\langle\chi_+^{A_j}| + (1 - r_{\text{prep}}) \cdot \frac{1}{2}I$, the channel changes this to $r_{\text{prep}} \cdot \sigma_{C_j} |\chi_+^{A_j}\rangle\langle\chi_+^{A_j}| \sigma_{C_j}^\dagger + (1 - r_{\text{prep}}) \cdot \frac{1}{2}I$, and just before Alice's measurement in the $|\chi_\pm^{A_j}\rangle$ basis, measurement error changes the state to $r_{\text{prep}} r_{\text{meas}} \cdot \sigma_{C_j} |\chi_+^{A_j}\rangle\langle\chi_+^{A_j}| \sigma_{C_j}^\dagger + (1 - r_{\text{prep}} r_{\text{meas}}) \cdot \frac{1}{2}I$. This can be viewed as the SPAM-free result — namely, σ_{C_j} applied to $|\chi_+^{A_j}\rangle$ — passed through the depolarizing channel with retention parameter $r = r_{\text{prep}} \cdot r_{\text{meas}}$. In other words:

Given C , for each $j \in [n]$ independently, Alice gets the random readout she normally would (namely $\Xi(C)$) with probability $r = r_{\text{prep}} \cdot r_{\text{meas}}$, and a uniformly random bit from $\{0, 1\}$ with probability $\delta := 1 - r$. In other words, she receives $\text{BSC}_{\frac{\delta}{2}} \circ \Xi(C)$.

It is easy to show that the reductions described after Definition 2.3.3 in Section 2.3 continue to hold. Thus we finally obtain:

Theorem 3.0.4. *The task of estimating the error rates π of an n -qubit Pauli channel with SPAM governed by retention rates $r_{\text{prep}}, r_{\text{meas}}$ reduces (with a factor $O(n \log(n/\epsilon))$ loss of efficiency) to the task of Individual Recovery of $\pi(\mathbf{0}^n)$ under noise channel $\text{BSC}_{\frac{\delta}{2}} \circ \mathbf{Z}_{\frac{1}{3}}$, where $\delta = 1 - r_{\text{prep}} \cdot r_{\text{meas}}$.*

Remark 3.0.5. As in [FO21], the above Theorem and our main Theorem 1.1.1 extend to the case of learning the Pauli error rates of a *general* n -qubit channel \mathcal{E} . These are defined as the error rates of the Pauli channel formed from \mathcal{E} by Pauli twirling, namely $\rho \mapsto \mathbf{E}_{T \sim \{0,1,2,3\}^n} \sigma_T^\dagger \mathcal{E}(\sigma_T \rho \sigma_T^\dagger) \sigma_T$. The details are exactly as in [FO21, Section 6.1], with the only change to the statement of Theorem 1.1.1 being that Alice now randomly prepares qubits in one of the six pure states $|\chi_\pm^j\rangle$, rather than just the three $|\chi_+^j\rangle$.

Chapter 4

Individual recovery for combined noise channels

The Individual Recovery problem is well understood separately for BSC and BEC [PSW17, DOS17]. As noted in [FO21], the Z-channel behaves similarly to BEC in the context of Population Recovery. However, handling the combined channel $\text{BSC}_{\frac{\delta}{2}} \circ \text{Z}_{\frac{1}{3}}$ from Theorem 3.0.4 will require more work. We will follow the methodology from [DOS17].

4.1 Generating functions

Recall our goal is Individual Recovery of $\mathcal{D}(\mathcal{O}^n)$, given samples from an unknown probability distribution \mathcal{D} on $\{0, 1\}^n$, masked by a binary noise channel Λ . As described in [DOS17], we can assume without loss of generality that \mathcal{D} is symmetric and assigns equal probability mass to all strings with the same Hamming weight. This is because, given an arbitrary distribution \mathcal{D} , we can randomly permute each sample's coordinates, which is equivalent to sampling a symmetric distribution \mathcal{D}^{sym} , where $\mathcal{D}^{\text{sym}}(\mathcal{O}^n) = \mathcal{D}(\mathcal{O}^n)$. Therefore, it suffices to be able to estimate in the symmetric setting.

Given this, let us introduce the row vector $p = [p_0 \ p_1 \ \cdots \ p_n]$, where p_i denotes the total probability mass \mathcal{D} has on Hamming weight i . Thus the learner's goal is to estimate p_0 . Note also that the learner only needs to observe the Hamming weights of the samples it obtains (this is true even when Λ is the BEC). Therefore, we can think of the learner as obtaining samples from $\{0, 1, \dots, n\}$ distributed according to the probability row vector $q = [q_0 \ q_1 \ \cdots \ q_n]$, where

$$q = pA^{(\Lambda)}, \quad A_{ij}^{(\Lambda)} = \Pr[\text{a weight } i \text{ string becomes a weight } j \text{ string under } \Lambda \text{ noise}]. \quad (4.1)$$

The rows of each matrix $A^{(\Lambda)}$ can be nicely expressed using generating functions. For example, [DOS17, Propositions 2.1, 2.2] are the following:

Proposition 4.1.1. *For the A matrices associated with the BEC_λ and BSC_b channels, and with*

z an arbitrary complex number:

$$\sum_{j=0}^n A_{ij}^{(\text{BEC}_\lambda)} z^j = (\lambda + (1 - \lambda)z)^i \quad (4.2)$$

$$\sum_{j=0}^n A_{ij}^{(\text{BSC}_b)} z^j = (b + (1 - b)z)^i ((1 - b) + bz)^{n-i} \quad (4.3)$$

Also, because both the BEC_λ and the Z_λ channels convert 1's to non-1's with the same probability, λ , it is easy to see:

Fact 4.1.2. *The Z-channel and BEC have the same A-matrices, $A^{(Z_\lambda)} = A^{(\text{BEC}_\lambda)}$.*

We would now like to derive the generating function for our combined channel, $\text{BSC}_{\frac{\delta}{2}} \circ Z_{\frac{1}{3}}$. Let us make a general definition:

Definition 4.1.3. For parameters $0 \leq \tau_1, \tau_2 < 1$, we define the *ZFlip* channel $\text{ZFlip}_{\tau_1, \tau_2} : \{0, 1\} \rightarrow \{0, 1\}$ to be the concatenated channel $\text{BSC}_{\frac{1-\tau_2}{2}} \circ Z_{1-\tau_1}$.

The slightly strange parameterization makes formulas simpler, as τ_1, τ_2 may be thought of as “retention rates”. For the Pauli error estimation problem with overall SPAM retention parameter $r = r_{\text{prep}} \cdot r_{\text{meas}}$, we care about

$$\tau_1 = \frac{2}{3}, \quad \tau_2 = r. \quad (4.4)$$

To compute the generating functions for the ZFlip channel, we start with the following simple fact:

Proposition 4.1.4. *Given channels Λ_1, Λ_2 , the “A-matrix” (as in Equation (4.1)) for the concatenated channel satisfies $A^{(\Lambda_2 \circ \Lambda_1)} = A^{(\Lambda_1)} \cdot A^{(\Lambda_2)}$.*

Proof. We have

$$\begin{aligned} A_{ij}^{(\Lambda_2 \circ \Lambda_1)} &= \Pr[\text{a weight } i \text{ string becomes a weight } j \text{ string under } \Lambda_2 \circ \Lambda_1 \text{ noise}] \\ &= \sum_{k=0}^n \Pr[\text{weight } i \rightarrow \text{weight } k \text{ under } \Lambda_1 \text{ noise}] \cdot \Pr[\text{weight } k \rightarrow \text{weight } j \text{ under } \Lambda_2] \\ &= \sum_{k=0}^n A_{ik}^{(\Lambda_1)} A_{kj}^{(\Lambda_2)} = (A^{(\Lambda_1)} \cdot A^{(\Lambda_2)})_{ij}. \quad \square \end{aligned}$$

Now we can derive the generating function for the $\text{ZFlip}_{\tau_1, \tau_2}$ channel:

Proposition 4.1.5. *Given $0 \leq \tau_1, \tau_2 \leq 1$, write $b = \frac{1-\tau_2}{2}$. Then*

$$G_i(z) := \sum_{j=0}^n A_{ij}^{(\text{ZFlip}_{\tau_1, \tau_2})} z^j = ((1 - b) + bz)^n \cdot ((1 - \tau_1) + \tau_1 z)^i, \quad \text{where } w := \frac{b + (1 - b)z}{(1 - b) + bz}. \quad (4.5)$$

Proof. For brevity, write $A^{(1)} = A^{(Z_{1-\tau_1})}$, which from Fact 4.1.2 and Equation (4.2) we know has

$$\sum_{k=0}^n A_{ik}^{(1)} z^k = ((1 - \tau_1) + \tau_1 z)^i. \quad (4.6)$$

And for brevity, write $A^{(2)} = A^{(\text{BSC}_b)}$, which from Equation (4.3) we know has

$$\sum_{j=0}^n A_{kj}^{(2)} z^j = (b + (1 - b)z)^k ((1 - b) + bz)^{n-k} = ((1 - b) + bz)^n \cdot w^k. \quad (4.7)$$

Now Proposition 4.1.4 tells us that $\sum_{j=0}^n A_{ij}^{(\text{ZFlip}_{\tau_1, \tau_2})} z^j$ equals

$$\sum_{j=0}^n \sum_{k=0}^n A_{ik}^{(1)} A_{kj}^{(2)} z^j = \sum_{k=0}^n A_{ik}^{(1)} \left(\sum_{j=0}^n A_{kj}^{(2)} z^j \right) = ((1 - b) + bz)^n \cdot \sum_{k=0}^n A_{ik}^{(1)} w^k \quad (4.8)$$

$$= ((1 - b) + bz)^n \cdot ((1 - \tau_1) + \tau_1 w)^i, \quad (4.9)$$

as claimed. \square

Recall now that our task is to (with high probability) estimate p_0 to additive precision ϵ , given samples drawn from distribution $q = pA^{(\Lambda)}$, where Λ is the ZFlip channel of interest from Definition 4.1.3. The work [DOS17] precisely lower-bounds the sample complexity of this task, and gives an algorithmic upper bound matching to within polynomial factors:

Theorem 4.1.6. (*Combination of [DOS17, Props. 3.1, 3.2, and subsequent, slight notation change].*) For binary noise channel Λ , define

$$\eta_\Lambda(\epsilon) = \min_{\substack{\text{probability vectors } p, p' \\ |p_0 - p'_0| > \epsilon}} \|pA^{(\Lambda)} - p'A^{(\Lambda)}\|_1. \quad (4.10)$$

Then the sample complexity of the Individual Recovery task is $\Omega(\frac{1}{\eta_\Lambda(2\epsilon)})/\sqrt{n}$. Moreover, the task can be solved with $\text{poly}(n, \frac{1}{\eta_\Lambda(\epsilon)})$ samples and running time.

Finally, if the generating function $\sum_{j=0}^n A_{ij}^{(\Lambda)} z^j$ is $G_i(z)$, then

$$\eta_\Lambda(\epsilon) \geq \min_{\substack{c \in \Delta \\ c_0 > \epsilon}} \max_{\substack{z \in \mathbb{C} \\ |z|=1}} \left| \sum_{j=0}^n c_j G_j(z) \right|, \quad (4.11)$$

where $\Delta := \{c = (c_0, c_1, \dots, c_n) : \sum_j c_j = 0, \sum_j |c_j| \leq 2\}$.

In the above theorem, we need to understand $G_j(z)$'s values for z on the complex unit circle. Recall from Proposition 4.1.5 that for our ZFlip channel of interest, with the Möbius transformation $w = \frac{b+(1-b)z}{(1-b)+bz}$, we have

$$G_j(z) = ((1 - b) + bz)^n \cdot ((1 - \tau_1) + \tau_1 w)^j = \left(\frac{1 - 2b}{(1 - b) - bw} \right)^n \cdot ((1 - \tau_1) + \tau_1 w)^j, \quad (4.12)$$

where we solved for z in terms of w . It is easy to see that in the formula for w , the numerator and denominator of w have the same squared length when $|z| = 1$; i.e., we have $|w|^2 = 1$. Since Möbius transformations map circles to circles, we conclude that the locus of w for $|z| = 1$ is also the unit circle, $|w| = 1$.¹ Now writing $w = e^{i\theta}$ for $\theta \in (-\pi, \pi]$, we have

$$G_j(\theta) = \left(\frac{\tau_2}{(1-b) - be^{i\theta}} \right)^n \cdot ((1 - \tau_1) + \tau_1 e^{i\theta})^j, \quad (4.13)$$

where recall $b = \frac{1-\tau_2}{2}$. Using the notation $Q(v) = \sum_{j=0}^n c_j v^j$, Theorem 4.1.6 tells us that for $\Lambda = \text{ZFlip}_{\tau_1, \tau_2}$,

$$\eta_\Lambda(\epsilon) \geq \min_Q \max_{-\pi < \theta \leq \pi} \left| \frac{\tau_2}{(1-b) - be^{i\theta}} \right|^n \cdot |Q((1 - \tau_1) + \tau_1 e^{i\theta})|, \quad (4.14)$$

where the minimum is over real-coefficient polynomials of degree at most n satisfying $Q(0) > \epsilon$, $Q(1) = 0$, and $L(Q) := \sum_j |c_j| \leq 2$. Finally, note that

$$\begin{aligned} |(1-b) - be^{i\theta}|^2 &= (1-b)^2 + b^2 + 2b(1-b)\cos\theta \\ &= \frac{1+\tau_2^2}{2} - \frac{1-\tau_2^2}{2}\cos\theta = \frac{1+\tau_2^2}{2} - \frac{1-\tau_2^2}{2}(1-2\sin^2(\theta/2)) = \tau_2^2 + (1-\tau_2^2)\sin^2(\theta/2). \end{aligned} \quad (4.15)$$

Thus

$$\left| \frac{\tau_2}{(1-b) - be^{i\theta}} \right|^n = \left(\frac{\tau_2^2}{\tau_2^2 + (1-\tau_2^2)\sin^2(\theta/2)} \right)^{n/2} = \left(1 + \frac{1-\tau_2^2}{\tau_2^2}\sin^2(\theta/2) \right)^{-n/2}. \quad (4.16)$$

Putting everything together, we conclude:

Theorem 4.1.7. *Individual Recovery of $\pi(\mathbf{0}^n)$ under the noise channel $\text{ZFlip}_{\tau_1, \tau_2}$ can be accomplished with $\text{poly}(n, 1/\eta(\epsilon))$ samples and running time, where the function $\eta(\epsilon)$ satisfies*

$$\eta(\epsilon) \geq \min_Q \max_{-\pi < \theta \leq \pi} \left(1 + \frac{1-\tau_2^2}{\tau_2^2}\sin^2(\theta/2) \right)^{-n/2} \cdot |Q((1 - \tau_1) + \tau_1 e^{i\theta})| \quad (4.17)$$

and the minimum is over real-coefficient polynomials Q of degree at most n satisfying $Q(0) > \epsilon$, $Q(1) = 0$, and $L(Q) \leq 2$.

4.2 Lower-bounding $\eta(\epsilon)$

In order to prove Theorem 1.1.1, we need to find a lower bound on the two terms specified in Theorem 4.1.7 above. We will prove the following bound:

¹Unless $b = \frac{1}{2}$, in which case w is constantly 1. This corresponds to the trivial case of maximal SPAM, $\tau_2 = 0$, which we henceforth exclude.

Theorem 4.2.1. For $0 < \epsilon, \tau_1, \tau_2 < 1$, and $n \in \mathbb{N}$, we have

$$\eta(\epsilon) \geq \max \left\{ \epsilon^{O\left(\frac{1}{\tau_1}\right)}, \exp \left(-O \left(\frac{\ln^{2/3}(1/\epsilon) \cdot (n(1 - \tau_2^2))^{1/3}}{(\tau_1 \tau_2)^{2/3}} \right) \right) \right\}, \quad (4.18)$$

where $\eta(\epsilon)$ characterizes the sample complexity and runtime of solving Individual Recovery of $\pi(\mathbf{0}^n)$ under the noise channel $\text{ZFlip}_{\tau_1, \tau_2}$.

Proof. Note that for $-\pi < \theta < \pi$, $\theta^2/16 \leq \sin^2(\theta/2) \leq \theta^2/4$. Using the fact that $e^{-x} \leq (1+x)^{-1}$ for all $x \geq 0$, we have that

$$\left(1 + \frac{1 - \tau_2^2}{\tau_2^2} \sin^2(\theta/2) \right)^{-1} \geq \exp \left(-\frac{1 - \tau_2^2}{4\tau_2^2} \theta^2 \right), \quad (4.19)$$

and therefore,

$$\eta(\epsilon) \geq \min_Q \max_{-\pi < \theta \leq \pi} \exp \left(-\frac{1 - \tau_2^2}{8\tau_2^2} \theta^2 n \right) \cdot |Q((1 - \tau_1) + \tau_1 e^{i\theta})|. \quad (4.20)$$

Next, fix an arbitrary vector $c = [c_0 \ c_1 \ \cdots \ c_n] \in \Delta$ with $c_0 > \epsilon$. Let Q_c denote the polynomial Q with coefficients c . We now need to establish a lower bound for the modulus of Q_c on the circle $\partial D_{\tau_1}(1 - \tau_1)$ of radius τ_1 centered at $1 - \tau_1$. Actually, we will prove a lower bound for a slightly modified polynomial

$$\tilde{Q}_c(z) = \frac{1}{c_0} Q_c(z) = \sum_{i=0}^n \tilde{c}_i z^i, \quad (4.21)$$

where $\tilde{c}_0 = 1$, and $|\tilde{c}_i| \leq \frac{1}{c_0}$ for all i . In our proof, we will only need \tilde{Q}_c to satisfy the following two properties:

For $M \geq 1$ and $0 < \gamma \leq 1$,

$$\left| \tilde{Q}_c \left(\frac{1}{4M} \right) \right| \geq \frac{1}{2}, \quad \text{and} \quad |z| \leq 1 - \gamma \Rightarrow |\tilde{Q}_c(z)| \leq M/\gamma. \quad (4.22)$$

The first property holds, since, if we set M to $\frac{1}{c_0}$, and let $z_0 = \frac{1}{4M}$, then

$$|\tilde{Q}_c(z_0)| \geq 1 - \sum_{i=1}^n M(z_0)^i = 1 - \frac{Mz_0(1 - z_0^n)}{1 - z_0} \geq 1 - \frac{Mz_0}{1 - z_0} \geq \frac{1}{2}, \quad (4.23)$$

as desired. The second property holds, since, for $0 < \gamma \leq 1$ and for $z \in \mathbb{C}$ such that $|z| \leq 1 - \gamma$, we have that

$$|\tilde{Q}_c(z)| \leq \sum_{i=0}^n |\tilde{c}_i| |z|^i \leq \sum_{i=0}^n \frac{1}{c_0} |z|^i \leq \sum_{i=0}^n M |z|^i = \frac{M(1 - |z|^{n+1})}{1 - |z|} \leq \frac{M}{\gamma}. \quad (4.24)$$

Since our proof only requires these two properties, we will prove our lower bound for all polynomials that satisfy them. We thus define the following set of functions:

Definition 4.2.2. Let $0 < \kappa \leq 1$ and $\lambda \geq 1$. $\mathcal{C}_{\kappa,\lambda}$ is the set of all analytic functions f that are continuous on the closed unit disk and satisfy

$$\left| f\left(\frac{1}{4M}\right) \right| \geq \kappa, \quad \text{and} \quad |z| \leq 1 - \gamma \Rightarrow |f(z)| \leq \frac{\lambda M}{1 - |z|}, \quad (4.25)$$

where $0 < \gamma \leq 1$.

We will prove our lower bound for all polynomials in the set $\mathcal{C}_{\frac{1}{2},1}$, on an arc of the circle $\partial D_{\tau_1}(1 - \tau_1)$ with central angle $\theta \in (0, \pi)$. This arc has endpoints α and β , where

$$\alpha = \tau_1 [\cos(\theta/2) + i \sin(\theta/2)] + 1 - \tau_1, \quad \text{and} \quad \beta = \tau_1 [\cos(\theta/2) - i \sin(\theta/2)] + 1 - \tau_1. \quad (4.26)$$

Before delving into the proof, we must establish several lemmas. The first is derived from the Hadamard three-line theorem. We define

$$I_t = \left\{ z \in \mathbb{C} \mid \arg\left(\frac{\alpha - z}{z - \beta}\right) = t \right\}, \quad (4.27)$$

following the notation used in [BE97]. Note that

$$\arg\left(\frac{\alpha - z}{z - \beta}\right) = \arg(\alpha - z) - \arg(z - \beta). \quad (4.28)$$

$\arg(\alpha - z)$ represents the argument of the vector from point z to point α , while $\arg(z - \beta)$ represents the argument of the vector from point β to point z . The difference between these two arguments is 0 when z lies on the chord between α and β . Therefore, I_0 corresponds to this chord. Furthermore, by the inscribed angle theorem, the difference between these two arguments is $\frac{\theta}{2}$ when z lies on the smaller arc of $\partial D_{\tau_1}(1 - \tau_1)$ with endpoints α and β , as shown below. Therefore, $I_{\theta/2}$ corresponds to this arc.

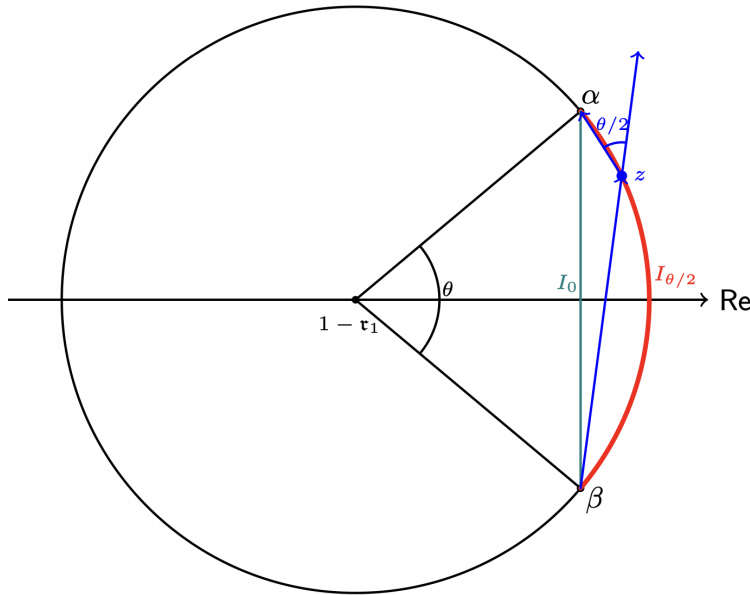


Figure 4.1: Chord I_0 and arc $I_{\theta/2}$ on circle $\partial D_{\tau_1}(1 - \tau_1)$.

We can again apply the inscribed angle theorem on the larger circle with center $1 - 2\tau_1$ and radius $2\tau_1 \cos(\theta/4)$ to determine the arc that $I_{\theta/4}$ corresponds to, as shown below.

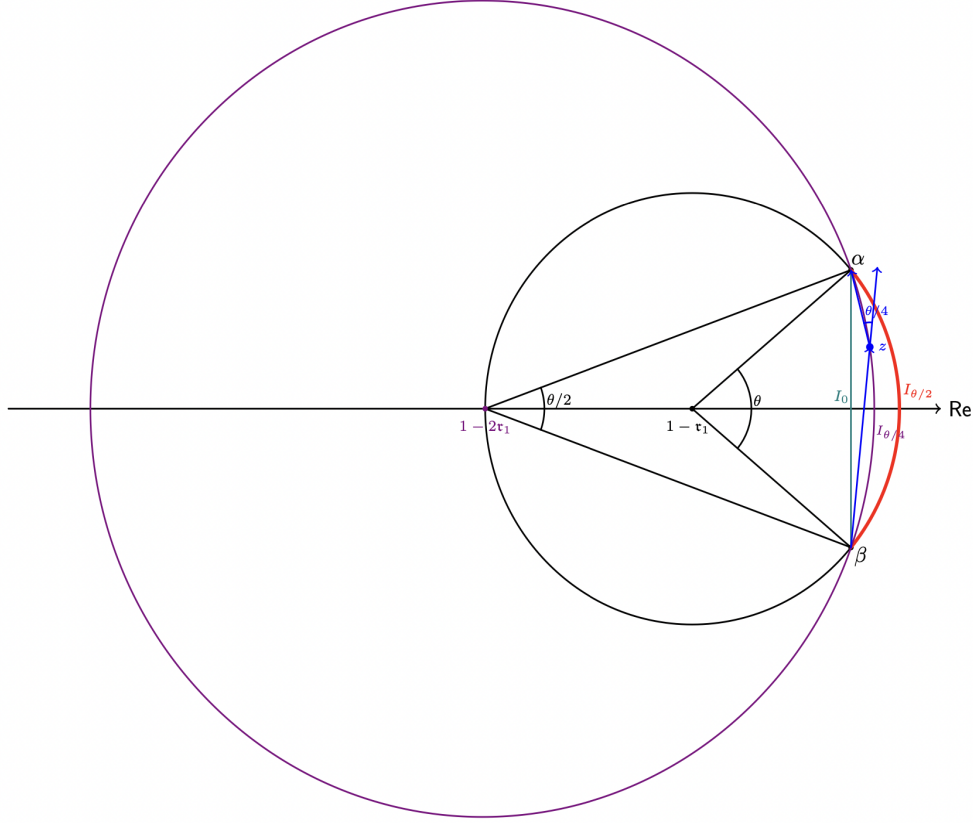


Figure 4.2: Arc $I_{\theta/4}$ on circle $\partial D_{2\tau_1 \cos(\theta/4)}(1 - 2\tau_1)$.

Therefore, $I_{\theta/4}$ is the arc with endpoints α, β on the circle $\partial D_{2\tau_1 \cos(\theta/4)}(1 - 2\tau_1)$.

In the following lemma, $I_0, I_{\theta/2}$, and $I_{\theta/4}$ are mapped to three infinite vertical lines on the complex plane. Then, the Hadamard three-line theorem is applied to upper-bound the maximum modulus of a function on $I_{\theta/4}$ using the maximum modulus of the same function on I_0 and $I_{\theta/2}$. This technique can be applied to any complex circle, not just $\partial D_{\tau_1}(1 - \tau_1)$.

Lemma 4.2.3. (Modified Lemma 4.3 in [BE97]). *Consider the complex circle with radius r and center c . Let $0 < \theta < \pi$, $\alpha = r [\cos(\theta/2) + i \sin(\theta/2)] + c$, and $\beta = r [\cos(\theta/2) - i \sin(\theta/2)] + c$. Let $I_t = \left\{ z \in \mathbb{C} \mid \arg \left(\frac{\alpha - z}{z - \beta} \right) = t \right\}$.*

Suppose g is an analytic function in the open region bounded by $I_{\theta/4}$ and $I_{\theta/2}$, and suppose g is continuous on the closed region between I_0 and $I_{\theta/2}$. Then

$$\max_{z \in I_{\theta/4}} |g(z)| \leq \left(\max_{z \in I_0} |g(z)| \right)^{1/2} \left(\max_{z \in I_{\theta/2}} |g(z)| \right)^{1/2}. \quad (4.29)$$

Proof. We can apply Lemma 4.2 in [BE97], which stems from the Hadamard three-line theorem:

Lemma 4.2.4. (Lemma 4.2 in [BE97]). *Let $a > 0$ and*

$$E_a := \{z \in \mathbb{C} \mid 0 \leq \text{Im}(z) \leq a\}. \quad (4.30)$$

Suppose g is an analytic function in the interior of $E_{a/2}$, and suppose g is continuous on $E_{a/2} \cup \infty$. Then

$$\max_{\{z \mid \text{Im}(z)=t/4\}} |g(z)| \leq \left(\max_{\{z \mid \text{Im}(z)=0\}} |g(z)| \right)^{1/2} \left(\max_{\{z \mid \text{Im}(z)=t/2\}} |g(z)| \right)^{1/2}. \quad (4.31)$$

Let $w := \log \left(\frac{\alpha-z}{z-\beta} \right)$. Given $z \in I_{\theta/4}$, note that

$$\log \left(\frac{\alpha-z}{z-\beta} \right) = \log \left(\left| \frac{\alpha-z}{z-\beta} \right| \right) + i \arg \left(\frac{\alpha-z}{z-\beta} \right) = \log \left(\left| \frac{\alpha-z}{z-\beta} \right| \right) + \frac{i\theta}{4} \in \{z \mid \text{Im}(z) = \theta/4\}. \quad (4.32)$$

Similarly, given $z \in I_0$, $w \in \{z \mid \text{Im}(z) = 0\}$, and given $z \in I_{\theta/2}$, $w \in \{z \mid \text{Im}(z) = \theta/2\}$. Now, we define a function h such that $h(y) = g(w^{-1}(y))$. We thus have that

$$\begin{aligned} \max_{z \in I_{\theta/4}} |g(z)| &= \max_{\{w \mid \text{Im}(w)=\theta/4\}} |h(w)| \\ &\leq \left(\max_{\{w \mid \text{Im}(w)=0\}} |h(w)| \right)^{1/2} \left(\max_{\{w \mid \text{Im}(w)=\theta/2\}} |h(w)| \right)^{1/2} \quad (\text{by Lemma 4.2.4}) \\ &= \left(\max_{z \in I_0} |g(z)| \right)^{1/2} \left(\max_{z \in I_{\theta/2}} |g(z)| \right)^{1/2}. \end{aligned} \quad (4.33)$$

□

The next theorem we will need is only a slight generalization of [BE97, Lemma 4.1], but we prefer to give a full proof here, as the one in [BE97] has a significant typo.

Theorem 4.2.5. *Let $0 < \delta \leq \frac{1}{4}$, $0 < a < \pi$, $\lambda \geq 1$, and $M \geq 1$. Let g be a continuous function on the unit disk, analytic on the interior, satisfying the following properties:*

$$|g(\delta)| \geq 1, \quad \text{and} \quad |z| \leq 1 - \gamma \Rightarrow |g(z)| \leq \frac{\lambda M}{\gamma} \quad (4.34)$$

for $0 < \gamma \leq 1$.

Let Γ be the circle with center δ and radius $1 - \delta$. Let J be the closed arc of Γ with midpoint 1 and arc length a . Then

$$\max_{z \in J} |g(z)| \geq \left(\frac{\delta}{\lambda M} \right)^{O(1/a)}. \quad (4.35)$$

Proof. Let $2m$ be the smallest even integer greater than or equal to $\frac{4\pi}{a}$. We define $2m$ equally spaced points on Γ such that the arc length between two adjacent points is approximately (but not larger than) $\frac{a}{2}$:

$$\eta_k := \delta + (1 - \delta) \xi^k, \quad k \in \mathbb{Z}, -m \leq k \leq m, \quad (4.36)$$

where $\xi := \exp(\frac{2\pi i}{2m})$ is the first $(2m)$ -th root of unity. Note that the arc with endpoints η_{-1} and η_1 that passes through η_0 is entirely contained within arc J . Also note that η_{-m} and η_m correspond to the same point.

First, we will show that there exists a constant $c > 0$ such that for all $k \in \{1, \dots, m-1\}$,

$$|z| \leq 1 - c\delta(ka)^2 \quad (4.37)$$

when z is on the arc with endpoints η_k and η_{k+1} or η_{-k} and $\eta_{-(k+1)}$.

This is because WLOG, assume that z is on the arc with endpoints η_k and η_{k+1} . Then z is of the form

$$\delta + (1 - \delta)e^{i\theta}, \quad \theta \in \left[\frac{\pi k}{m}, \frac{\pi(k+1)}{m} \right]. \quad (4.38)$$

This means that

$$|z|^2 = 1 - 2\delta(1 - \delta)(1 - \cos\theta). \quad (4.39)$$

Since $|z| \leq 1$,

$$|z| \leq \frac{1 + |z|^2}{2} = 1 - \delta(1 - \delta)(1 - \cos\theta) \leq 1 - \frac{3}{4}\delta(1 - \cos\theta). \quad (4.40)$$

Since $\theta \in [0, \pi]$,

$$1 - \cos\theta \geq \frac{2}{\pi^2}\theta^2 \geq \frac{2k^2}{m^2}. \quad (4.41)$$

Since $2m \leq c_1 \frac{4\pi}{a}$ for some $c_1 > 0$:

$$1 - \cos\theta \geq \frac{c_1}{2\pi^2} \cdot (ka)^2. \quad (4.42)$$

Combining Equation (4.40) with Equation (4.42) gives us Equation (4.37).

Now, we define the following function

$$h(z) := \prod_{j=-m}^{m-1} g(\delta + \xi^j(z - \delta)). \quad (4.43)$$

Note that when $z \in \Gamma$, the $2m$ points plugged into g to evaluate $h(z)$ are equally spaced on Γ . The arc length between two adjacent points is

$$\frac{2\pi}{2m}(1 - \delta) > \frac{a}{3}(1 - \delta) \geq \frac{a}{4}. \quad (\text{since } 2m < \frac{4\pi}{a} + 2 \leq \frac{6\pi}{a})$$

Therefore, the arc length between two adjacent points is strictly greater than a fourth of the arc length of J . This means that at most four of these points lie on arc J .

Consider the remaining $2m-4$ points. Each of these points lie on a distinct arc with endpoints η_j and η_{j+1} , or endpoints η_{-j} and $\eta_{-(j+1)}$, for $j = \{2, \dots, m-1\}$. By Equation (4.37) and the properties of g , any point z on the arc with endpoints η_j and η_{j+1} , or endpoints η_{-j} and $\eta_{-(j+1)}$, satisfies

$$|g(z)| \leq \frac{\lambda M}{c\delta(ja)^2}. \quad (4.44)$$

Putting this all together,

$$\begin{aligned}
\max_{z \in \Gamma} |h(z)| &\leq \left(\max_{z \in J} |g(z)| \right)^4 \cdot \prod_{k=2}^{m-1} \left(\frac{\lambda M}{c\delta(ka)^2} \right)^2 \\
&= \left(\max_{z \in J} |g(z)| \right)^4 \left(\frac{\lambda M}{\delta} \right)^{2(m-2)} \left(\frac{1}{c_1 a} \right)^{4(m-2)} ((m-1)!)^{-4} \\
&\leq \left(\max_{z \in J} |g(z)| \right)^4 \left(\frac{\lambda M}{\delta} \right)^{2(m-2)} \left(\frac{m}{c_1 \pi} \right)^{4(m-2)} \left(\frac{e}{m-1} \right)^{4(m-1)} \\
&\quad \text{(by Stirling's approximation, and since } a \geq \frac{2\pi}{m} \text{)} \\
&\leq \left(\max_{z \in J} |g(z)| \right)^4 \left(\frac{\lambda M e}{c_1 \delta \pi} \right)^{2(m-2)} \left(\frac{m}{m-1} \right)^{4(m-1)} e^{2m} \\
&\leq \left(\max_{z \in J} |g(z)| \right)^4 \left(\frac{\lambda M e}{c_1 \delta \pi} \right)^{2(m-2)} e^{2m+4} \quad \text{(since } (1 + \frac{1}{n})^n \rightarrow e \text{ as } n \rightarrow \infty \text{)} \\
&= \left(\max_{z \in J} |g(z)| \right)^4 \left(\frac{\lambda M}{\delta} \right)^{O(1/a)}. \tag{4.45}
\end{aligned}$$

Now, by the Maximum Modulus principle,

$$|g(\delta)|^{2m} = |h(\delta)| \leq \max_{z \in \Gamma} |h(z)| \leq \left(\frac{\lambda M}{\delta} \right)^{O(1/a)} \left(\max_{z \in J} |g(z)| \right)^4. \tag{4.46}$$

So

$$\left(\max_{z \in J} |g(z)| \right)^4 \geq \left(\frac{\delta}{\lambda M} \right)^{O(1/a)} |g(\delta)|^{2m} \geq \left(\frac{\delta}{\lambda M} \right)^{O(1/a)} (1) = \left(\frac{\delta}{\lambda M} \right)^{O(1/a)}. \tag{4.47}$$

Therefore

$$\max_{z \in J} |g(z)| \geq \left(\frac{\delta}{\lambda M} \right)^{O(1/a)}. \quad \square$$

Note that if we instead have that $g(\delta) \geq \kappa$ for some $0 < \kappa \leq 1$, then

$$\max_{z \in J} |g(z)| \geq \left(\frac{\delta \kappa}{\lambda M} \right)^{O(1/a)}. \tag{4.48}$$

This follows from replacing g with g/κ and M with M/κ in Theorem 4.2.5.

In our proof, we will use the following corollary, which provides a lower bound for g on the arc of a circle with a slightly smaller radius that has a midpoint less than 1.

Corollary 4.2.6. *Let $0 < \delta \leq \frac{1}{4}$, $0 < a < \pi$, $\lambda \geq 1$, $M \geq 1$, and $0 < \kappa \leq 1$. Let $0 \leq \mu < 1 - \delta$. Let g be a continuous function on the unit disk, analytic on the interior, satisfying the following properties:*

$$|g(\delta)| \geq \kappa, \quad \text{and} \quad |z| \leq 1 - \gamma \Rightarrow |g(z)| \leq \frac{\lambda M}{\gamma}, \tag{4.49}$$

where $\gamma \leq 1 - \delta$.

Let Γ_μ be the circle with center δ and radius $1 - \delta - \mu$. Let P be the closed arc of Γ_μ that is symmetric with respect to the real axis and has arc length a . Then

$$\max_{z \in P} |g(z)| \geq \left(\frac{\delta \kappa}{\lambda M} \right)^{O(1/a)}. \quad (4.50)$$

Proof. Consider the function

$$h(z) = g \left(\left(\frac{1 - \delta - \mu}{1 - \delta} \right) (z - \delta) + \delta \right). \quad (4.51)$$

First, note that $h(\delta) = g(\delta)$. Since $|g(\delta)| \geq \kappa$, we have that $|h(\delta)| \geq \kappa$.

Next, assume $|z| \leq 1 - \gamma$.

$$\begin{aligned} \left| \left(\frac{1 - \delta - \mu}{1 - \delta} \right) (z - \delta) + \delta \right| &\leq \left(\frac{1 - \delta - \mu}{1 - \delta} \right) |z| + \delta \left(1 - \frac{1 - \delta - \mu}{1 - \delta} \right) \\ &\quad \text{(by triangle inequality)} \\ &\leq \left(1 - \frac{\mu}{1 - \delta} \right) (1 - \gamma) + \delta \left(\frac{\mu}{1 - \delta} \right) \\ &= 1 - \gamma - \left(\frac{\mu}{1 - \delta} \right) (1 - \gamma - \delta) \\ &\leq 1 - \gamma. \quad \text{(since } 1 - \gamma - \delta \geq 0 \text{)} \end{aligned}$$

Therefore,

$$|h(z)| = \left| g \left(\left(\frac{1 - \delta - \mu}{1 - \delta} \right) (z - \delta) + \delta \right) \right| \leq \frac{\lambda M}{\gamma}. \quad (4.52)$$

So $|z| \leq 1 - \gamma \Rightarrow |h(z)| \leq \frac{\lambda M}{\gamma}$.

As defined in Theorem 4.2.5, let Γ be the circle with center δ and radius $1 - \delta$. When $z \in P$, $\left(\frac{1 - \delta - \mu}{1 - \delta} \right) (z - \delta) + \delta$ lies on the arc of Γ with midpoint 1 and the same central angle as P . Its arc length equals $\frac{1 - \delta}{1 - \delta - \mu} a \geq a$. Therefore, we can apply Theorem 4.2.5 to get

$$\max_{z \in P} |g(z)| = \max_{z \in J} |h(z)| \geq \left(\frac{\delta \kappa}{\lambda M} \right)^{O(1/a)}. \quad \square$$

We will now prove our main result, which generalizes Theorem 3.1 in [BE97] to complex circles beyond the unit circle.

Theorem 4.2.7. (Generalized Theorem 3.1 in [BE97]). *Let $0 < \theta < \pi$, $0 < \tau_1 \leq 1$, and $M \geq 1$. Let A be an arc of $\partial D_{\tau_1}(1 - \tau_1)$ with central angle θ that is symmetric with respect to the real axis, passing through the point 1. Then*

$$\max_{z \in A} |f(z)| \geq \left(\frac{1}{M} \right)^{O\left(\frac{1}{\tau_1 \theta}\right)} \quad (4.53)$$

for all $f \in \mathcal{C}_{\frac{1}{2}, 1}$.

Proof. Note that the two endpoints of A are α and β , as defined in Equation (4.26).

We define a new function g , where

$$g(z) = (z - \alpha)(z - \beta)f(z). \quad (4.54)$$

In this proof, we will derive a lower bound for the maximum modulus of g on arc A , and then translate that to a lower bound for f . Note that since $f \in \mathcal{C}_{\frac{1}{2},1}$,

$$|z| \leq 1 - \gamma \Rightarrow |g(z)| \leq \frac{M|(z - \alpha)(z - \beta)|}{\gamma} \leq \frac{4M}{\gamma} \quad \text{for } 0 < \gamma \leq 1, \quad (4.55)$$

and

$$\left|g\left(\frac{1}{4M}\right)\right| \geq \left(\frac{9}{32}\right) \left|f\left(\frac{1}{4M}\right)\right| \geq \left(\frac{9}{32}\right) \left(\frac{1}{2}\right) \geq \frac{1}{8}. \quad (4.56)$$

Therefore, $g \in \mathcal{C}_{\frac{1}{8},4}$.

First, we wish to find a lower bound on $|g(z)|$ for $z \in I_0$. Recall that I_0 is the chord with endpoints α and β , as defined in Lemma 4.2.3. We split into two cases: when $0 < \tau_1 \leq \frac{3}{4}$, and when $\frac{3}{4} < \tau_1 \leq 1$.

Case 1. $0 < \tau_1 \leq \frac{3}{4}$. Note that on I_0 , $|(z - \alpha)(z - \beta)|$ is maximized when z is the midpoint of I_0 , i.e. $z = \tau_1 \cos(\theta/2) + 1 - \tau_1$. With this value of z ,

$$|(z - \alpha)(z - \beta)| = |(-\tau_1 \sin(\theta/2)i)(\tau_1 \sin(\theta/2)i)| = \tau_1^2 \sin^2(\theta/2). \quad (4.57)$$

Note that $1 - |z|$ is minimized when z is either α or β . WLOG, assume $z = \alpha$. Then

$$1 - |z| = 1 - \sqrt{1 - 2\tau_1(1 - \tau_1)(1 - \cos(\theta/2))}. \quad (4.58)$$

Now, the Taylor series of $1 - \sqrt{1 - x}$ is at least $\frac{x}{2}$ if x is non-negative. Since $2\tau_1(1 - \tau_1)(1 - \cos(\theta/2)) \geq 0$ when $0 \leq \theta \leq \pi$, we have that

$$1 - |z| \geq \tau_1(1 - \tau_1)(1 - \cos(\theta/2)). \quad (4.59)$$

So for $z \in I_0$,

$$\begin{aligned} |g(z)| &\leq \frac{M\tau_1^2 \sin^2(\theta/2)}{\tau_1(1 - \tau_1)(1 - \cos(\theta/2))} \\ &= \frac{M\tau_1 \sin^2(\theta/2)}{(1 - \tau_1)(1 - \cos(\theta/2))} \\ &= \frac{M\tau_1 \sin^2(\theta/2)}{(1 - \tau_1)(2 \sin^2(\theta/4))} \\ &= \frac{2M\tau_1 \cos^2(\theta/4)}{1 - \tau_1} \\ &\leq \frac{2M\tau_1}{1 - \tau_1} \quad (\text{since } \cos^2(\theta/4) \leq 1 \text{ for } 0 \leq \theta \leq \pi) \\ &\leq 8M\tau_1. \end{aligned} \quad (4.60)$$

Case 2. $\frac{3}{4} < \mathfrak{r}_1 \leq 1$.

WLOG, consider a point $z \in I_0$ whose distance from α is $\tau \in [0, \sin(\theta/2)]$.

Note that $z = \mathfrak{r}_1 \cos(\theta/2) + 1 - \mathfrak{r}_1 + (\mathfrak{r}_1 \sin(\theta/2) - \tau)i$. This means that

$$\begin{aligned} |z| &= \sqrt{(1 - \mathfrak{r}_1 + \mathfrak{r}_1 \cos(\theta/2))^2 + (\mathfrak{r}_1 \sin(\theta/2) - \tau)^2} \\ &= \sqrt{1 - 2\mathfrak{r}_1(1 - \mathfrak{r}_1)(1 - \cos(\theta/2)) - 2\tau\mathfrak{r}_1 \sin(\theta/2) + \tau^2}. \end{aligned} \quad (4.61)$$

Again, by Taylor series expansion, $1 - \sqrt{1 - x} \geq \frac{x}{2}$ for non-negative x . Therefore:

$$\begin{aligned} 1 - |z| &\geq \mathfrak{r}_1(1 - \mathfrak{r}_1)(1 - \cos(\theta/2)) + \tau\mathfrak{r}_1 \sin(\theta/2) - \frac{\tau^2}{2} \\ &\geq \tau\mathfrak{r}_1 \sin(\theta/2) - \frac{\tau^2}{2} \\ &\geq \tau\mathfrak{r}_1 \sin(\theta/2) - \frac{\tau \sin(\theta/2)}{2} \\ &\geq \frac{\tau \sin(\theta/2)}{4}. \end{aligned} \quad (4.62)$$

Since $|(z - \alpha)(z - \beta)| = \tau|(z - \beta)| \leq 2\tau$, we have that for $z \in I_0$,

$$|g(z)| \leq \frac{8M}{\sin(\theta/2)}. \quad (4.63)$$

Now, we can apply Lemma 4.2.3 to Equation (4.60) and Equation (4.63). Let $L = \max_{z \in I_{\theta/2}} |g(z)| = \max_{z \in A} |g(z)|$, since $I_{\theta/2}$ and A correspond to the same arc. Then:

$$\max_{z \in I_{\theta/4}} |g(z)| \leq \begin{cases} (8M\mathfrak{r}_1 L)^{1/2}, & \text{when } 0 < \mathfrak{r}_1 \leq \frac{3}{4} \\ \left(\frac{8ML}{\sin(\theta/2)}\right)^{1/2}, & \text{when } \frac{3}{4} < \mathfrak{r}_1 \leq 1 \end{cases}, \quad (4.64)$$

Next, we define G to be the open region bounded by $I_{\theta/4}$ and A . By the Maximum Modulus principle, we have that

$$\max_{z \in G} |g(z)| \leq \begin{cases} \max \left\{ L, (8M\mathfrak{r}_1 L)^{1/2} \right\}, & \text{when } 0 < \mathfrak{r}_1 \leq \frac{3}{4} \\ \max \left\{ L, \left(\frac{8ML}{\sin(\theta/2)}\right)^{1/2} \right\}, & \text{when } \frac{3}{4} < \mathfrak{r}_1 \leq 1 \end{cases}. \quad (4.65)$$

Now, we want to define an arc that passes through the region G , so that we can apply Corollary 4.2.6 and obtain a lower bound for the maximum modulus of g on this arc. Note that $I_{\theta/4}$'s point of intersection with the real axis is

$$1 - 2\mathfrak{r}_1 + 2\mathfrak{r}_1 \cos(\theta/4) \leq 1 - 2\mathfrak{r}_1 + 2\mathfrak{r}_1 \left(1 - \frac{\theta^2}{64}\right) = 1 - \frac{\mathfrak{r}_1 \theta^2}{32}. \quad (4.66)$$

Therefore, we define Γ_P as the circle with center $\frac{1}{4M}$ and radius $1 - \frac{1}{4M} - \frac{\mathfrak{r}_1 \theta^2}{64}$, as its point of intersection with the real axis is $1 - \frac{\mathfrak{r}_1 \theta^2}{64} \in (1 - \frac{\mathfrak{r}_1 \theta^2}{32}, 1)$. The choice of 64 in the denominator is arbitrary; any constant exceeding 32 is valid. We will now consider the arc $P := \Gamma_P \cap G$.

Before applying Corollary 4.2.6, we first want to show that P has arc length at least $\frac{\tau_1 \theta}{c}$ for some constant $c \geq 1$.

Let $a := \tau_1 \theta$. Recall that the length of the chord $|I_0| = 2\tau_1 \sin(\theta/2)$. Therefore,

$$\begin{aligned} |I_0| &= \frac{2 \sin(\theta/2)}{\theta} a \\ &\geq \frac{2}{\pi} a && \text{(for } 0 < \theta \leq \pi) \\ &\geq \frac{a}{c_1}. && \text{(for some } c_1 \geq 1) \end{aligned}$$

So it suffices to show that P 's length is at least $\frac{|I_0|}{c_1}$ for some $c_1 \geq 1$.

Note that $I_{\theta/4}$ has radius $2\tau_1 \cos(\theta/4) \leq 2\tau_1$. The radius of P is

$$1 - \frac{1}{4M} - \frac{\tau_1 \theta^2}{64} \geq \frac{3}{4} - \frac{\tau_1 \theta^2}{64} \geq \frac{3}{4} - \frac{\tau_1}{4}. \quad (4.67)$$

We split into two cases: $\tau_1 \leq \frac{1}{3}$, and $\tau_1 > \frac{1}{3}$.

Case 1. $\tau_1 \leq \frac{1}{3}$. In this case, we have that $\frac{3}{4} - \frac{\tau_1}{4} \geq 2\tau_1$. This means that $I_{\theta/4}$ has a smaller radius (which corresponds to greater curvature) than P , and so the endpoints of arc P are on arc A , and not on arc $I_{\theta/4}$.

In the worst case, P appears as a straight chord within G . Let $1 - t$ be the point at which this chord intersects the real axis ($t = \frac{\tau_1 \theta^2}{64}$ based on how we've defined P).

Then, by the Pythagorean Theorem, $(\tau_1 - t)^2 + \left(\frac{1}{2}|P|\right)^2 = \tau_1^2$. This means that

$$|P| = 2\sqrt{t(2\tau_1 - t)} \geq 2\sqrt{t(2\tau_1 - \tau_1)} = 2\sqrt{\tau_1 t}. \quad (4.68)$$

Note that I_0 intersects with the real axis at the point $1 - \tau_1 + \tau_1 \cos(\theta/2)$. Since $\cos(\theta/2) \geq 1 - \frac{(\theta/2)^2}{2} = 1 - \frac{\theta^2}{8}$, this means that the point of intersection with the real axis is $\geq 1 - \frac{\tau_1 \theta^2}{8} = 1 - 8t$. Therefore,

$$|I_0| \leq 2\sqrt{t(16\tau_1 - t)} \leq 2\sqrt{16\tau_1 t} = 4|P|. \quad (4.69)$$

Case 2. $\tau_1 > \frac{1}{3}$. In this case, $I_{\theta/4}$ has a larger radius than arc P , and so the endpoints of arc P are on arc $I_{\theta/4}$ and not on arc A .

Note that $I_{\theta/4}$'s radius is $2\tau_1 \cos(\theta/4) \leq 2$, and P 's radius is $1 - \frac{1}{4} - \frac{\tau_1 \theta^2}{64} \geq \frac{1}{2}$. In the worst case, we assume P has radius $\frac{1}{2}$ and $I_{\theta/4}$ has radius 2. Recall that with $t = \frac{\tau_1 \theta^2}{64}$, P intersects the real axis at point $1 - t$, and $I_{\theta/4}$ intersects the real axis at point $1 - 2t$. Therefore, P intersects $I_{\theta/4}$ at the points

$$x = \frac{-3t^2 - 5t + 3}{2t + 3}, y = \pm \frac{\sqrt{-t(t-1)(t+3)(t+4)}}{2t + 3}. \quad (4.70)$$

Note that we can approximate $\frac{1}{2}|P|$ by calculating the length of the hypotenuse between point $1 - t$ and point $(x, +y)$. By the Pythagorean theorem,

$$\left(\frac{1}{2}|P|\right)^2 \geq (1 - t - x)^2 + y^2 = \frac{t(t+4)}{2t+3} \geq t. \quad (4.71)$$

Therefore, $|P| \geq 2\sqrt{t}$. Recall that

$$|I_0| \leq 2\sqrt{16\tau_1 t} \leq 2\sqrt{16t} = 4|P|. \quad (4.72)$$

Therefore, in both cases, we've established that the arc length of P is at least $\frac{a}{c}$ for some $c \geq 1$. We can now apply Corollary 4.2.6 with our function $g \in \mathcal{C}_{\frac{1}{8},4}$, and with $\delta = \frac{1}{4M}$ and $\mu = \frac{\tau_1 \theta^2}{64}$, which gives us:

$$\max_{z \in P} |g(z)| \geq \left(\frac{1}{M}\right)^{o\left(\frac{1}{\tau_1 \theta}\right)}. \quad (4.73)$$

We can now combine Equation (4.65) and Equation (4.73) to derive our lower bound for $L = \max_{z \in A} |g(z)|$. When $\tau_1 \leq \frac{3}{4}$, we have that

$$\max \left\{ L, (8M\tau_1 L)^{1/2} \right\} \geq \max_{z \in P} |g(z)| \geq \left(\frac{1}{M}\right)^{o\left(\frac{1}{\tau_1 \theta}\right)}. \quad (4.74)$$

If $L \geq (8M\tau_1 L)^{1/2}$, then we have that $L \geq \left(\frac{1}{M}\right)^{o\left(\frac{1}{\tau_1 \theta}\right)}$. Otherwise, we have that

$$L \geq \left(\frac{1}{M}\right)^{o\left(\frac{1}{\tau_1 \theta}\right)} \left(\frac{1}{\tau_1}\right) \geq \left(\frac{1}{M}\right)^{o\left(\frac{1}{\tau_1 \theta}\right)}. \quad (4.75)$$

When $\tau_1 > \frac{3}{4}$, we have that

$$\max \left\{ L, \left(\frac{8ML}{\sin(\theta/2)}\right)^{1/2} \right\} \geq \max_{z \in P} |g(z)| \geq \left(\frac{1}{M}\right)^{o\left(\frac{1}{\tau_1 \theta}\right)}. \quad (4.76)$$

If $L \geq \left(\frac{8ML}{\sin(\theta/2)}\right)^{1/2}$, then we have that $L \geq \left(\frac{1}{M}\right)^{o\left(\frac{1}{\tau_1 \theta}\right)}$. Otherwise, we have that

$$L \geq \left(\frac{1}{M}\right)^{o\left(\frac{1}{\tau_1 \theta}\right)} \sin(\theta/2). \quad (4.77)$$

Note that for $0 < \theta < \pi$:

$$\sin(\theta/2) \geq \left(\frac{1}{4}\right)^{\left(\frac{1}{\theta}\right)} \geq \left(\frac{1}{M}\right)^{o\left(\frac{1}{\tau_1 \theta}\right)}. \quad (4.78)$$

Therefore, in both cases, we have that

$$L \geq \left(\frac{1}{M}\right)^{o\left(\frac{1}{\tau_1 \theta}\right)}. \quad (4.79)$$

Putting everything together, we have that

$$\max_{z \in A} |f(z)| \geq \frac{1}{4} \max_{z \in A} |g(z)| = \frac{L}{4} \geq \left(\frac{1}{M}\right)^{o\left(\frac{1}{\tau_1 \theta}\right)}. \quad \square$$

Note that the following corollary holds for Theorem 4.2.7:

Corollary 4.2.8. (Generalized Corollary 3.2 of [BE97]). *Let $Q(z)$ be a polynomial with complex coefficients, $\sum_{j=0}^n b_j z^j$, such that $|b_0| = 1$ and all coefficients $|b_j| \leq M$. Let $0 < \tau_1 \leq 1$, and let A be an arc of $\partial D_{\tau_1}(1 - \tau_1)$ with central angle $0 < \theta < \pi$ that is symmetric with respect to the real axis and passes through the point 1. Then there is some $w \in A$ such that*

$$|Q(w)| \geq \left(\frac{1}{M}\right)^{O\left(\frac{1}{\tau_1 \theta}\right)}. \quad (4.80)$$

Proof. Polynomials of this form are part of set $\mathcal{C}_{\frac{1}{2}, 1}$, so Theorem 4.2.7 can be applied. \square

Now, we're finally ready to prove Theorem 4.2.1. Let

$$\theta_0 = c \cdot \frac{\tau_2^{2/3} \ln^{1/3}(1/\epsilon)}{(\tau_1(1 - \tau_2^2)n)^{1/3}}, \quad \text{for some constant } c \in (0, 1). \quad (4.81)$$

Note that

$$\begin{aligned} \eta(\epsilon) &\geq \max_{-\pi < \theta \leq \pi} \exp\left(-\frac{1 - \tau_2^2}{8\tau_2^2} \theta^2 n\right) \cdot |Q_c(\tau_1 e^{i\theta} + 1 - \tau_1)| \\ &\geq \max_{-\theta_0 < \theta \leq \theta_0} \exp\left(-\frac{1 - \tau_2^2}{8\tau_2^2} \theta^2 n\right) \cdot |Q_c(\tau_1 e^{i\theta} + 1 - \tau_1)| \\ &\geq \exp\left(-\frac{1 - \tau_2^2}{8\tau_2^2} \theta_0^2 n\right) \cdot \max_{-\theta_0 < \theta \leq \theta_0} |Q_c(\tau_1 e^{i\theta} + 1 - \tau_1)|. \end{aligned} \quad (4.82)$$

If $\theta_0 \leq \frac{\pi}{2}$, we can apply Corollary 4.2.8 to our modified polynomial \tilde{Q}_c , with $M = \frac{1}{c_0}$. Note that $Q_c(z) = c_0 \tilde{Q}_c(z) > \epsilon \tilde{Q}_c(z)$, so:

$$\begin{aligned} \eta(\epsilon) &\geq \exp\left(-\frac{1 - \tau_2^2}{8\tau_2^2} \theta_0^2 n\right) \cdot \exp\left(-O\left(\frac{1}{\tau_1 \theta_0}\right) \ln(1/\epsilon)\right) \\ &\geq \exp\left(-\frac{1 - \tau_2^2}{8\tau_2^2} \theta_0^2 n - O\left(\frac{1}{\tau_1 \theta_0}\right) \ln(1/\epsilon)\right). \end{aligned} \quad (4.83)$$

Plugging in the value of θ_0 , we get that

$$\eta(\epsilon) \geq \exp\left(-O\left(\frac{\ln^{2/3}(1/\epsilon) \cdot (n(1 - \tau_2^2))^{1/3}}{(\tau_1 \tau_2)^{2/3}}\right)\right). \quad (4.84)$$

Otherwise, if $\theta_0 > \frac{\pi}{2}$, meaning that $n \ll \frac{\tau_2^2 \ln(1/\epsilon)}{\tau_1(1 - \tau_2^2)}$, we plug in $\frac{\pi}{2}$ instead to get

$$\begin{aligned} \eta(\epsilon) &\geq \exp\left(-\frac{1 - \tau_2^2}{32\tau_2^2} \pi^2 n\right) \cdot \exp\left(-O\left(\frac{2}{\tau_1 \pi}\right) \ln(1/\epsilon)\right) \\ &\geq \exp\left(-O\left(\frac{1}{\tau_1}\right) \ln(1/\epsilon)\right) \\ &\geq \epsilon^{O\left(\frac{1}{\tau_1}\right)}. \end{aligned}$$

Therefore, we have shown that

$$\eta(\epsilon) \geq \max \left\{ \epsilon^{O\left(\frac{1}{\tau_1}\right)}, \exp \left(-O \left(\frac{\ln^{2/3}(1/\epsilon) \cdot (n(1 - \tau_2^2))^{1/3}}{(\tau_1 \tau_2)^{2/3}} \right) \right) \right\}. \quad \square$$

With our choice of τ_1, τ_2 from Definition 4.1.3, It is not hard to see that the $\theta_0 \leq \frac{\pi}{2}$ case corresponds to $\delta \gg \frac{\ln(1/\epsilon)}{n}$, while $\theta_0 > \frac{\pi}{2}$ corresponds to $\delta \ll \frac{\ln(1/\epsilon)}{n}$. Therefore, by plugging in the parameters from Definition 4.1.3 into Theorem 4.2.1, and by Theorem 3.0.4, we successfully prove Theorem 1.1.1.

Remark 4.2.9. The $\text{poly}(n/\epsilon)$ factor in the low-SPAM case in Theorem 1.1.1 arises from the $\text{poly}\left(n, \frac{1}{\eta_\Lambda(\epsilon)}\right)$ factor in Theorem 4.1.6. We aim to tighten this dependence in the future.

Chapter 5

Conclusion

In this thesis, we investigate Pauli error estimation in practical quantum learning settings, presenting the first algorithm for this problem that is entanglement-free and robust to state preparation and measurement errors. To handle SPAM, we extend the reduction to Population Recovery introduced in [FO21] to the combined erasure and bit-flip channel: a previously unstudied noise model. We extend complex analysis tools to prove a runtime and sample complexity on the order of $\exp(n^{1/3})$ for an n -qubit channel. We also argue that this scaling is essentially optimal: no SPAM-tolerant algorithm can achieve asymptotically fewer channel uses.

In doing so, this thesis resolves a key open question posed in [FO21]. From a practical standpoint, it brings Pauli error estimation closer to feasibility in real-world quantum set-ups, where preparation and measurement errors are unavoidable. We hope that the techniques presented here can enable SPAM-tolerant approaches to other quantum learning problems.

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